Research Article

Some Geometric Properties of the Domain of the Double Sequential Band Matrix $B(\tilde{r}, \tilde{s})$ in the Sequence Space $\ell(p)$

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The sequence space $\ell(p)$ was introduced by Maddox (1967). Quite recently, the sequence space $\ell(\tilde{B}, p)$ of nonabsolute type has been introduced and studied which is the domain of the double sequential band matrix $B(\tilde{r}, \tilde{s})$ in the sequence space $\ell(p)$ by Nergiz and Başar (2012). The main purpose of this paper is to investigate the geometric properties of the space $\ell(\tilde{B}, p)$, like rotundity and Kadec-Klee and the uniform Opial properties. The last section of the paper is devoted to the conclusion.

1. Introduction

By ω , we denote the space of all real-valued sequences. Any vector subspace of ω is called a *sequence space*. We write ℓ_∞ , c , and c_0 for the spaces of all bounded, convergent, and null sequences, respectively. Also by bs , cs , ℓ_1 , and ℓ_p ; we denote the spaces of all bounded, convergent, absolutely convergent, and p -absolutely convergent series, respectively, where $1 < p < \infty$.

Assume here and after that (p_k) is a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$. Then, the linear space $\ell(p)$ was defined by Maddox [1] (see also Simons [2] and Nakano [3]) as follows:

$$\ell(p) = \left\{ x = (x_k) \in \omega : \sum_k |x_k|^{p_k} < \infty \right\} \quad (1)$$

$$(0 < p_k \leq H < \infty)$$

which is complete paranormed space paranormed by

$$g(x) = \left(\sum_k |x_k|^{p_k} \right)^{1/M}. \quad (2)$$

For simplicity in notation, here and in what follows, the summation without limits runs from 1 to ∞ .

Quite recently, Nergiz and Başar [4] have introduced the space $\ell(\tilde{B}, p)$ of nonabsolute type which consists of all sequences whose $B(\tilde{r}, \tilde{s})$ -transforms are in the space $\ell(p)$, where $B(\tilde{r}, \tilde{s}) = \{b_{nk}(r_k, s_k)\}$ is defined by

$$b_{nk}(r_k, s_k) = \begin{cases} r_k, & k = n, \\ s_k, & k = n - 1, \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

for all $k, n \in \mathbb{N}$, where $\tilde{r} = (r_k)$ and $\tilde{s} = (s_k)$ are the convergent sequences. We should record that the double sequential band matrices were used for determining its fine spectrum over some sequence spaces by Kumar and Srivastava in [5, 6], Panigrahi and Srivastava in [7], and Akhmedov and El-Shabrawy in [8]. The reader may refer to Nergiz and Başar [4, 9] for relevant terminology and additional references on the space $\ell(\tilde{B}, p)$, since the present paper is a natural continuation of them. Here and after, for short we write $\tilde{B}$ instead of $B(\tilde{r}, \tilde{s})$. In the special case $p_k = p$ for all $k \in \mathbb{N}$, the space $\ell(\tilde{B}, p)$ is reduced to the space $(\ell_p)_{\tilde{B}}$; that is,

$$(\ell_p)_{\tilde{B}} := \left\{ (x_k) \in \omega : \sum_k |s_{k-1}x_{k-1} + r_k x_k|^p < \infty \right\}, \quad (4)$$

$$(0 < p < \infty).$$

2. The Rotundity of the Space $\ell(\tilde{B}, p)$

The rotundity of Banach spaces is one of the most important geometric property in functional analysis. For details, the reader may refer to [10–12]. In this section, we characterize the rotundity of the space $\ell(\tilde{B}, p)$ and give some results related to this concept.

Definition 1. Let $S(X)$ be the unit sphere of a Banach space X . Then, a point $x \in S(X)$ is called an extreme point if $2x = y + z$ implies $y = z$ for every $y, z \in S(X)$. A Banach space X is said to be rotund (strictly convex) if every point of $S(X)$ is an extreme point.

Definition 2. A Banach space X is said to have Kadec-Klee property (or property (H)) if every weakly convergent sequence on the unit sphere is convergent in norm.

Definition 3. A Banach space X is said to have

- (i) the Opial property if every sequence (x_n) weakly convergent to $x_0 \in X$ satisfies

$$\liminf_{n \rightarrow \infty} \|x_n - x_0\| < \liminf_{n \rightarrow \infty} \|x_n + x\| \quad (5)$$

for every $x \in X$ with $x \neq x_0$;

- (ii) the uniform Opial property if for each $\epsilon > 0$, there exists an $r > 0$ such that

$$1 + r \leq \liminf_{n \rightarrow \infty} \|x_n + x\| \quad (6)$$

for each $x \in X$ with $\|x\| \geq \epsilon$ and each sequence (x_n) in X such that $x_n \rightarrow 0$ and $\liminf_{n \rightarrow \infty} \|x_n\| \geq 1$.

Definition 4. Let X be a real vector space. A functional $\sigma : X \rightarrow [0, \infty)$ is called a modular if

- (i) $\sigma(x) = 0$ if and only if $x = \theta$;
- (ii) $\sigma(\alpha x) = \sigma(x)$ for all scalars α with $|\alpha| = 1$;
- (iii) $\sigma(\alpha x + \beta y) \leq \sigma(x) + \sigma(y)$ for all $x, y \in X$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$;
- (iv) the modular σ is called convex if $\sigma(\alpha x + \beta y) \leq \alpha \sigma(x) + \beta \sigma(y)$ for all $x, y \in X$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$.

A modular σ on X is called

- (a) right continuous if $\lim_{\alpha \rightarrow 1^+} \sigma(\alpha x) = \sigma(x)$ for all $x \in X_\sigma$.
- (b) left continuous if $\lim_{\alpha \rightarrow 1^-} \sigma(\alpha x) = \sigma(x)$ for all $x \in X_\sigma$.
- (c) continuous if it is both right and left continuous, where

$$X_\sigma = \left\{ x \in X : \lim_{\alpha \rightarrow 0^+} \sigma(\alpha x) = 0 \right\}. \quad (7)$$

We define σ_p on $\ell(\tilde{B}, p)$ by $\sigma_p(x) = \sum_k |s_{k-1}x_{k-1} + r_k x_k|^{p_k}$. If $p_k \geq 1$ for all $k \in \mathbb{N} = \{1, 2, 3, \dots\}$, by the convexity of the function $t \mapsto |t|^{p_k}$ for each $k \in \mathbb{N}$, σ_p is a convex modular on $\ell(\tilde{B}, p)$.

Proposition 5. The modular σ_p on $\ell(\tilde{B}, p)$ satisfies the following properties with $p_k \geq 1$ for all $k \in \mathbb{N}$:

- (i) if $0 < \alpha \leq 1$, then $\alpha^M \sigma_p(x/\alpha) \leq \sigma_p(x)$ and $\sigma_p(\alpha x) \leq \alpha \sigma_p(x)$.
- (ii) If $\alpha \geq 1$, then $\sigma_p(x) \leq \alpha^M \sigma_p(x/\alpha)$.
- (iii) If $\alpha \geq 1$, then $\sigma_p(x) \leq \alpha \sigma_p(x/\alpha)$.
- (iv) The modular σ_p is continuous on the space $\ell(\tilde{B}, p)$.

Proof. Consider the modular σ_p on $\ell(\tilde{B}, p)$.

- (i) Let $0 < \alpha \leq 1$, then $\alpha^M/\alpha^{p_k} \leq 1$. So, we have

$$\begin{aligned} \alpha^M \sigma_p\left(\frac{x}{\alpha}\right) &= \alpha^M \sum_k \frac{1}{\alpha^{p_k}} |s_{k-1}x_{k-1} + r_k x_k|^{p_k} \\ &= \sum_k \frac{\alpha^M}{\alpha^{p_k}} |s_{k-1}x_{k-1} + r_k x_k|^{p_k} \\ &\leq \sum_k |s_{k-1}x_{k-1} + r_k x_k|^{p_k} = \sigma_p(x), \\ \sigma_p(\alpha x) &= \sum_k \alpha^{p_k} |s_{k-1}x_{k-1} + r_k x_k|^{p_k} \\ &\leq \alpha \sum_k |s_{k-1}x_{k-1} + r_k x_k|^{p_k} = \alpha \sigma_p(x). \end{aligned} \quad (8)$$

- (ii) Let $\alpha \geq 1$. Then, $\alpha^M/\alpha^{p_k} \geq 1$ for all $p_k \geq 1$. So, we have

$$\sigma_p(x) \leq \frac{\alpha^M}{\alpha^{p_k}} \sigma_p(x) = \alpha^M \sigma_p\left(\frac{x}{\alpha}\right). \quad (9)$$

- (iii) Let $\alpha \geq 1$. Then, $\alpha/\alpha^{p_k} \geq 1$ for all $p_k \geq 1$. So, we have

$$\begin{aligned} \sigma_p(x) &= \sum_k |s_{k-1}x_{k-1} + r_k x_k|^{p_k} \\ &\leq \sum_k \frac{\alpha}{\alpha^{p_k}} |s_{k-1}x_{k-1} + r_k x_k|^{p_k} = \alpha \sigma_p\left(\frac{x}{\alpha}\right). \end{aligned} \quad (10)$$

- (iv) By (ii) and (iii), one can immediately see for $\alpha > 1$ that

$$\sigma_p(x) \leq \alpha \sigma_p(x) \leq \sigma_p(\alpha x) \leq \alpha^M \sigma_p(x). \quad (11)$$

By passing to limit as $\alpha \rightarrow 1^+$ in (11), we have $\lim_{\alpha \rightarrow 1^+} \sigma_p(\alpha x) = \sigma_p(x)$. Hence, σ_p is right continuous. If $0 < \alpha < 1$, by (i) we have

$$\alpha^M \sigma_p(x) \leq \sigma_p(\alpha x) \leq \alpha \sigma_p(x). \quad (12)$$

By letting $\alpha \rightarrow 1^-$ in (12), we observe that $\lim_{\alpha \rightarrow 1^-} \sigma_p(\alpha x) = \sigma_p(x)$. Hence, σ_p is also left continuous, and so, it is continuous. $\square$

Proposition 6. For any $x \in \ell(\tilde{B}, p)$, the following statements hold:

- (i) if $\|x\| < 1$, then $\sigma_p(x) \leq \|x\|$.
- (ii) If $\|x\| > 1$, then $\sigma_p(x) \geq \|x\|$.
- (iii) $\|x\| = 1$ if and only if $\sigma_p(x) = 1$.
- (iv) $\|x\| < 1$ if and only if $\sigma_p(x) < 1$.
- (v) $\|x\| > 1$ if and only if $\sigma_p(x) > 1$.

Proof. Let $x \in \ell(\tilde{B}, p)$.

- (i) Let $\epsilon > 0$ be such that $0 < \epsilon < 1 - \|x\|$. By the definition of $\|\cdot\|$, there exists an $\alpha > 0$ such that $\|x\| + \epsilon > \alpha$ and $\sigma_p(x) \leq 1$. From Parts (i) and (ii) of Proposition 5, we obtain

$$\sigma_p(x) \leq \sigma_p\left[\left(\|x\| + \epsilon\right) \frac{x}{\alpha}\right] \leq (\|x\| + \epsilon) \sigma_p\left(\frac{x}{\alpha}\right) \leq \|x\| + \epsilon. \quad (13)$$

Since ϵ is arbitrary, we have (i).

- (ii) If we choose $\epsilon > 0$ such that $0 < \epsilon < 1 - (1/\|x\|)$, then $1 < (1 - \epsilon)\|x\| < \|x\|$. By the definition of $\|\cdot\|$ and Part (i) of Proposition 5, we have

$$1 < \sigma_p\left[\frac{x}{(1 - \epsilon)\|x\|}\right] \leq \frac{1}{(1 - \epsilon)\|x\|} \sigma_p(x). \quad (14)$$

So, $(1 - \epsilon)\|x\| < \sigma_p(x)$ for all $\epsilon \in (0, 1 - (1/\|x\|))$. This implies that $\|x\| < \sigma_p(x)$.

- (iii) Since σ_p is continuous, by Theorem 1.4 of [12] we directly have (iii).
- (iv) This follows from Parts (i) and (iii).
- (v) This follows from Parts (ii) and (iii). $\square$

Now, we consider the space $\ell(\tilde{B}, p)$ equipped with the Luxemburg norm given by

$$\|x\| = \inf \left\{ \alpha > 0 : \sigma_p\left(\frac{x}{\alpha}\right) \leq 1 \right\}. \quad (15)$$

Theorem 7. $\ell(\tilde{B}, p)$ is a Banach space with Luxemburg norm.

Proof. Let $S_x = \{\alpha > 0 : \sigma_p(x/\alpha) \leq 1\}$ and $\|x\| = \inf S_x$ for all $x \in \ell(\tilde{B}, p)$. Then, $S_x \subset (0, \infty)$. Therefore, $\|x\| \geq 0$ for all $x \in \ell(\tilde{B}, p)$.

For $x = \theta$, $\sigma_p(\theta) = 0$ for all $\alpha > 0$. Hence, $S_0 = (0, \infty)$ and $\|\theta\| = \inf S_0 = \inf(0, \infty) = 0$.

Let $x \neq \theta$ and $Y = \{kx : k \in \mathbb{C} \text{ and } x \in \ell(\tilde{B}, p)\}$ be a nonempty subset of $\ell(\tilde{B}, p)$. Since $Y \subsetneq S[\ell(\tilde{B}, p)]$, there exists $k_1 \in \mathbb{C}$ such that $k_1 x \notin S[\ell(\tilde{B}, p)]$. Obviously, $k_1 \neq 0$. We assume that $0 < \alpha < 1/k_1$ and $\alpha \in S_x$. Then, $(x/\alpha) \in S[\ell(\tilde{B}, p)]$. Since $|k_1 \alpha| < 1$, we get

$$k_1 x = k_1 \alpha \frac{x}{\alpha} \in S[\ell(\tilde{B}, p)] \quad (16)$$

which contradicts the assumption. Hence, we obtain that if $\alpha \in S_x$, then $\alpha > 1/|k_1|$. This means that $\|x\| \geq 1/|k_1| > 0$. Thus, we conclude that $\|x\| = 0$ if and only if $x = \theta$.

Now, let $k \neq 0$ and $\alpha \in S_{kx}$. Then, we have

$$\sigma_p\left(\frac{kx}{\alpha}\right) \leq 1, \quad \frac{kx}{\alpha} \in S[\ell(\tilde{B}, p)]. \quad (17)$$

Therefore, we obtain

$$\frac{|k|}{\alpha} x = \frac{|k|}{k} \times \frac{kx}{\alpha} \in S[\ell(\tilde{B}, p)], \quad \frac{\alpha}{|k|} \in S_x. \quad (18)$$

That is, $\|x\| \leq \alpha/|k|$ and $|k|\|x\| \leq \alpha$ for all $\alpha \in S_{kx}$. So, $|k|\|x\| \leq \|kx\|$.

If we take $1/k$ and kx instead of k and x , respectively, then we obtain that

$$\left| \frac{1}{kx} \right| \|kx\| \leq \left\| \frac{1}{k} kx \right\| = \|x\|, \quad \|kx\| \leq |k| \|x\|. \quad (19)$$

Hence, we get $\|kx\| = |k|\|x\|$. This also holds when $k = 0$.

To prove the triangle inequality, let $x, y \in \ell(\tilde{B}, p)$ and $\epsilon > 0$ be given. Then, there exist $\alpha \in S_x$ and $\beta \in S_y$ such that $\alpha < \|x\| + \epsilon$ and $\beta < \|y\| + \epsilon$. Since $S[\ell(\tilde{B}, p)]$ is convex,

$$\begin{aligned} \frac{x}{\alpha} &\in S[\ell(\tilde{B}, p)], \quad \frac{y}{\beta} \in S[\ell(\tilde{B}, p)], \\ \frac{(x+y)}{\alpha+\beta} &= \frac{\alpha}{\alpha+\beta} \left(\frac{x}{\alpha}\right) + \frac{\beta}{\alpha+\beta} \left(\frac{y}{\beta}\right) \in S[\ell(\tilde{B}, p)]. \end{aligned} \quad (20)$$

Therefore, $\alpha + \beta \in S_{x+y}$. Then, we have $\|x + y\| \leq \alpha + \beta < \|x\| + \|y\| + 2\epsilon$. Since $\epsilon > 0$ was arbitrary, we obtain $\|x + y\| \leq \|x\| + \|y\|$. Hence, $\|x\| = \inf\{\alpha > 0 : \sigma_p(x/\alpha) \leq 1\}$ is a norm on $\ell(\tilde{B}, p)$.

Now, we need to show that every Cauchy sequence in $\ell(\tilde{B}, p)$ is convergent according to the Luxemburg norm. Let $\{x_k^{(n)}\}$ be a Cauchy sequence in $\ell(\tilde{B}, p)$ and $\epsilon \in (0, 1)$. Thus, there exists n_0 such that $\|x^{(n)} - x^{(m)}\| < \epsilon$ for all $n, m \geq n_0$. By Part (i) of Proposition 6, we have

$$\sigma_p(x^{(n)} - x^{(m)}) \leq \|x^{(n)} - x^{(m)}\| < \epsilon \quad (21)$$

for all $n, m \geq n_0$. This implies that

$$\sum_k \left| \left[\tilde{B}(x^{(n)} - x^{(m)}) \right]_k \right|^{p_k} < \epsilon. \quad (22)$$

Then, for each fixed k and for all $n, m \geq n_0$,

$$\left| \left[\tilde{B}(x^{(n)} - x^{(m)}) \right]_k \right| = \left| (\tilde{B}x^{(n)})_k - (\tilde{B}x^{(m)})_k \right| < \epsilon. \quad (23)$$

Hence, the sequence $\{(\tilde{B}x^{(n)})_k\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there is a $(\tilde{B}x)_k \in \mathbb{R}$ such that $(\tilde{B}x^{(m)})_k \rightarrow (\tilde{B}x)_k$ as $m \rightarrow \infty$. Therefore, as $m \rightarrow \infty$ by (22), we have

$$\sum_k \left| \left[\tilde{B}(x^{(n)} - x) \right]_k \right|^{p_k} < \epsilon \quad (24)$$

for all $n \geq n_0$.

Now, we have to show that (x_k) is an element of $\ell(\tilde{B}, p)$. Since $(\tilde{B}x^{(m)})_k \rightarrow (\tilde{B}x)_k$ as $m \rightarrow \infty$, we have

$$\lim_{m \rightarrow \infty} \sigma_p(x^{(n)} - x^{(m)}) = \sigma_p(x^{(n)} - x). \quad (25)$$

Then, we see by (21) that $\sigma_p(x^{(n)} - x) \leq \|x^{(n)} - x\| < \epsilon$ for all $n \geq n_0$. This implies that $x^{(n)} \rightarrow x$ as $n \rightarrow \infty$. So, we have $x = x^{(n)} - (x^{(n)} - x) \in \ell(\tilde{B}, p)$. Therefore, the sequence space $\ell(\tilde{B}, p)$ is complete with respect to Luxemburg norm. This completes the proof. $\square$

Theorem 8. *The space $\ell(\tilde{B}, p)$ is rotund if and only if $p_k > 1$ for all $k \in \mathbb{N}$.*

Proof. Let $\ell(\tilde{B}, p)$ be rotund and choose $k \in \mathbb{N}$ such that $p_k = 1$ for $k < 3$. Consider the following sequences given by

$$\begin{aligned} x &= \left(0, \frac{1}{r_1}, \frac{-s_1}{r_1 r_2}, \frac{s_1 s_2}{r_1 r_2 r_3}, \dots\right), \\ y &= \left(0, 0, \frac{1}{r_2}, \frac{-s_2}{r_2 r_3}, \frac{s_2 s_3}{r_2 r_3 r_4}, \dots\right). \end{aligned} \quad (26)$$

Then, obviously $x \neq y$ and

$$\sigma_p(x) = \sigma_p(y) = \sigma_p\left(\frac{x+y}{2}\right) = 1. \quad (27)$$

By Part (iii) of Proposition 6, $x, y, (x+y)/2 \in S[\ell(\tilde{B}, p)]$ which leads us to the contradiction that the sequence space $\ell(\tilde{B}, p)$ is not rotund. Hence, $p_k > 1$ for all $k \in \mathbb{N}$.

Conversely, let $x \in S[\ell(\tilde{B}, p)]$ and $v, z \in S[\ell(\tilde{B}, p)]$ with $x = (v+z)/2$. By convexity of σ_p and Part (iii) of Proposition 6, we have

$$1 = \sigma_p(x) \leq \frac{\sigma_p(v) + \sigma_p(z)}{2} \leq \frac{1}{2} + \frac{1}{2} = 1, \quad (28)$$

which gives that $\sigma_p(v) = \sigma_p(z) = 1$, and

$$\sigma_p(x) = \frac{\sigma_p(v) + \sigma_p(z)}{2}. \quad (29)$$

Also, we obtain from (29) that

$$\begin{aligned} \sum_k |s_{k-1}x_{k-1} + r_k x_k|^{p_k} &= \frac{1}{2} \left(\sum_k |s_{k-1}v_{k-1} + r_k v_k|^{p_k} \right. \\ &\quad \left. + \sum_k |s_{k-1}z_{k-1} + r_k z_k|^{p_k} \right). \end{aligned} \quad (30)$$

Since $x = (v+z)/2$, we have

$$\begin{aligned} &\sum_k |s_{k-1}(v_{k-1} + z_{k-1}) + r_k(v_k + z_k)|^{p_k} \\ &= \frac{1}{2} \left(\sum_k |s_{k-1}v_{k-1} + r_k v_k|^{p_k} + \sum_k |s_{k-1}z_{k-1} + r_k z_k|^{p_k} \right). \end{aligned} \quad (31)$$

This implies that

$$\begin{aligned} &|s_{k-1}(v_{k-1} + z_{k-1}) + r_k(v_k + z_k)|^{p_k} \\ &= \frac{1}{2} |s_{k-1}v_{k-1} + r_k v_k|^{p_k} + \frac{1}{2} |s_{k-1}z_{k-1} + r_k z_k|^{p_k} \end{aligned} \quad (32)$$

for all $k \in \mathbb{N}$. Since the function $t \mapsto |t|^{p_k}$ is strictly convex for all $k \in \mathbb{N}$, it follows by (32) that $v_k = z_k$ for all $k \in \mathbb{N}$. Hence, $v = z$. That is, the sequence space $\ell(\tilde{B}, p)$ is rotund. $\square$

Theorem 9. *Let $x \in \ell(\tilde{B}, p)$. Then, the following statements hold:*

- (i) $0 < \alpha < 1$ and $\|x\| > \alpha$ imply $\sigma_p(x) > \alpha^M$.
- (ii) $\alpha \geq 1$ and $\|x\| < \alpha$ imply $\sigma_p(x) < \alpha^M$.

Proof. Let $x \in \ell(\tilde{B}, p)$.

- (i) Suppose that $\|x\| > \alpha$ with $0 < \alpha < 1$. Then, $\|x/\alpha\| > 1$. By Part (ii) of Proposition 6, $\|x/\alpha\| > 1$ implies $\sigma_p(x/\alpha) \geq \|x/\alpha\| > 1$. That is, $\sigma_p(x/\alpha) > 1$. Since $0 < \alpha < 1$, by Part (i) of Proposition 5, we get $\alpha^M \sigma_p(x/\alpha) \leq \sigma_p(x)$. Thus, we have $\alpha^M < \sigma_p(x)$.
- (ii) Let $\|x\| < \alpha$ and $\alpha \geq 1$. Then, $\|x/\alpha\| < 1$. By Part (i) of Proposition 6, $\|x/\alpha\| < 1$ implies $\sigma_p(x/\alpha) \leq \|x/\alpha\| < 1$. That is, $\sigma_p(x/\alpha) < 1$. If $\alpha = 1$, then $\sigma_p(x/\alpha) = \sigma_p(x) < 1 = \alpha^M$. If $\alpha > 1$, then by Part (ii) of Proposition 5, we have $\sigma_p(x) \leq \alpha^M \sigma_p(x/\alpha)$. This means that $\sigma_p(x) < \alpha^M$. $\square$

Theorem 10. *Let (x_n) be a sequence in $\ell(\tilde{B}, p)$. Then, the following statements hold:*

- (i) $\lim_{n \rightarrow \infty} \|x_n\| = 1$ implies $\lim_{n \rightarrow \infty} \sigma_p(x_n) = 1$.
- (ii) $\lim_{n \rightarrow \infty} \sigma_p(x_n) = 0$ implies $\lim_{n \rightarrow \infty} \|x_n\| = 0$.

Proof. Let (x_n) be a sequence in $\ell(\tilde{B}, p)$.

- (i) Let $\lim_{n \rightarrow \infty} \|x_n\| = 1$ and $\epsilon \in (0, 1)$. Then, there exists $n_0 \in \mathbb{N}$ such that $1 - \epsilon < \|x_n\| < \epsilon + 1$ for all $n \geq n_0$. By Parts (i) and (ii) of Theorem 9, $1 - \epsilon < \|x_n\|$ implies $\sigma_p(x_n) > (1 - \epsilon)^M$ and $\|x_n\| < \epsilon + 1$ implies $\sigma_p(x_n) < (1 + \epsilon)^M$ for all $n \geq n_0$. This means $\epsilon \in (0, 1)$ and for all $n \geq n_0$ there exists $n_0 \in \mathbb{N}$ such that $(1 - \epsilon)^M < \sigma_p(x_n) < (1 + \epsilon)^M$. That is, $\lim_{n \rightarrow \infty} \sigma_p(x_n) = 1$.
- (ii) We assume that $\lim_{n \rightarrow \infty} \|x_n\| \neq 0$ and $\epsilon \in (0, 1)$. Then, there exists a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| > \epsilon$ for all $k \in \mathbb{N}$. By Part (i) of Theorem 9, $0 < \epsilon < 1$ and $\|x_{n_k}\| > \epsilon$ imply $\sigma_p(x_{n_k}) > \epsilon^M$. Thus, $\lim_{n \rightarrow \infty} \sigma_p(x_n) \neq 0$ for all $k \in \mathbb{N}$. Hence, we obtain that $\lim_{n \rightarrow \infty} \sigma_p(x_n) = 0$ implies $\lim_{n \rightarrow \infty} \|x_n\| = 0$. $\square$

Theorem 11. *Let $x \in \ell(\tilde{B}, p)$ and $(x^{(n)}) \subset \ell(\tilde{B}, p)$. If $\sigma_p(x^{(n)}) \rightarrow \sigma_p(x)$ as $n \rightarrow \infty$ and $x_k^{(n)} \rightarrow x_k$ as $n \rightarrow \infty$ for all $k \in \mathbb{N}$, then $x^{(n)} \rightarrow x$ as $n \rightarrow \infty$.*

Proof. Let $\epsilon > 0$ be given. Since $\sigma_p(x) = \sum_k |(\tilde{B}x)_k|^{p_k} < \infty$, there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0+1}^{\infty} |(\tilde{B}x)_k|^{p_k} < \frac{\epsilon}{3(2^{M+1})}. \quad (33)$$

It follows from the fact

$$\lim_{n \rightarrow \infty} \left[\sigma_p(x^{(n)}) - \sum_{k=1}^{k_0} |(\tilde{B}x^{(n)})_k|^{p_k} \right] = \sigma_p(x) - \sum_{k=1}^{k_0} |(\tilde{B}x)_k|^{p_k} \quad (34)$$

that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $k \in \mathbb{N}$,

$$\begin{aligned} & \sigma_p(x_{n_k}) - \sum_{k=1}^{k_0} |(\tilde{B}x^{(n)})_k|^{p_k} \\ & < \sigma_p(x) - \sum_{k=1}^{k_0} |(\tilde{B}x)_k|^{p_k} + \frac{\epsilon}{3(2^M)}, \end{aligned} \quad (35)$$

and for all $n \geq n_0$,

$$\sum_{k=1}^{k_0} |\{\tilde{B}(x^{(n)} - x)\}_k|^{p_k} < \frac{\epsilon}{3}. \quad (36)$$

Therefore, we obtain from (33), (35), and (36) that

$$\begin{aligned} \sigma_p(x_n - x) &= \sum_{k=1}^{\infty} |\{\tilde{B}(x^{(n)} - x)\}_k|^{p_k} \\ &< \sum_{k=1}^{k_0} |\{\tilde{B}(x^{(n)} - x)\}_k|^{p_k} \\ &\quad + \sum_{k=k_0+1}^{\infty} |\{\tilde{B}(x^{(n)} - x)\}_k|^{p_k} \\ &< \frac{\epsilon}{3} + 2^M \left[\sum_{k=k_0+1}^{\infty} |(\tilde{B}x^{(n)})_k|^{p_k} \right. \\ &\quad \left. + \sum_{k=k_0+1}^{\infty} |(\tilde{B}x)_k|^{p_k} \right] \end{aligned}$$

$$\begin{aligned} &< \frac{\epsilon}{3} + 2^M \left[\sigma_p(x_n) - \sum_{k=1}^{k_0} |(\tilde{B}x^{(n)})_k|^{p_k} \right. \\ &\quad \left. + \sum_{k=k_0+1}^{\infty} |(\tilde{B}x)_k|^{p_k} \right] \\ &< \frac{\epsilon}{3} + 2^M \left[\sigma_p(x) - \sum_{k=1}^{k_0} |(\tilde{B}x)_k|^{p_k} \right. \\ &\quad \left. + \frac{\epsilon}{3(2^M)} + \sum_{k=k_0+1}^{\infty} |(\tilde{B}x)_k|^{p_k} \right] \\ &< \frac{\epsilon}{3} + 2^M \left[2 \sum_{k=k_0+1}^{\infty} |(\tilde{B}x)_k|^{p_k} + \frac{\epsilon}{3(2^M)} \right] \\ &< \frac{\epsilon}{3} + 2^M \left[2 \frac{\epsilon}{3(2^{M+1})} + \frac{\epsilon}{3(2^M)} \right] = \epsilon. \end{aligned} \quad (37)$$

This means that $\sigma_p(x^{(n)} - x) \rightarrow 0$ as $n \rightarrow \infty$. By Part (ii) of Theorem 10, $\sigma_p(x^{(n)} - x) \rightarrow 0$ as $n \rightarrow \infty$ implies $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$. Hence, $x_n \rightarrow x$ as $n \rightarrow \infty$. $\square$

Theorem 12. *The sequence space $\ell(\tilde{B}, p)$ has the Kadec-Klee property.*

Proof. Let $x \in S[\ell(\tilde{B}, p)]$ and $(x^{(n)}) \subset \ell(\tilde{B}, p)$ such that $\|x^{(n)}\| \rightarrow 1$ and $x^{(n)} \xrightarrow{w} x$ are given. By Part (ii) of Theorem 10, we have $\sigma_p(x^{(n)}) \rightarrow 1$ as $n \rightarrow \infty$. Also $x \in S[\ell(\tilde{B}, p)]$ implies $\|x\| = 1$. By Part (iii) of Proposition 6, we obtain $\sigma_p(x) = 1$. Therefore, we have $\sigma_p(x^{(n)}) \rightarrow \sigma_p(x)$ as $n \rightarrow \infty$.

Since $x^{(n)} \xrightarrow{w} x$ and $q_k : \ell(\tilde{B}, p) \rightarrow \mathbb{R}$ defined by $q_k(x) = x_k$ is continuous, $x_k^{(n)} \rightarrow x_k$ as $n \rightarrow \infty$ for all $k \in \mathbb{N}$. Therefore, $x^{(n)} \rightarrow x$ as $n \rightarrow \infty$.

Since any weakly convergent sequence in $\ell(\tilde{B}, p)$ is convergent, the sequence space $\ell(\tilde{B}, p)$ has the Kadec-Klee property. $\square$

Theorem 13. *For any $1 < p < \infty$, the space $(\ell_p)_{\tilde{B}}$ has the uniform Opial property.*

Proof. Let $\epsilon > 0$ and $\epsilon_0 \in (0, \epsilon)$ be given such that $1 + (\epsilon^p/2) > (1 + \epsilon_0^p)^p$. Also let $x \in (\ell_p)_{\tilde{B}}$ and $\|x\| \geq \epsilon$. There exists $k_1 \in \mathbb{N}$ such that

$$\sum_{k=k_1+1}^{\infty} |(\tilde{B}x)_k|^p < \left(\frac{\epsilon_0}{4}\right)^p. \quad (38)$$

Hence, we have

$$\left\| \sum_{k=k_1+1}^{\infty} x_k e_k \right\| < \frac{\epsilon_0}{4}. \quad (39)$$

Furthermore, we have

$$\begin{aligned}\epsilon^p &\leq \sum_{k=1}^{k_1} |(\tilde{B}x)_k|^p + \sum_{k=k_1+1}^{\infty} |(\tilde{B}x)_k|^p \\ &< \sum_{k=1}^{k_1} |(\tilde{B}x)_k|^p + \left(\frac{\epsilon_0}{4}\right)^p \\ &< \sum_{k=1}^{k_1} |(\tilde{B}x)_k|^p + \frac{\epsilon^p}{4},\end{aligned}\quad (40)$$

which yields that

$$\frac{3\epsilon^p}{4} < \sum_{k=1}^{k_1} |(\tilde{B}x)_k|^p. \quad (41)$$

For any weakly null sequence $(x^{(m)}) \subset S[(\ell_p)_{\tilde{B}}]$, since $x_k^{(m)} \rightarrow 0$ as $m \rightarrow \infty$ for each $k \in \mathbb{N}$, there exists $m_0 \in \mathbb{N}$ such that for all $m > m_0$,

$$\left\| \sum_{k=1}^{k_1} x_k^{(m)} e_k \right\| < \frac{\epsilon^p}{4}. \quad (42)$$

Therefore, for all $m > m_0$,

$$\begin{aligned}\|x^{(m)} + x\| &= \left\| \sum_{k=1}^{k_1} (x_k^{(m)} + x_k) e_k + \sum_{k=k_1+1}^{\infty} (x_k^{(m)} + x_k) e_k \right\| \\ &\geq \left\| \sum_{k=1}^{k_1} x_k e_k + \sum_{k=k_1+1}^{\infty} x_k^{(m)} e_k \right\| \\ &\quad - \left\| \sum_{k=1}^{k_1} x_k^{(m)} e_k \right\| - \left\| \sum_{k=k_1+1}^{\infty} x_k e_k \right\| \\ &\geq \left\| \sum_{k=1}^{k_1} x_k e_k + \sum_{k=k_1+1}^{\infty} x_k^{(m)} e_k \right\| - \frac{\epsilon^p}{4} - \frac{\epsilon^p}{4}.\end{aligned}\quad (43)$$

Moreover,

$$\begin{aligned}&\left\| \sum_{k=1}^{k_1} x_k e_k + \sum_{k=k_1+1}^{\infty} x_k^{(m)} e_k \right\|^p \\ &= \sum_{k=1}^{k_1} |(\tilde{B}x)_k e_k|^p + \sum_{k=k_1+1}^{\infty} |(\tilde{B}x^{(m)})_k e_k|^p \\ &\geq \frac{3\epsilon^p}{4} + \left(1 - \frac{\epsilon^p}{4}\right) \\ &= 1 + \frac{\epsilon^p}{2} \\ &> (1 + \epsilon_0)^p.\end{aligned}\quad (44)$$

Then, we have

$$\begin{aligned}\|x^{(m)} + x\| &\geq \left\| \sum_{k=1}^{k_1} x_k e_k + \sum_{k=k_1+1}^{\infty} x_k^{(m)} e_k \right\| - \frac{\epsilon^p}{2} \\ &\geq 1 + \epsilon_0 - \frac{\epsilon^p}{2} \\ &> 1 + \frac{\epsilon_0^p}{2}.\end{aligned}\quad (45)$$

This means that $(\ell_p)_{\tilde{B}}$ has the uniform Opial property. $\square$

3. Conclusion

The sequence spaces $bv(u, p)$ and $bv_{\infty}(u, p)$ of nonabsolute type consisting of all sequences $x = (x_k)$ such that $\{u_k(x_k - x_{k-1})\}$ is in the Maddox' spaces $\ell(p)$ and $\ell_{\infty}(p)$ were introduced by Başar et al. [13], where $u = (u_k)$ is a sequence such that $u_k \neq 0$ for all $k \in \mathbb{N}$ and the rotundity of the space $bv(u, p)$ was examined.

The sequence space $a^r(u, p)$ of nonabsolute type consisting of all sequences $x = (x_k)$ such that $A^r x = \{\sum_{k=0}^n (1 + r^k) x_k / (n+1)\} \in \ell(p)$ was studied by Aydın and Başar [14], and some results related to the rotundity of the space $a^r(u, p)$ were given.

Quite recently, the sequence space $\tilde{\ell}(p)$ of nonabsolute type consisting of all sequences $x = (x_k)$ such that $B(r, s)x = (sx_{k-1} + rx_k) \in \ell(p)$ was defined by Aydın and Başar [15], and emphasized the rotundity of the space $\tilde{\ell}(p)$ together with some related results.

Although the sequence spaces $a^r(u, p)$ and $\ell(\tilde{B}, p)$ are not comparable, since the double sequential band matrix $B(\tilde{r}, \tilde{s})$ reduces to the generalized difference matrix $B(r, s)$ in the special case $\tilde{r} = re$ and $\tilde{s} = se$, the new space $\ell(\tilde{B}, p)$ is more general than the space $\tilde{\ell}(p)$. Similarly, the sequence space $\ell(\tilde{B}, p)$ is also reduced to the space $bv(u, p)$ in the case $\tilde{r} = (u_k)$ and $\tilde{s} = (-u_k)$. So, the results on the space $\ell(\tilde{B}, p)$ are much more comprehensive than the results on the space $bv(u, p)$. Additionally, the corresponding theorems on the Kadec-Klee property of the space $\ell(\tilde{B}, p)$ and the uniform Opial property of the space $(\ell_p)_{\tilde{B}}$ were not given by Başar et al. [13] and Aydın and Başar [15] which make the present paper significant.

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