Research Article

Boundedness for a Class of Generalized Commutators of Fractional Hardy Operators with a Rough Kernel

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The authors consider the generalized commutator of fractional Hardy operator with a rough kernel as follows: $\mathcal{H}_{\Omega, A, \beta}^m f(x) = 1/(|x|^{n-\beta}) \int_{|y| < |x|} (\Omega(x-y)/|x-y|^{m-1}) R_m(A; x, y) f(y) dy$, where $\Omega \in L^1(\mathbb{S}^{n-1})$, $0 \leq \beta < n$, and $R_m(A; x, y) = A(x) - \sum_{|\gamma| < m} (1/\gamma!) D^\gamma A(y) (x-y)^\gamma$ with $m \in \mathbb{Z}^+$. The authors prove that $\mathcal{H}_{\Omega, A, \beta}^m$ is bounded on Herz type space and λ -Central Morrey space with $m \geq 1$, which is an open problem for $m > 2$.

1. Introduction

It is well known that the C-Z singular integrals and their commutators have been studied a lot by many mathematicians; please see [1] or [2] for more details. For the generalizations of the commutators of singular integrals, Cohen [3] studied the following generalized commutator T_A^2 which is defined by

$$T_A^2 f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}} (A(x) - A(y) - \nabla A(y)(x-y)) f(y) dy, \quad (1)$$

where $\Omega \in L^1(\mathbb{S}^{n-1})$ is homogeneous of degree zero and satisfies the moment condition

$$\int_{\mathbb{S}^{n-1}} \Omega(x) x^\gamma d\sigma(x) = 0 \quad (2)$$

with $|\gamma| = 1$. Cohen [3] proved that if $\Omega \in \text{Lip}_1(\mathbb{S}^{n-1})$ and $\nabla A \in \text{BMO}$, then T_A^2 is bounded on $L^p(\mathbb{R}^n)$ with $1 < p < \infty$. Later, Cohen and Gosselin [4] considered another type of generalized commutator as follows:

$$T_A^m f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A; x, y) f(y) dy, \quad (3)$$

where $R_m(A; x, y)$ ($m \in \mathbb{Z}^+$) is defined by $R_m(A; x, y) = A(x) - \sum_{|\gamma| < m} (1/\gamma!) D^\gamma A(y) (x-y)^\gamma$, the m th remainder of Taylor series of the function A at y about x , and Ω satisfies the following moment condition:

$$\int_{\mathbb{S}^{n-1}} \Omega(x) x^\gamma d\sigma(x) = 0, \quad (4)$$

with $|\gamma| = m-1$. Obviously, if we choose $m = 1$, T_A^m becomes $[A, T]$, the commutator of T generalized by A and T . Furthermore, T_A^m becomes T_A^2 if we choose $m = 2$.

Cohen and Gosselin proved that if $m \geq 2$, $\Omega \in \text{Lip}_1(\mathbb{S}^{n-1})$, and the function A has derivatives of order $m-1$ in $\text{BMO}(\mathbb{R}^n)$, then the operator T_A^m is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Later, T_A^m was studied by many mathematicians; please see [5, 6] or [7] for more details. Recently, Wang and Zhang [8] gave a new proof of Wu's theorem in [9] by using the $W^{1,p}$ estimate for the elliptic equation of divergence form with partially BMO coefficients and the L^p boundedness of the Cohen-Gosselin type generalized commutators proved by Yan in [6]. Furthermore, the method used in [8] is much simpler than that in [9]. Recently, Yu and Tao [7] proved that T_A^m is bounded on λ -Central Morrey space.

Let f be a nonnegative integral on $\mathbb{R}^+$: then the Hardy operator is defined by

$$Hf(x) = \frac{1}{x} \int_0^x f(t) dt, \quad x \neq 0. \quad (5)$$

In 1920, Hardy [10] proved the following inequality:

$$\|Hf\|_{L^p(\mathbb{R}^+)} \leq \frac{p}{p-1} \|f\|_{L^p(\mathbb{R}^+)}, \quad (6)$$

where $1 < p < \infty$ and the constant $p/(p-1)$ is the best possible.

In 2007, Fu et al. [11] introduced the n -dimensional fractional type Hardy operator $\mathcal{H}_\beta$ as follows:

$$\mathcal{H}_\beta f(x) = \frac{1}{|x|^{n-\beta}} \int_{|t|<|x|} f(t) dt, \quad x \in \mathbb{R}^n \setminus \{0\}, \quad (7)$$

where $-n < \beta < n$ and f is a locally integrable function on $\mathbb{R}^n$.

Obviously, when $\beta = 0$, $\mathcal{H}_0$ is just the n -dimensional Hardy operator $\mathcal{H}$ which was proposed by Christ and Grafakos in [12].

In [11], the authors gave the characterization of the $\text{CBMO}^q(\mathbb{R}^n)$ by the boundedness of the commutator of the fractional type Hardy operator $[\mathcal{H}_\beta, b]$ on Herz type spaces. Here the $\text{CBMO}^q(\mathbb{R}^n)$ space is defined by the following.

Definition 1 (see [13]). Let $1 \leq q < \infty$. A function $f \in L_{\text{loc}}^q(\mathbb{R}^n)$ is said to belong to the homogeneous Central BMO space $\text{CBMO}^q(\mathbb{R}^n)$ if

$$\|b\|_{\text{CBMO}^q(\mathbb{R}^n)} := \sup_{r>0} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x) - f_B|^q dx \right)^{1/q} < \infty, \quad (8)$$

where $f_B = (1/|B(0,r)|) \int_{B(0,r)} f(x) dx$.

From [14], we know that $\text{BMO}(\mathbb{R}^n) \subset \text{CBMO}^q(\mathbb{R}^n)$ for $1 \leq q < \infty$.

The $\text{CBMO}^q(\mathbb{R}^n)$ space can be regarded as the space of bounded mean oscillation, a local version of $\text{BMO}(\mathbb{R}^n)$ at the origin. But the famous John-Nirenberg inequality no longer holds in $\text{CBMO}^q(\mathbb{R}^n)$.

Now we are interested in the following generalized commutator of Hardy operator:

$$\mathcal{H}_A^m f(x) = \frac{1}{|x|^n} \int_{|y|<|x|} \frac{1}{|x-y|^{m-1}} f(y) R_m(A; x, y) dy, \quad (9)$$

where $R_m(A; x, y) = A(x) - \sum_{|\gamma|<m} (1/\gamma!) D^\gamma A(y) (x-y)^\gamma$ and $m \in \mathbb{Z}^+$.

In 2010, Lu and Zhao [15] proved that when $m = 2$, $\mathcal{H}_A^2$ is bounded on Herz type space and Morrey-Herz type space. Later, Gao and Yu [16] proved that $\mathcal{H}_A^2$ is bounded on λ -Central Morrey spaces. However, we would like to point out

that the method used in [15, 16] cannot apply to the case when $m > 2$. An interesting question is whether the boundedness of $\mathcal{H}_A^m$ on Herz type space or λ -Central Morrey space still holds with $m > 2$. In this paper, we will use a different method to answer this question. Furthermore, we will consider the generalized commutator of fractional Hardy operator with a rough kernel as follows:

$$\begin{aligned} & \mathcal{H}_{\Omega, A, \beta}^m f(x) \\ &= \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} \frac{\Omega(x-y)}{|x-y|^{m-1}} f(y) R_m(A; x, y) dy, \end{aligned} \quad (10)$$

where $m \in \mathbb{Z}^+$, $0 \leq \beta < n$, and $\Omega \in L^r(\mathbb{S}^{n-1})$.

In [17], we prove that $\mathcal{H}_{\Omega, A, \beta}^m$ is bounded from L^p to L^q with $1/p - 1/q = \beta/n$. Furthermore, we study the endpoint estimates of $\mathcal{H}_{\Omega, A, \beta}^m$ on H^1 spaces with $\Omega \in \text{Lip}_1(\mathbb{S}^{n-1})$ in [17]. In this paper, we will prove that $\mathcal{H}_{\Omega, A, \beta}^m$ is bounded on Herz type space and λ -Central Morrey space when $\Omega \in L^r(\mathbb{S}^{n-1})$.

2. Boundedness of $\mathcal{H}_{\Omega, A, \beta}^m$ on Herz Type Spaces

In this section, we will give the boundedness of $\mathcal{H}_{\Omega, A, \beta}^m$ on Herz type spaces. First we introduce some notations that will be used throughout this paper.

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $C_k = B_k \setminus B_{k-1}$, and $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$: here χ_{C_k} is the characteristic function of the set C_k .

Definition 2 (see [18]). Let $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$. Then the homogeneous Herz type space $\dot{K}_q^{\alpha, p}(\mathbb{R}^n)$ is defined by

$$\dot{K}_q^{\alpha, p}(\mathbb{R}^n) = \left\{ f \in L_{\text{loc}}^q(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha, p}(\mathbb{R}^n)} < \infty \right\}, \quad (11)$$

where $\|f\|_{\dot{K}_q^{\alpha, p}(\mathbb{R}^n)}$ is defined as

$$\|f\|_{\dot{K}_q^{\alpha, p}(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f \chi_k\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p} \quad (12)$$

with the usual modifications made when $p = \infty$ or $q = \infty$.

Now we show our main results in this section.

Theorem 3. Suppose $m \geq 2$, $\Omega \in L^r(\mathbb{S}^{n-1})$ with $1 < r < \infty$, and A has derivatives of order $m-1$ in CBMO^{p_2} with $n < p_2 < \infty$. Let $0 < s \leq p < \infty$, $1 < q, p_1 < \infty$, $1/q = 1/p_1 + 1/p_2 - \beta/n$ with $0 \leq \beta < n$. If $1/r' - 1/q - \beta/n > 0$, $r > p_1'$, $\alpha_2 = \alpha_1 + n/p_2$ and α_2 satisfies the following condition:

$$\alpha_2 + n/q - n/r' - 1/r - n/p_2 + \beta < 0, \quad (13)$$

then there exists a constant C , such that

$$\|\mathcal{H}_{\Omega, A, \beta}^m f\|_{\dot{K}_q^{\alpha_1, s}} \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\text{CBMO}^{p_2}} \|f\|_{\dot{K}_{p_1}^{\alpha_2, p}}. \quad (14)$$

For $m = 1$, $\mathcal{H}_{\Omega, A, \beta}^1$ is just the commutator of Hardy operator; that is, $\mathcal{H}_{\Omega, A, \beta}^1 = \mathcal{H}_{\Omega, \beta}^A f(x) = A(x) \mathcal{H}_{\Omega, \beta} f(x) - \mathcal{H}_{\Omega, \beta}(Af)(x)$. We have the following theorem of $\mathcal{H}_{\Omega, \beta}^A$ on Herz type space.

Theorem 4. Suppose $\Omega \in L^r(\mathbb{S}^{n-1})$ with $1 < r < \infty$ and $A \in CBMO^{p_2}$. Let $0 < s \leq p < \infty$, $1 < q, p_1, p_2 < \infty$, $1/q = 1/p_1 + 1/p_2 - \beta/n$ with $0 \leq \beta < n$. If $1/r' - 1/q - \beta/n > 0$, $r > p_1'$, $\alpha_2 = \alpha_1 + n/p_2$, and α_2 satisfies (13), then there exists a constant C , such that

$$\|\mathcal{H}_{\Omega, \beta}^A f\|_{\dot{K}_{q, s}^{\alpha_1}} \leq C \|A\|_{CBMO^{p_2}} \|f\|_{\dot{K}_{p_1}^{\alpha_2, p}}. \quad (15)$$

Remark 5. Comparing Theorems 3 and 4, we find that the restrictions on α_1 and α_2 are more rigid in Theorem 4 than in Theorem 3, which indicates that $\mathcal{H}_{\Omega, A, \beta}^m$ with $m \geq 2$ has better properties than the commutators.

In order to prove Theorems 3 and 4, we need the following lemmas.

Lemma 6 (see [19]). Let $1 < p_1, p_2 < \infty$, $0 \leq \beta < n$, and $\beta/n = 1/p_1 - 1/p_2$. If $\Omega \in L^r(\mathbb{S}^{n-1})$ with $r > p_1'$, then there exists a constant C independent of f , such that

$$\|\mathcal{H}_{\Omega, \beta} f\|_{L^{p_2}} \leq C \|f\|_{L^{p_1}}, \quad (16)$$

where $\mathcal{H}_{\Omega, \beta}$ is defined by

$$\mathcal{H}_{\Omega, \beta} f(x) = \frac{1}{|x|^{n-\beta}} \int_{|y| < |x|} \Omega(x-y) f(y) dy. \quad (17)$$

By checking [19] carefully, one can draw the conclusion that if one replaces $\mathcal{H}_{\Omega, \beta} f(x)$ by $\mathcal{H}_{|\Omega|, \beta} |f|(x)$, then (16) still holds.

Lemma 7. Let $m \geq 1$, $1 < p_1 < p_2 < \infty$ and $0 \leq \beta < n$. If A has derivatives of order $m-1$ in $L^r(\mathbb{R}^n)$ with $1/p_2 = 1/p_1 + 1/r - \beta/n$ and $r > p_1'$, then one has

$$\|\mathcal{H}_{\Omega, A, \beta}^m f\|_{L^{p_2}} \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{L^r} \|f\|_{L^{p_1}}, \quad (18)$$

where the constant C is independent of f and A .

Proof. From [20, p. 241], we have the following estimates:

$$\begin{aligned} \frac{R_m(A; x, y)}{|x-y|^{m-1}} &\leq \frac{R_{m-1}(A; x, y)}{|x-y|^{m-1}} + C \sum_{|\gamma|=m-1} |D^\gamma A(y)| \\ &\leq C \sum_{|\gamma|=m-1} ((D^\gamma A)^*(x) + (D^\gamma A)^*(y)), \end{aligned} \quad (19)$$

where $m \geq 1$ and $(f)^*$ is the Hardy-Littlewood maximal function of f .

Thus we obtain

$$\begin{aligned} &|\mathcal{H}_{\Omega, A, \beta}^m f(x)| \\ &= \left| \frac{1}{|x|^{n-\beta}} \int_{|y| < |x|} \frac{\Omega(x-y) f(y)}{|x-y|^{m-1}} R_m(A; x, y) dy \right| \\ &\leq C \frac{1}{|x|^{n-\beta}} \int_{|y| < |x|} |f(y)| |\Omega(x-y)| \\ &\quad \times \sum_{|\gamma|=m-1} ((D^\gamma A)^*(x) + (D^\gamma A)^*(y)) dy \\ &\leq C \sum_{|\gamma|=m-1} [(D^\gamma A)^*(x) \mathcal{H}_{|\Omega|, \beta} |f|(x) \\ &\quad + \mathcal{H}_{|\Omega|, \beta} ((D^\gamma A)^* |f|)(x)]. \end{aligned} \quad (20)$$

By the above estimates, we can get

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} |\mathcal{H}_{\Omega, A, \beta}^m f(x)|^q dx \right)^{1/q} \\ &\leq C \sum_{|\gamma|=m-1} \left(\left(\int_{\mathbb{R}^n} |(D^\gamma A)^*(x) \mathcal{H}_{|\Omega|, \beta} |f|(x)|^q dx \right)^{1/q} \right. \\ &\quad \left. + \left(\int_{\mathbb{R}^n} |\mathcal{H}_{|\Omega|, \beta} ((D^\gamma A)^* |f|)(x)|^q dx \right)^{1/q} \right) \\ &\leq C \sum_{|\gamma|=m-1} (I + II). \end{aligned} \quad (21)$$

For the term I , let $1/q = 1/r + 1/l = 1/r + 1/p - \beta/n$; then by the Hölder inequality, Lemma 6, and the boundedness of Hardy-Littlewood maximal function on L^p spaces, we obtain

$$\begin{aligned} I &\leq \left(\int_{\mathbb{R}^n} (D^\gamma A)^*(x)^r dx \right)^{1/r} \left(\int_{\mathbb{R}^n} |\mathcal{H}_{|\Omega|, \beta} |f|(x)|^l dx \right)^{1/l} \\ &\leq C \|(D^\gamma A)^*\|_{L^r} \|f\|_{L^p} \\ &\leq C \|D^\gamma A\|_{L^r} \|f\|_{L^p}. \end{aligned} \quad (22)$$

For the term II , let $1/q = 1/t - \beta/n = 1/r + 1/p - \beta/n$; then by the Hölder inequality and Lemma 6, we have

$$\begin{aligned} II &\leq C \left(\int_{\mathbb{R}^n} |(D^\gamma A)^*(x) f(x)|^t dx \right)^{1/t} \\ &\leq C \|(D^\gamma A)^*\|_{L^r} \|f\|_{L^p} \\ &\leq C \|D^\gamma A\|_{L^r} \|f\|_{L^p}. \end{aligned} \quad (23)$$

Combining the estimates of I and II , we finish the proof of Lemma 7. $\square$

Lemma 8 (see [4]). Let b be a function on $\mathbb{R}^n$ with m th order derivatives in $L_{\text{loc}}^q(\mathbb{R}^n)$ for some $q > n$. Then

$$|R_m(b; x, y)| \leq C_{m,n} |x - y|^m \times \sum_{|\gamma|=m} \left(\frac{1}{|\bar{Q}(x, y)|} \int_{\bar{Q}(x, y)} |D^\gamma b(z)|^q dz \right)^{1/q}, \quad (24)$$

where $\bar{Q}(x, y)$ is the cube centered at x having diameter $5\sqrt{n}|x - y|$.

Lemma 9 (see [5]). Suppose that $f \in \dot{CBMO}^q(\mathbb{R}^n)$, $1 \leq q < \infty$, and $r_1, r_2 > 0$; then

$$\left(\frac{1}{|B(0, r_1)|} \int_{B(0, r_1)} |f(x) - f_{B(0, r_2)}|^q dx \right)^{1/q} \leq C \left(1 + \left| \log \left(\frac{r_1}{r_2} \right) \right| \right) \|f\|_{\dot{CBMO}^q}. \quad (25)$$

Proof of Theorem 3. To prove Theorem 3, first we split each f as

$$f(x) = \sum_{i=-\infty}^{+\infty} f(x) \chi_i(x) = \sum_{i=-\infty}^{+\infty} f_i(x); \quad (26)$$

then we have

$$\begin{aligned} & \left\| \mathcal{H}_{\Omega, A, \beta}^m f \chi_k \right\|_{L^q}^q \\ &= \int_{C_k} \left(\int_{B(0, |x|)} \frac{|f(y)| |\Omega(x - y)|}{|x - y|^{m-1}} R_m(A; x, y) dy \right)^q \\ & \quad \times |x|^{(\beta-n)q} dx \\ &\leq \int_{C_k} \left(\sum_{i=-\infty}^k \int_{C_i} \frac{|\Omega(x - y) f(y)|}{|x - y|^{m-1}} R_m(A; x, y) dy \right)^q \\ & \quad \times |x|^{(\beta-n)q} dx \\ &\leq C \int_{C_k} \left(\sum_{i=-\infty}^{k-3} \int_{C_i} \frac{|\Omega(x - y) f(y)|}{|x - y|^{m-1}} R_m(A; x, y) dy \right)^q \\ & \quad \times |x|^{(\beta-n)q} dx \\ & \quad + C \int_{C_k} \left(\sum_{i=k-2}^k \int_{C_i} \frac{|\Omega(x - y) f(y)|}{|x - y|^{m-1}} R_m(A; x, y) dy \right)^q \\ & \quad \times |x|^{(\beta-n)q} dx \\ &= C(I_1 + I_2). \end{aligned} \quad (27)$$

For the term I_1 , we denote $A_k(x) = A(x) - \sum_{|\gamma|=m-1} (1/\gamma!) m_{B_k}(D^\gamma A) x^\gamma$; then it is easy to check $R_m(A; x, y) = R_m(A_k; x, y)$. By the fact that $x \in C_k$,

$y \in C_i$ with $i \leq k - 3$, we have $|x - y| \sim |x| \sim 2^k$. As $p_2 > n$, then by Lemmas 8 and 9, we obtain

$$\begin{aligned} & |R_m(A_k; x, y)| \\ &\leq |R_{m-1}(A_k; x, y)| + \sum_{|\gamma|=m-1} \frac{1}{\gamma!} |D^\gamma A_k(x)| |x - y|^{m-1} \\ &\leq C|x - y|^{m-1} \sum_{|\gamma|=m-1} \left\{ \left(\frac{1}{|\bar{Q}(x, y)|} \int_{\bar{Q}(x, y)} |D^\gamma A_k(z)|^{p_2} dz \right)^{1/p_2} \right. \\ & \quad \left. + |D^\gamma A_k(y)| \right\} \\ &\leq C|x - y|^{m-1} \sum_{|\gamma|=m-1} (\|D^\gamma A\|_{\dot{CBMO}^{p_2}} + |D^\gamma A_k(y)|). \end{aligned} \quad (28)$$

As $|x - y| \sim |x| \sim 2^k$ and $1 - 1/p_1 - 1/p_2 - 1/r = 1/r' - 1/q - \beta/n > 0$, then by the Hölder inequality, we have

$$\begin{aligned} I_1 &\leq C \int_{C_k} \left(\sum_{i=-\infty}^{k-3} \int_{C_i} |\Omega(x - y) f(y)| \right. \\ & \quad \times \sum_{|\gamma|=m-1} (\|D^\gamma A\|_{\dot{CBMO}^{p_2}} + |D^\gamma A_k(y)|) dy \Big)^q \\ & \quad \times |x|^{(\beta-n)q} dx \\ &\leq C 2^{-k(n-\beta)q} \int_{C_k} \left(\sum_{i=-\infty}^{k-3} \int_{C_i} |f(y)| |\Omega(x - y)| \right. \\ & \quad \times \sum_{|\gamma|=m-1} (\|D^\gamma A\|_{\dot{CBMO}^{p_2}} + |D^\gamma A_k(y)|) dy \Big)^q dx \\ &\leq C 2^{-k(n-\beta)q} \int_{C_k} \sum_{i=-\infty}^{k-3} \left(\int_{C_i} |f(y)|^{p_1} dy \right)^{q/p_1} \\ & \quad \times \left(\int_{C_i} |\Omega(x - y)|^r dy \right)^{q/r} \\ & \quad \times \left(\sum_{|\gamma|=m-1} \int_{C_i} (\|D^\gamma A\|_{\dot{CBMO}^{p_2}} + |D^\gamma A_k(y)|)^{p_2} dy \right)^{q/p_2} \\ & \quad \times |C_i|^{q(1-1/p_1-1/p_2-1/r)} dx. \end{aligned} \quad (29)$$

As

$$\begin{aligned} & \left(\int_{C_i} (\|D^\gamma A\|_{\text{CBMO}^{p_2}} + |D^\gamma A_k(y)|)^{p_2} dy \right)^{1/p_2} \\ & \leq \left(\int_{C_i} (\|D^\gamma A\|_{\text{CBMO}^{p_2}} \right. \\ & \quad \left. + |D^\gamma A_k(y) - m_{B_k}(D^\gamma A)(y)|)^{p_2} dy \right)^{1/p_2} \end{aligned} \quad (30)$$

$$\leq C \|D^\gamma A\|_{\text{CBMO}^{p_2}} |C_i| |k-i|,$$

$$\begin{aligned} & \int_{C_i} |\Omega(x-y)|^r dy \\ & \leq \int_{|x|-2^i}^{|x|+2^i} \int_{\mathbb{S}^{n-1}} |\Omega(y')|^r d\sigma(y') r^{n-1} dr \leq C 2^{kn-k+i}, \end{aligned}$$

we obtain the following estimates:

$$\begin{aligned} I_1 & \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\text{CBMO}^{p_2}}^q |k-i|^q 2^{-knq+k\beta q} \\ & \quad \times 2^{in(1-1/p_1-1/p_2-1/r)q} \\ & \quad \times 2^{((k(n-1)+i)/r)q+(in/p_2)q} \\ & \quad \times \int_{C_k} \sum_{i=-\infty}^{k-3} \|f_i\|_{L^{p_1}}^q dx \\ & \leq C \sum_{i=-\infty}^{k-3} \left(|k-i| 2^{in(1-1/p_1-1/p_2-1/r)} \right. \\ & \quad \times 2^{-kn+k\beta+(k(n-1)+i)/r+in/p_2+kn/q} \\ & \quad \left. \times \|f_i\|_{L^{p_1}} \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\text{CBMO}^{p_2}} \right)^q. \end{aligned} \quad (31)$$

For the term I_2 , we choose $\phi \in C_0^\infty(\mathbb{R}^n)$ satisfying $\text{supp } \phi \subset B(0, 4)$ as well as $\phi \equiv 1$ in $B(0, 2)$ and we set $L = \max\{\|D^\gamma \phi\|_{L^\infty}, |\gamma| \leq m-1\}$. Let $y_0 \in B_{k+4}$ and $A_k^\phi(x) = R_{m-1}(A_k; x, y_0)\phi(2^{-k}x)$. Then it is easy to see that $R_m(A; x, y) = R_m(A_k^\phi; x, y) = R_m(A_k; x, y)$ for $x \in B_k$ and $y \in B_i$ with $k-2 \leq i \leq k$. Thus we get

$$\mathcal{H}_{\Omega, A, \beta}^m f_i(x) = \mathcal{H}_{\Omega, A_k^\phi, \beta}^m f_i(x) = \mathcal{H}_{\Omega, A_k, \beta}^m f_i(x). \quad (32)$$

Thus by Lemma 7, we have

$$\begin{aligned} I_2 & \leq C \sum_{i=k-2}^k \int_{C_k} \left(\frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} \frac{|\Omega(x-y)f_i(y)|}{|x-y|^{m-1}} \right. \\ & \quad \left. \times R_m(A_k^\phi; x, y) dy \right)^q dx \end{aligned} \quad (33)$$

$$\begin{aligned} & \leq C \sum_{i=k-2}^k \left\| \mathcal{H}_{\Omega, A_k^\phi, \beta}^m f_i \right\|_{L^q}^q \\ & \leq C \sum_{i=k-2}^k \|f_i\|_{L^{p_1}}^q \sum_{|\gamma|=m-1} \|D^\gamma A_k^\phi\|_{L^{p_2}}^q. \end{aligned}$$

From [5, p.80], we have the following estimates:

$$\|D^\gamma A_k^\phi\|_{L^{p_2}} \leq C 2^{kn/p_2} \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\text{CBMO}^{p_2}}, \quad (34)$$

where C is dependent on L . Thus we get

$$I_2 \leq C \sum_{i=k-2}^k \|f_i\|_{L^{p_1}}^q 2^{knq/p_2} \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\text{CBMO}^{p_2}}^q. \quad (35)$$

As $0 < s \leq p < \infty$, we obtain the following estimates:

$$\begin{aligned} & \left\{ \sum_{k=-\infty}^{+\infty} 2^{k\alpha_1 s} \left\| \mathcal{H}_{\Omega, A, \beta}^m f \chi_k \right\|_{L^q}^s \right\}^{1/s} \\ & \leq \left\{ \sum_{k=-\infty}^{+\infty} 2^{k\alpha_1 p} \left\| \mathcal{H}_{\Omega, A, \beta}^m f \chi_k \right\|_{L^q}^p \right\}^{1/p} \\ & \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\text{CBMO}^{p_2}} \\ & \quad \times \left\{ \sum_{k=-\infty}^{+\infty} 2^{k\alpha_1 p} \right. \\ & \quad \times \sum_{i=-\infty}^{k-3} (|k-i| 2^{in(1-1/p_1-1/p_2-1/r)} \|f_i\|_{L^{p_1}} \\ & \quad \left. \times 2^{-kn+k\beta+(k(n-1)+i)/r+in/p_2+kn/q})^p \right\}^{1/p} \\ & \quad + C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\text{CBMO}^{p_2}} \\ & \quad \times \left\{ \sum_{k=-\infty}^{+\infty} 2^{k\alpha_1 p} \sum_{i=k-2}^k \|f_i\|_{L^{p_1}}^p 2^{(kn/p_2)p} \right\}^{1/p} \\ & = C(A+B). \end{aligned} \quad (36)$$

For the term A , we have

$$\begin{aligned}
A &= \sum_{|y|=m-1} \|D^y A\|_{\dot{\text{CBMO}}^{p_2}} \\
&\times \left\{ \sum_{k=-\infty}^{+\infty} 2^{k\alpha_1 p} \right. \\
&\quad \times \sum_{i=-\infty}^{k-3} (|k-i| 2^{in(1-1/p_1-1/p_2-1/r)} 2^{-kn+k\beta} \\
&\quad \times 2^{(k(n-1)+i)/r+in/p_2+kn/q} \|f_i\|_{L^{p_1}})^p \left. \right\}^{1/p} \\
&\leq C \sum_{|y|=m-1} \|D^y A\|_{\dot{\text{CBMO}}^{p_2}} \\
&\times \left\{ \sum_{k=-\infty}^{+\infty} \sum_{i=-\infty}^{k-3} (|k-i| 2^{(k-i)(n/q-n/r'+\beta-1/r-n/p_2)} \right. \\
&\quad \times 2^{(k-i)(\alpha_1+n/p_2)} \\
&\quad \times 2^{i(\alpha_1+n/p_2)} \|f_i\|_{L^{p_1}})^p \left. \right\}^{1/p}. \tag{37}
\end{aligned}$$

When $0 < p \leq 1$, by condition (13), we get

$$\begin{aligned}
A^p &\leq C \sum_{|y|=m-1} \|D^y A\|_{\dot{\text{CBMO}}^{p_2}}^p \\
&\times \sum_{k=-\infty}^{+\infty} \left(\sum_{i=-\infty}^{k-3} 2^{(k-i)(n/q-n/r'-1/r-n/p_2+\beta)} \right. \\
&\quad \times 2^{(k-i)(\alpha_1+n/p_2)} \\
&\quad \times |k-i| 2^{i(\alpha_1+n/p_2)} \|f_i\|_{L^{p_1}} \left. \right)^p \\
&\leq C \sum_{|y|=m-1} \|D^y A\|_{\dot{\text{CBMO}}^{p_2}}^p \\
&\times \sum_{k=-\infty}^{+\infty} 2^{i\alpha_2 p} \left(\sum_{i=-\infty}^{k-3} |k-i| \right. \\
&\quad \times 2^{(k-i)(n/q-n/r'-1/r-n/p_2+\beta+\alpha_2)} \\
&\quad \times \|f_i\|_{L^{p_1}} \left. \right)^p \\
&\leq C \sum_{|y|=m-1} \|D^y A\|_{\dot{\text{CBMO}}^{p_2}}^p \|f\|_{\dot{K}_{p_1}^{\alpha_2, p}}^p. \tag{38}
\end{aligned}$$

When $p \geq 1$, by the Hölder inequality and condition (13), we have

$$\begin{aligned}
A^p &\leq C \sum_{|y|=m-1} \|D^y A\|_{\dot{\text{CBMO}}^{p_2}}^p \\
&\times \sum_{k=-\infty}^{+\infty} 2^{k\alpha_1 p} \left(\sum_{i=-\infty}^{k-3} |k-i| \right. \\
&\quad \times 2^{(k-i)(n/q-n/r'-1/r-n/p_2+\beta)} \\
&\quad \times 2^{kn/p_2} \|f_i\|_{L^{p_1}} \left. \right)^p \\
&\leq C \sum_{|y|=m-1} \|D^y A\|_{\dot{\text{CBMO}}^{p_2}}^p \\
&\times \sum_{k=-\infty}^{+\infty} \left(\sum_{i=-\infty}^{k-3} |k-i| 2^{k\alpha_1} 2^{(k-i)(n/q-n/r'-1/r-n/p_2+\beta)} \right. \\
&\quad \times 2^{kn/p_2} \|f_i\|_{L^{p_1}} \left. \right)^p \\
&\leq C \sum_{|y|=m-1} \|D^y A\|_{\dot{\text{CBMO}}^{p_2}}^p \\
&\times \sum_{k=-\infty}^{+\infty} \left(\sum_{i=-\infty}^{k-3} |k-i| 2^{i\alpha_2} 2^{(k-i)(n/q-n/r'-1/r-n/p_2+\beta+\alpha_2)} \right. \\
&\quad \times \|f_i\|_{L^{p_1}} \left. \right)^p \\
&\leq C \sum_{|y|=m-1} \|D^y A\|_{\dot{\text{CBMO}}^{p_2}}^p \\
&\times \sum_{k=-\infty}^{+\infty} \sum_{i=-\infty}^{k-3} 2^{i\alpha_2 p} \|f_i\|_{L^{p_1}}^p 2^{p(k-i)(n/q-n/r'-1/r-n/p_2+\beta+\alpha_2)/p} \\
&\quad \times \left(\sum_{i=-\infty}^{k-3} |k-i|^{p'} 2^{((k-i)/p')(n/q-n/r'-1/r-n/p_2+\beta+\alpha_2)p'} \right)^{p/p'} \\
&\leq C \sum_{|y|=m-1} \|D^y A\|_{\dot{\text{CBMO}}^{p_2}}^p \\
&\times \sum_{i=-\infty}^{+\infty} \sum_{k=i+3}^{\infty} 2^{ip(\alpha_1+n/p_2)} 2^{(k-i)(n/q-n/r'-1/r-n/p_2+\beta+\alpha_2)} \|f_i\|_{L^{p_1}} \\
&\quad \times \left(\sum_{i=-\infty}^{k-3} |k-i|^{p'} 2^{(k-i)(n/q-n/r'-1/r-n/p_2+\beta+\alpha_2)} \right)^{p/p'}.
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{|y|=m-1} \|D^y A\|_{\dot{CBMO}^{p_2}}^p \sum_{i=-\infty}^{+\infty} 2^{i\alpha_2 p} \|f_i\|_{L^{p_1}}^p \\
&\leq C \sum_{|y|=m-1} \|D^y A\|_{\dot{CBMO}^{p_2}}^p \|f\|_{\dot{K}_{p_1}^{\alpha_2, p}}^p.
\end{aligned} \quad (39)$$

For the term B , we have the following estimates:

$$\begin{aligned}
B &\leq C \sum_{|y|=m-1} \|D^y A\|_{\dot{CBMO}^{p_2}} \left\{ \sum_{k=-\infty}^{+\infty} 2^{k\alpha_1 p} \sum_{i=k-2}^k 2^{knp/p_2} \|f_i\|_{L^{p_1}}^p \right\}^{1/p} \\
&\leq C \sum_{|y|=m-1} \|D^y A\|_{\dot{CBMO}^{p_2}} \left(\sum_{i=-\infty}^{+\infty} 2^{i\alpha_2 p} \|f_i\|_{L^{p_1}}^p \right)^{1/p} \\
&\leq C \sum_{|y|=m-1} \|D^y A\|_{\dot{CBMO}^{p_2}} \|f\|_{\dot{K}_{p_1}^{\alpha_2, p}}.
\end{aligned} \quad (40)$$

Combining the estimates of A and B , we finish the proof of Theorem 3. $\square$

Proof of Theorem 4. The proof of Theorem 4 is quite similar and much easier than Theorem 3 and we omit the details here. $\square$

3. Boundedness of $\mathcal{H}_{\Omega, A, \beta}^m$ on λ -Central Morrey Spaces

In [21], Wiener gave a way to describe the behavior of a function at the infinity. Later, Beurling [22] extended Wiener's idea and introduced a pair of dual Banach spaces, A^q and $B^{q'}$, with $1/q + 1/q' = 1$. In [23], Feichtinger proved that B^q can be described as

$$\|f\|_{B^q} = \sup_{k \geq 0} (2^{-kn/q} \|f\chi_k\|_{L^q}) < \infty, \quad (41)$$

where χ_0 is the characterization of the unit ball $\{x \in \mathbb{R}^n : |x| \leq 1\}$ and χ_k is defined as in Section 2.

Now by duality, the space A^q , which is called the Beurling algebra, can be described by

$$\|f\|_{A^q} = \sum_{k=0}^{\infty} 2^{kn/q'} \|f\chi_k\|_{L^q} < \infty. \quad (42)$$

Later, Chen and Lau [24] as well as García-Cuerva [25] introduced atomic spaces $HA^q(\mathbb{R}^n)$ associated with the Beurling algebra A^q and the dual space of $HA^q(\mathbb{R}^n)$ can be described by

$$\begin{aligned}
&\|f\|_{\dot{CBMO}^q} \\
&:= \sup_{R \geq 1} \left(\frac{1}{|B(0, R)|} \int_{B(0, R)} |f(x) - f_{B(0, R)}|^q dx \right)^{1/q} < \infty;
\end{aligned} \quad (43)$$

here the $\dot{CBMO}^q$ can be regarded as the inhomogeneous central BMO spaces.

In 2000, Alvarez et al. [26] introduced the λ -Central bounded mean oscillation space and λ -Central Morrey space, respectively.

Definition 10 (see [26]). Given $\lambda < 1/n$, $1 < q < \infty$, then a function $f \in L_{\text{loc}}^q(\mathbb{R}^n)$ is said to belong to the λ -Central bounded mean oscillation space $\dot{CBMO}^{q, \lambda}(\mathbb{R}^n)$ if

$$\begin{aligned}
&\|f\|_{\dot{CBMO}^{q, \lambda}} \\
&:= \sup_{R > 0} \left(\frac{1}{|B(0, R)|^{1+\lambda q}} \int_{B(0, R)} |f(x) - f_{B(0, R)}|^q dx \right)^{1/q} < \infty.
\end{aligned} \quad (44)$$

Definition 11 (see [26]). Let $\lambda \in \mathbb{R}$ and $1 < q < \infty$. Then the λ -Central Morrey space $\dot{E}^{q, \lambda}(\mathbb{R}^n)$ is defined by

$$\|f\|_{\dot{E}^{q, \lambda}} = \sup_{R > 0} \left(\frac{1}{|B(0, R)|^{1+\lambda q}} \int_{B(0, R)} |f(x)|^q dx \right)^{1/q} < \infty. \quad (45)$$

From [27], we know that if $1 < q_1 < q_2 < \infty$, we obtain $\dot{E}^{q_2, \lambda} \subset \dot{E}^{q_1, \lambda}$ for $\lambda \in \mathbb{R}$ and $\dot{CBMO}^{q_2, \lambda} \subset \dot{CBMO}^{q_1, \lambda}$ for $\lambda < 1/n$. Furthermore, when $\lambda < -1/q$, $\dot{E}^{q, \lambda}$ reduces to $\{0\}$ and $\dot{CBMO}^{q, \lambda}$ reduces to the space of constant functions. When $\lambda = -1/q$, $\dot{CBMO}^{q, \lambda}$ coincides with $L^q(\mathbb{R}^n)$ modulo constant and $\dot{E}^{q, \lambda} = L^q$.

In 2011, Fu et al. [19] proved the boundedness of the commutator of fractional Hardy operator with a rough kernel on λ -Central Morrey space. Later, Fu et al. [28] proved the boundedness of the weighted Hardy operator and its commutator on λ -Central Morrey space. In this paper, we will give the boundedness of $\mathcal{H}_{\Omega, A, \beta}^m$ on λ -Central Morrey space with $m \geq 1$.

Our results can be stated as follows.

Theorem 12. Suppose $m \geq 2$, $n < p_2 < \infty$, $1 < p_1 < \infty$, $1/q = 1/p_1 + 1/p_2 - \beta/n$ with $0 \leq \beta < n$ and $\lambda = \lambda_1 + \lambda_2 + \beta/n$. Let $\Omega \in L'(\mathbb{S}^{n-1})$ with $1/r' > \beta/n + 1/q$, and A has derivatives of order $m-1$ in $\dot{CBMO}^{p_2, \lambda_2}$. If $r > p_1'$, $\lambda_1 > -1/p_1$, and $q\lambda + 1 > 0$, then one has

$$\|\mathcal{H}_{\Omega, A, \beta}^m f\|_{\dot{E}^{q, \lambda}} \leq C \sum_{|y|=m-1} \|D^y A\|_{\dot{CBMO}^{p_2, \lambda_2}} \|f\|_{\dot{E}^{p_1, \lambda_1}}, \quad (46)$$

where the constant C is independent of f and A .

For the case $m = 1$, we have the following theorem.

Theorem 13. Suppose $1 < p_1, p_2 < \infty$, $1/q = 1/p_1 + 1/p_2 - \beta/n$ with $0 \leq \beta < n$ and $\lambda = \lambda_1 + \lambda_2 + \beta/n$. Let $\Omega \in L'(\mathbb{S}^{n-1})$ with $1/r' > \beta/n + 1/q$ and $A \in \dot{CBMO}^{p_2, \lambda_2}$. If $\lambda_1 > -1/p_1$, $r > p_1'$, and $q\lambda + 1 > 0$, then one has

$$\|\mathcal{H}_{\Omega, \beta}^A f\|_{\dot{E}^{q, \lambda}} \leq C \|A\|_{\dot{CBMO}^{p_2, \lambda_2}} \|f\|_{\dot{E}^{p_1, \lambda_1}}, \quad (47)$$

where the constant C is independent of f and A .

In order to prove Theorems 12 and 13, by a standard argument, we have the following lemma.

Lemma 14 (see [16]). *Suppose $f \in \dot{CBMO}^{p,\lambda}$, $1 \leq p < \infty$, $\lambda < 1/n$, and $r_1, r_2 \in \mathbb{R}^+$; then*

$$\left(\frac{1}{|B(0, r_1)|^{1+p\lambda}} \int_{B(0, r_1)} |f(x) - f_{B(0, r_2)}|^p dx \right)^{1/p} \leq C \left(1 + \left| \log \frac{r_1}{r_2} \right| \right) \|f\|_{\dot{CBMO}^{p,\lambda}}. \quad (48)$$

Proof of Theorem 12. For any $R > 0$, we denote $B(0, R)$ by B and $B(0, kR)$ by kB for any $k \in \mathbb{Z}^+$. Thus we have the following estimates:

$$\begin{aligned} & \frac{1}{|B|} \int_B |\mathcal{H}_{\Omega, A, \beta}^m f(x)|^q dx \\ & \leq \frac{1}{|B|} \int_B \left| \frac{1}{|x|^{n-\beta}} \int_{B(0, |x|)} \frac{|\Omega(x-y)f(y)|}{|x-y|^{m-1}} \right. \\ & \quad \times |R_m(A; x, y)| dy \Big|^q dx \\ & = \frac{1}{|B|} \sum_{k=-\infty}^0 \int_{2^k B \setminus 2^{k-1} B} \left| \frac{1}{|x|^{n-\beta}} \sum_{i=-\infty}^k \int_{2^i B \setminus 2^{i-1} B} \frac{|\Omega(x-y)f(y)|}{|x-y|^{m-1}} \right. \\ & \quad \times |R_m(A; x, y)| dy \Big|^q dx \\ & \leq \frac{1}{|B|} \sum_{k=-\infty}^0 \int_{2^k B \setminus 2^{k-1} B} \left| \frac{1}{|x|^{n-\beta}} \sum_{i=k-2}^{k-3} \int_{2^i B \setminus 2^{i-1} B} \frac{|\Omega(x-y)f(y)|}{|x-y|^{m-1}} \right. \\ & \quad \times |R_m(A; x, y)| dy \Big|^q dx \\ & \quad + \frac{1}{|B|} \\ & \quad \times \sum_{k=-\infty}^0 \int_{2^k B \setminus 2^{k-1} B} \left| \frac{1}{|x|^{n-\beta}} \sum_{i=k-2}^k \int_{2^i B \setminus 2^{i-1} B} \frac{|\Omega(x-y)f(y)|}{|x-y|^{m-1}} \right. \\ & \quad \times |R_m(A; x, y)| dy \Big|^q dx \\ & = I + II. \end{aligned} \quad (49)$$

For the term I , let $\tilde{A}(x) = A(x) - \sum_{|\gamma|=m-1} (1/\gamma!) m_{2^k B}(D^\gamma A) x^\gamma$. Then it is easy to see

$R_m(A; x, y) = R_m(\tilde{A}; x, y)$. As $p_2 > n$, then by Lemmas 8 and 9, we have

$$\begin{aligned} |R_m(\tilde{A}; x, y)| & \leq |R_{m-1}(\tilde{A}; x, y)| \\ & \quad + \sum_{|\gamma|=m-1} \frac{1}{\gamma!} |D^\gamma \tilde{A}(y)| |x-y|^{m-1} \\ & \leq C |x-y|^{m-1} \sum_{|\gamma|=m-1} \left[|\tilde{Q}|^{\lambda_2} \|D^\gamma A\|_{\dot{CBMO}^{p_2, \lambda_2}} \right. \\ & \quad \left. + |D^\gamma(\tilde{A})(y)| \right], \end{aligned} \quad (50)$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having diameter $5\sqrt{n}|x-y|$.

As $x \in 2^k B \setminus 2^{k-1} B$ and $y \in 2^i B \setminus 2^{i-1} B$ with $i \leq k-3$, we have $|x-y| \sim |x| \sim |2^k B|^{1/n}$ and $|x-y| \geq C_1 |2^i B|^{1/n}$.

Thus by the Hölder inequality and the condition $1-1/p_1-1/p_2-1/r=1/r'-\beta/n-1/q > 0$, we have

$$\begin{aligned} I & \leq C |B|^{-1} \\ & \quad \times \sum_{k=-\infty}^0 \int_{2^k B \setminus 2^{k-1} B} \left| \frac{1}{|x|^{n-\beta}} \right. \\ & \quad \times \sum_{i=-\infty}^{k-3} \int_{2^i B \setminus 2^{i-1} B} \frac{|\Omega(x-y)f(y)|}{|x-y|^{m-1}} \\ & \quad \times |x-y|^{m-1} \\ & \quad \times \sum_{|\gamma|=m-1} \left(|\tilde{Q}|^{\lambda_2} \|D^\gamma A\|_{\dot{CBMO}^{p_2, \lambda_2}} \right. \\ & \quad \left. \left. + |D^\gamma \tilde{A}(y)| \right) dy \right|^q dx \\ & \leq C |B|^{-1} \\ & \quad \times \sum_{k=-\infty}^0 \int_{2^k B \setminus 2^{k-1} B} \left| \frac{1}{|x|^{n-\beta}} \right. \\ & \quad \times \sum_{i=-\infty}^{k-3} \int_{2^i B \setminus 2^{i-1} B} |\Omega(x-y)f(y)| \\ & \quad \times \sum_{|\gamma|=m-1} \left(|\tilde{Q}|^{\lambda_2} \right. \\ & \quad \times \|D^\gamma A\|_{\dot{CBMO}^{p_2, \lambda_2}} \\ & \quad \left. \left. + |D^\gamma \tilde{A}(y)| \right) dy \right|^q dx \end{aligned}$$

$$\begin{aligned}
&\leq C|B|^{-1} \\
&\quad \times \sum_{k=-\infty}^0 \int_{2^k B \setminus 2^{k-1} B} \frac{1}{|x|^{(n-\beta)q}} \\
&\quad \times \left| \sum_{i=-\infty}^{k-3} \left(\int_{2^i B \setminus 2^{i-1} B} |\Omega(x-y)|^r dy \right)^{1/r} \right. \\
&\quad \times \left(\int_{2^i B \setminus 2^{i-1} B} \sum_{|\gamma|=m-1} (|x-y|^{\lambda_2 n} \right. \\
&\quad \times \|D^\gamma A\|_{\text{CBMO}^{p_2, \lambda_2}} \\
&\quad \left. \left. + |D^\gamma \tilde{A}(y)| \right)^{p_2} dy \right)^{1/p_2} \\
&\quad \times \left(\int_{2^i B \setminus 2^{i-1} B} |f(y)|^{p_1} dy \right)^{1/p_1} \\
&\quad \times \left| 2^i B \right|^{1-1/p_1-1/p_2-1/r} dx.
\end{aligned} \tag{51}$$

Note the following fact:

$$\begin{aligned}
&\left(\int_{2^i B \setminus 2^{i-1} B} \sum_{|\gamma|=m-1} \left(|x-y|^{\lambda_2 n} \|D^\gamma A\|_{\text{CBMO}^{p_2, \lambda_2}} \right. \right. \\
&\quad \left. \left. + |D^\gamma \tilde{A}(y)| \right)^{p_2} dy \right)^{1/p_2} \\
&\leq C \sum_{|\gamma|=m-1} \left[\|D^\gamma A\|_{\text{CBMO}^{p_2, \lambda_2}} |2^k B|^{\lambda_2} |2^i B|^{1/p_2} \right. \\
&\quad \left. + \left(\int_{2^i B} |D^\gamma A(y) - m_{\tilde{Q}}(D^\gamma A)|^{p_2} dy \right)^{1/p_2} \right] \\
&\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\text{CBMO}^{p_2, \lambda_2}} |2^i B|^{1/p_2} \left(|2^k B|^{\lambda_2} + |2^i B|^{\lambda_2} \right),
\end{aligned} \tag{52}$$

where the last inequality follows from Lemma 14 and the fact $|x-y| \geq C_1 |2^i B|^{1/n}$. Thus by the condition $1 + q\lambda > 0$ and $\lambda_1 + 1 > \lambda_1 + 1/p_1 > 0$, we have

$$\begin{aligned}
I &\leq C|B|^{-1} \\
&\quad \times \sum_{k=-\infty}^0 |2^k B|^{(\beta/n-1)q}
\end{aligned}$$

$$\begin{aligned}
&\times \int_{2^k B \setminus 2^{k-1} B} \left(\sum_{i=-\infty}^{k-3} \|f\|_{\dot{E}^{p_1, \lambda_1}} |2^i B|^{1/p_1 + \lambda_1} |2^i B|^{1/r} \right. \\
&\quad \times \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\text{CBMO}^{p_2, \lambda_2}} |2^i B|^{1/p_2} \\
&\quad \times \left(|2^k B|^{\lambda_2} + |2^i B|^{\lambda_2} \right) \\
&\quad \left. \times |2^i B|^{1-1/p_1-1/p_2-1/r} \right)^q dx \\
&\leq C \|f\|_{\dot{E}^{p_1, \lambda_1}}^q \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\text{CBMO}^{p_2, \lambda_2}}^q |B|^{-1} \\
&\quad \times \sum_{k=-\infty}^0 |2^k B|^{(\beta/n-1)q} |2^k B|^{\lambda_2 q} \\
&\quad \times \left(\sum_{i=-\infty}^{k-3} |2^i B|^{1/p_1 + \lambda_1} \right. \\
&\quad \times |2^i B|^{1/p_2} |2^i B|^{1/r} |2^i B|^{1-1/p_1-1/p_2-1/r} \left. \right)^q \\
&\leq C \|f\|_{\dot{E}^{p_1, \lambda_1}}^q \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\text{CBMO}^{p_2, \lambda_2}}^q \\
&\quad \times |B|^{-1 + \beta q/n - q + \lambda_2 q + 1 + \lambda_1 q + q} \\
&\quad \times \sum_{k=-\infty}^0 2^{nk\beta q/n - knq + kn + k\lambda_2 nq + k\lambda_1 nq + knq} \\
&\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\text{CBMO}^{p_2, \lambda_2}}^q \|f\|_{\dot{E}^{p_1, \lambda_1}}^q |B|^{q\lambda}.
\end{aligned} \tag{53}$$

To estimate the term II , we adopt some basic ideas from the estimates of the term I_2 in Theorem 3. First, we denote $R_m(A; x, y) = R_m(A_k^\phi; x, y) = R_m(A_k; x, y)$ for $x \in 2^k B$ and $y \in 2^i B$ with $k-2 \leq i \leq k$, where $A_k^\phi(x) = R_{m-1}(A_k; x, y_0)\phi(|x-y_0|^{-1}x)$ with $y_0 \in 2^i B \setminus 2^{i-1} B$. Here $A_k(x) = A(x) - \sum_{|\gamma|=m-1} (1/\gamma!) m_{2^k B}(D^\gamma A) x^\gamma$ and ϕ is defined as in Section 2.

As $\Omega \in L^r(\mathbb{S}^{n-1})$ with $r > p'_1$, by Lemma 7, we get

$$\begin{aligned}
II &\leq \frac{1}{|B|} \sum_{k=-\infty}^0 \sum_{i=k-2}^k \left\| \mathcal{H}_{\Omega, A_k^\phi, \beta}^m f_i \right\|_q^q \\
&\leq \frac{C}{|B|} \sum_{k=-\infty}^0 \sum_{i=k-2}^k \|f_i\|_{L^{p_1}}^q \sum_{|\gamma|=m-1} \|D^\gamma A_k^\phi\|_{L^{p_2}}^q,
\end{aligned} \tag{54}$$

where f_i is defined by $f_i(x) = f(x)\chi_{2^i B \setminus 2^{i-1} B}(x)$.

For $D^\nu A_k^\phi$, as $|x - y_0| \leq C|y_0|$, then by Lemmas 8 and 14, we have the following estimates:

$$\begin{aligned}
 & |D^\nu A_k^\phi(x)| \\
 & \leq \sum_{|\mu|+|\nu|=m-1} C_{\mu\nu} |R_{m-1-|\mu|}(D^\mu A_k; x, y_0)| \\
 & \quad \times |D^\nu \phi(|x - y_0|^{-1}x)| |x - y_0|^{-|\nu|} \\
 & \leq C \sum_{|\mu|+|\nu|=m-1} |x - y_0|^{m-1-|\mu|-|\nu|} \\
 & \quad \times \sum_{|\nu'|=m-1-|\mu|} \left(\frac{1}{|\bar{Q}(x, y_0)|} \int_{\bar{Q}(x, y_0)} |D^{\nu'}(D^\mu A_k)(z)|^{p_2} dz \right)^{1/p_2} \\
 & \quad \times \chi_{|x| \leq |x - y_0|}(x) \\
 & \leq C \sum_{|\nu|=m-1} \|D^\nu A\|_{\text{CBMO}^{p_2, \lambda_2}} |x - y_0|^{\lambda_2 n} \chi_{|x| \leq C|y_0|}(x).
 \end{aligned} \tag{55}$$

Thus we have

$$\begin{aligned}
 & \sum_{|\nu|=m-1} \|D^\nu A_k^\phi\|_{L^{p_2}}^q \\
 & \leq C \sum_{|\nu|=m-1} \|D^\nu A\|_{\text{CBMO}^{p_2, \lambda_2}}^q |2^i B|^{q\lambda_2} |2^i B|^{q/p_2}.
 \end{aligned} \tag{56}$$

By the above estimates, we obtain

$$\begin{aligned}
 II & \leq \frac{C}{|B|} \sum_{k=-\infty}^0 \sum_{i=k-2}^k \left(\frac{1}{|2^i B|^{1+p_1\lambda_1}} \int_{2^i B} |f(x)|^{p_1} dx \right)^{q/p_1} \\
 & \quad \times |2^i B|^{q/p_1 + q\lambda_1} \\
 & \quad \times \sum_{|\nu|=m-1} \|D^\nu A\|_{\text{CBMO}^{p_2, \lambda_2}}^q |2^i B|^{q\lambda_2 + q/p_2} \\
 & \leq C \sum_{k=-\infty}^0 \sum_{i=k-2}^k 2^{in(1+\lambda q)} \\
 & \quad \times \sum_{|\nu|=m-1} \|D^\nu A\|_{\text{CBMO}^{p_2, \lambda_2}}^q \|f\|_{\dot{E}^{p_1, \lambda_1}}^q |B|^{q\lambda} \\
 & \leq C \sum_{|\nu|=m-1} \|D^\nu A\|_{\text{CBMO}^{p_2, \lambda_2}}^q \|f\|_{\dot{E}^{p_1, \lambda_1}}^q |B|^{q\lambda}.
 \end{aligned} \tag{57}$$

Combining the estimates of I and II and by the definition of $\dot{E}^{q, \lambda}$, we finish the proof of Theorem 12. $\square$

Proof of Theorem 13. The proof of Theorem 13 is quite similar but much simpler than Theorem 12 and we omit the details here. $\square$

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