Research Article

Some Identities on the High-Order $q$-Euler Numbers and Polynomials with Weight 0

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We construct the $N$th order nonlinear ordinary differential equation related to the generating function of $q$-Euler numbers with weight 0. From this, we derive some identities on $q$-Euler numbers and polynomials of higher order with weight 0.

1. Introduction

As a well-known definition, the Euler polynomial $E_n(x)$ is given by

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$  \hspace{1cm} (1)

In the special case, $x = 0$, $E_n(0) = E_n$ is the $n$th Euler number.

From (1), we note that

$$E_0 = 1, \quad (E + 1)^n + E_n = 0, \quad \text{if } n > 0,$$  \hspace{1cm} (2)

with the usual convention of replacing $E^n$ by $E_n$ (see [1–16]).

In the viewpoint of the $q$-extension of (1) and (2), let us consider the following $q$-Euler number and polynomial:

$$\frac{2}{q e^t + 1} e^{xt} = \sum_{n=0}^{\infty} \bar{E}_{n,q}(x) \frac{t^n}{n!},$$  \hspace{1cm} (3)

$$\bar{E}_{0,q} = \frac{2}{1+q}, \quad q (\bar{E}_{q} + 1)^n + \bar{E}_{n,q} = 0, \quad \text{if } n > 0,$$  \hspace{1cm} (4)

with the usual convention of replacing $\bar{E}_{q}^n$ by $\bar{E}_{n,q}$.

Equation (3) is called the generating function of $q$-Euler polynomial with weight 0. In the case $x = 0$, $\bar{E}_{n,q}(0) = \bar{E}_{n,q}$ is the $n$th $q$-Euler number with weight 0 (see [5, 11]).

Throughout this paper, let $q$ be a complex number with $|q| < 1$. As $q \to 1$, we obtain (1) and (2) from (3) and (4).

The generating function of Eulerian polynomial $H_n(x | u)$ is defined by

$$\frac{1 - u}{e^t - u} e^{xt} = \sum_{n=0}^{\infty} H_n(x | u) \frac{t^n}{n!},$$  \hspace{1cm} (5)

where $u \in \mathbb{C}$ with $u \neq 1$. In the special case, $x = 0$, $H_n(0 | u) = H_n(u)$ is the $n$th Eulerian number (see [1–3]). Sometimes that is called the $n$th Frobenius-Euler number (see [9–11, 15]).

From (1) and (5), we note that $H_n(x | -1) = E_n(x)$. From (5), we have

$$H_0 (u) = 1, \quad H_n (1 | u) - u H_n (u) = (1 - u) \delta_{0,n},$$  \hspace{1cm} (6)

where $\delta_{n,k}$ is Kronecker symbol (see [9–11]).

For $N \in \mathbb{N}$, the $q$-Euler polynomial of order $N$ is defined by the generating function as follows:

$$C_{t,q}^N (t, x) = \left( \frac{2}{q e^t + 1} \right) \times \cdots \times \left( \frac{2}{q e^t + 1} \right) e^{xt}$$

$$= \sum_{n=0}^{\infty} E_{n,q}^{(N)} (x) \frac{t^n}{n!},$$  \hspace{1cm} (7)

In the special case, $x = 0$, $E_{n,q}^{(N)}(0) = \bar{E}_{n,q}^{(N)}$ is called the $n$th $q$-Euler number of order $N$ with weight 0 (see [5, 11]).
Abstract and Applied Analysis

In [9], Kim derived some identities between the sums of products of Frobenius-Euler polynomials and Frobenius-Euler polynomials of higher order. The main idea is to construct nonlinear ordinary differential equations with respect to \( t \) which are closely related to the generating function of Frobenius-Euler polynomial. In [3], Choi considered nonlinear ordinary differential equations with respect to \( u \) not \( t \).

In this paper, we construct nonlinear ordinary differential equations with respect to \( q \). The purpose of this paper is to give some new identities on the high order \( q \)-Euler numbers and polynomials with weight 0 by using the differential equations of \( q \).

2. Construction of Nonlinear Differential Equations

We define

\[
G = G(q) = \frac{1}{qe^t + 1},
\]

(8)

\[
G^N(t, x) = \underbrace{G \times \cdots \times G}_{N \text{-times}} e^{xt} \quad \text{for } N \in \mathbb{N}.
\]

From (7) and (8), we note that

\[
G^N_q(t, x) = 2^N G^N(t, x) = 2^N G^N e^{xt}.
\]

(9)

By differentiating (8) with respect to \( q \), we get

\[
\frac{dG}{dq} = -\frac{q e^t + 1 - 1}{q(q e^t + 1)^2} = -\frac{G + G^2}{q},
\]

(10)

\[
qG^{(1)} + G = G^2.
\]

By differentiating (10) with respect to \( q \), we get

\[
q^2 G^{(2)} + 4q G^{(1)} + 2G = 2! G^3,
\]

(11)

where \( G^{(N)} = \frac{d^N G}{dq^N} \).

By the derivative of (11) with respect to \( q \), we have

\[
q^3 G^{(3)} + 9q^2 G^{(2)} + 18q G^{(1)} + 3! G = 3! G^4.
\]

(12)

Continuing this process, we get

\[
(N-1)! G^N = \sum_{k=0}^{N-1} a_k(N) q^k G^{(k)}.
\]

(13)

Let us consider the derivative of (13) with respect to \( q \) to find the coefficient \( a_k(N) \) in (13).

By (10), we get

\[
q \frac{d}{dq} \left( (N-1)! G^N \right) = N! G^{N-1} q G^{(1)} = N! G^{N-1} (G + G^2) = N! G^{N+1} - N (N-1)! G^N.
\]

(14)

From (13) and (14), we get

\[
N! G^{N+1} = N (N-1)! G^N + \sum_{k=0}^{N-1} k a_k(N) q^k G^{(k)} + \sum_{k=1}^{N} a_{k-1}(N) q^k G^{(k)},
\]

(15)

where \( N! G^{N+1} = \sum_{k=0}^{N} a_k(N+1) q^k G^{(k)} \).

By comparing coefficients on both sides of (15), we obtain the following recurrence relations:

\[
a_0(N+1) = N a_0(N), \quad a_N(N+1) = a_{N-1}(N),
\]

(16)

\[
a_k(N+1) = N a_k(N) + k a_k(N) + a_k(N-1),
\]

(17)

for \( 1 \leq k \leq N - 1 \) and \( a_k(N) = 0 \).

From the first part of (16), we have

\[
a_0(N+1) = N a_0(N) = N(N-1) a_0(N-1) = \cdots = N! a_0(2).
\]

(18)

By (10) and (13), we have

\[
qG^{(1)} + G = G^2 = \sum_{k=0}^{1} a_k(2) q^k G^{(k)} = a_0(2) G + a_1(2) qG^{(1)}.
\]

(19)

From (18) and (19), we get

\[
a_0(2) = 1, \quad a_1(2) = 1, \quad a_0(N) = (N-1)!. \quad (20)
\]

From the second part of (16), we have

\[
a_N(N+1) = a_{N-1}(N) = \cdots = a_1(2) = 1.
\]

(21)

To find \( a_k(N) \) in (13) from (17), we set

\[
g(t, s) = \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} a_k(N) \frac{t^N}{N!} s^k,
\]

(22)

where \( |t| < 1 \) (see [9]).

From (17) and (22), we have

\[
\sum_{N \geq 1} \sum_{0 \leq k \leq N-1} a_{k+1}(N+1) \frac{t^N}{N!} s^k = \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} N a_{k-1}(N) \frac{t^N}{N!} s^k + g(t, s).
\]

(23)
From the left hand side of (23), we have

$$
\sum_{N \geq 1} \sum_{0 \leq k \leq N-1} a_{k+1} \frac{t^N}{N!} s^k
$$

$$
= \frac{1}{s} \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} a_k (N) \frac{t^{N-1}}{(N-1)!} s^k - a_0 (N) \frac{t^{N-1}}{(N-1)!}
$$

$$
= \frac{1}{s} \left( \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} a_k (N) \frac{t^{N-1}}{(N-1)!} s^k - a_0 (N) \frac{t^{N-1}}{(N-1)!} \right)
$$

$$
= \frac{1}{s} \left( g_t + \frac{1}{t-1} \right), \tag{24}
$$

where $g_t = \partial g/\partial t$. From the first term of the right hand side of (23), we have

$$
\sum_{N \geq 1} \sum_{0 \leq k \leq N-1} N a_{k+1} (N) \frac{t^N}{N!} s^k
$$

$$
= \frac{t}{s} \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} a_k (N) \frac{t^{N-1}}{(N-1)!} s^k
$$

$$
= \frac{t}{s} \left( \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} a_k (N) \frac{t^{N-1}}{(N-1)!} s^k - \sum_{N \geq 1} a_0 (N) \frac{t^{N-1}}{(N-1)!} \right)
$$

$$
= \frac{t}{s} \left( g_t + \frac{1}{t-1} \right). \tag{25}
$$

From the second term of the right hand side of (23), we have

$$
\sum_{N \geq 1} \sum_{0 \leq k \leq N-1} (k+1) a_{k+1} (N) \frac{t^N}{N!} s^k
$$

$$
= \sum_{N \geq 1} \sum_{1 \leq k \leq N} k a_k (N) \frac{t^{N-k}}{N!} s^{k-1}
$$

$$
= \sum_{N \geq 1} \sum_{1 \leq k \leq N} k a_k (N) \frac{t^{N-k}}{N!} s^{k-1} = g_s, \tag{26}
$$

where $g_s = \partial g/\partial s$.

From (22)–(26), we obtain the following initial value problem quasilinear first-order partial differential equation:

\[ (t-1) g_t + s g_s = -sg - 1, \quad |t| < 1, \]

\[ g(0, s) = 0, \quad s \in \mathbb{R}. \tag{27} \]

We consider Cauchy problem for the following first-order quasilinear partial differential equation:

\[ P(x, y, z) z_x + Q(x, y, z) z_y = R(x, y, z), \quad z(x_0(t), y_0(t)) = z_0(t), \quad t \in I, \tag{28} \]

where $I$ is some interval.

We know that (28) has a unique solution under some conditions as follows.

**Theorem A (see [17, page 65]).** Suppose that $P, Q$, and $R$ are of class $C^1$ in a domain $\Omega$ of $\mathbb{R}^3$ containing the point $(x_0, y_0, z_0)$ and suppose that

\[ P(x_0, y_0, z_0) \frac{dy_0(t_0)}{dt} - Q(x_0, y_0, z_0) \frac{dx_0(t_0)}{dt} \neq 0. \tag{29} \]

Then in a neighborhood $U$ of $(x_0, y_0)$ there exists a unique solution of (28) at every point of initial curve contained in $U$.

Since (27) satisfies (29) and regularity conditions, there exists a unique solution of (27). It is customary to write (27) in the form

\[ \frac{dt}{t-1} = \frac{ds}{s} = -\frac{dg}{s(g-1)}. \tag{30} \]

\[ t = 0, \quad s = p, \quad g = 0. \tag{31} \]

Since $dt/(t-1) = ds/s$ is separable, we get

\[ u_1(t, s, g) = \frac{1-t}{s}. \tag{32} \]

$u_1$ is a solution of partial differential equation of (27).

From (30), we get the linear equation

\[ \frac{dg}{ds} = \frac{1}{s} - g. \tag{33} \]

By the integrating factor method, we have

\[ u_2(t, s, g) = e^{\frac{t}{s}} g + E_i(s). \tag{34} \]

The exponential integral $E_i(s)$ is defined by

\[ E_i(s) = \int_{-\infty}^{s} \frac{e^r}{r} dr = \gamma + \ln |s| + \sum_{n=1}^{\infty} \frac{S^n}{n \cdot n!} (s \in \mathbb{R}, s \neq 0), \tag{35} \]

where $\gamma$ is Euler constant.

$u_2$ is another solution of partial differential equation of (27), and $u_1$ and $u_2$ are linearly independent.

From the parameterized initial conditions (31), (33), and (34), we get

\[ u_2 = E_i \left( \frac{1}{u_1} \right), \quad e^{\frac{t}{s}} g + E_i(x) = E_i \left( \frac{s}{1-t} \right). \tag{36} \]
Thus, from (35) and (36), we obtain the following unique solution of (27):

$$g(0) = e^{-s} \left( -\ln |1-t| + \sum_{n=1}^{\infty} \frac{s^n}{n!} \left( \left( \frac{1}{1-t} \right)^n - 1 \right) \right).$$

(37)

Moreover, if we choose another initial condition

$$g(0) = \sum_{N \geq 1} a_0(N) \frac{t^N}{N!} = \sum_{N \geq 1} \frac{t^N}{N!}$$

(38)

from (20) and (22), then (37) satisfies it.

We note that

$$\left( \frac{1}{1-t} \right)^n - 1 = \left( \sum_{l \geq 0} t^l \right) \times \cdots \times \left( \sum_{k \geq 0} t^k \right) - 1$$

\[= \sum_{N \geq 1} \left( \sum_{l+r+\cdots+N} t^N \right) \]

(39)

By (37) and (39), we get

$$g(0) = \left( \sum_{k \geq 0} \frac{(-1)^k}{k!} s^k \right) \left( \sum_{N \geq 1} \frac{t^N}{N!} \right)$$

$$+ \left( \sum_{k \geq 0} \frac{(-1)^k}{k!} \right)$$

$$\times \left( \sum_{n \geq 1} \frac{s^n}{n!} \sum_{N \geq 1} \left( \frac{n+N-1}{N} \right) t^N \right)$$

$$= \sum_{N \geq 1} \sum_{k \geq 0} \frac{(-1)^k}{N \cdot k!} \sum_{s \geq 0} \frac{(-1)^s}{s!} \sum_{k \geq 0} \frac{(-1)^k}{k!} s^k$$

$$+ \sum_{N \geq 1} \sum_{k \geq 0} \frac{(-1)^k}{k!} \sum_{l \geq 1} \sum_{k \geq 0} \frac{(-1)^k}{k!} \left( \frac{l+N-1}{N} \right) t^N s^k$$

$$+ \sum_{N \geq 1} \sum_{k \geq 0} \left( \sum_{l \geq 1} \left( \frac{l+N-1}{N} \right) t^N s^k \right).$$

(40)

It is known that

$$(-1)^l \left( \frac{l+N-1}{N} \right) = \frac{l}{N} \left( \frac{-N}{l} \right),$$

$$\sum_{k \geq 0} \frac{k}{l} \left( \frac{N}{l} \right) = \frac{k+N}{k}.$$ 

(41)

In the case of $k \geq N$ in (40), from (41), we get

$$\frac{(-1)^k}{N \cdot k!} + \sum_{l \geq 1} \frac{(-1)^{k-l}}{(k-l)! l!} \left( \frac{l+N-1}{N} \right)$$

$$= \frac{(-1)^k}{N \cdot k!} + \sum_{l \geq 1} \frac{1}{k!} \sum_{l \geq 1} \left( \frac{k}{l} \left( \frac{-N}{l} \right) \right)$$

(42)

$$= \frac{(-1)^k}{N \cdot k!} \left( 1 + \left( \frac{k-N}{k} \right) - 1 \right) = 0.$$

By (40) and (41), we get

$$g(0) = \sum_{N \geq 1} \sum_{k \geq 0} \frac{(-1)^k}{N \cdot k!} \left( \frac{k-N}{k} \right) t^N s^k$$

$$+ \sum_{N \geq 1} \left( \frac{-N}{k} \right)^2 t^N s^k,$$

(43)

where $\left( \frac{k-N}{k} \right) = (-1)^k \left( \frac{N-1}{k} \right)$. Thus, by (22) and (43), we get

$$a_k(N) = \left( \frac{-N}{k} \right)^2.$$

(44)

Therefore, by (13) and (44), we obtain the following theorem.

**Theorem 1.** For $q \in C$ with $|q| < 1$ and $N \in \mathbb{N}$, one can consider the following nonlinear $(N-1)$th order ordinary differential equation with respect to $q$:

$$G^N(q) = \frac{1}{(N-1)!} \sum_{k=0}^{N-1} (N-k-1)! \left( \frac{N-1}{k} \right)^2 q^k G^{(k)}$$

(45)

where $G^{(k)} = d^k G^{(0)}/dq^k$ and $G^{(N)}(q) = G(q) \times \cdots \times G(q).$

Then $G(q) = 1/(qe^q + 1)$ is a solution of (45).

Let us define $G^{(k)}(t, x) = G^{(k)}(q)ex^t$. Then we obtain the following corollary.

**Corollary 2.** For $N \in \mathbb{N}$, one considers

$$G^N(t, x) = \frac{1}{(N-1)!} \sum_{k=0}^{N-1} (N-k-1)! \left( \frac{N-1}{k} \right)^2 q^k G^{(k)}(t, x)$$

$$= \sum_{k=0}^{N-1} \frac{1}{k!} \left( \frac{N-1}{k} \right)^2 q^k G^{(k)}(t, x).$$

(46)

Then $G(t, x) = e^{xt}/(qe^q + 1)$ is a solution of (46).
3. Identities on the High-Order $q$-Euler Numbers and Polynomials with Weight 0

From (3), (7), and (8), we get

$$G^N(q) = \frac{1}{2^N} \left( \frac{2}{qe^2 + 1} \right) \times \cdots \times \left( \frac{2}{qe^2 + 1} \right) \overset{N\text{-times}}{=} \frac{1}{2^N} \sum_{n=0}^{\infty} E_n^{(N)} \frac{t^n}{n!}.$$  

$$G(q) = \frac{1}{2} \left( \frac{2}{qe^2 + 1} \right) = \frac{1}{2} \sum_{n=0}^{\infty} E_n(q) \frac{t^n}{n!}.$$  

From (47), we note that

$$G^{(k)} = \frac{d^k G(q)}{dq^k} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{d^k E_n(q)}{dq^k} \frac{t^n}{n!}. \quad (48)$$

Therefore, by (47), (48), and (45), we obtain the following theorem.

**Theorem 3.** For $N \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, one has

$$\widetilde{E}_n^{(N)} = \sum_{k=0}^{N-1} \frac{1}{k!} \binom{N-1}{k} q^k \frac{d^k E_n}{dq^k}.$$  

From (48), we get

$$G^{(k)}(t, x) = G^{(k)}(q) e^{xt} = \left( \sum_{n=0}^{\infty} \frac{d^n E_n(q)}{dq^n} \frac{t^n}{n!} \right) \left( \sum_{l=0}^{\infty} \frac{x^l}{l!} \right) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{(n! \cdot l!)} x^{n+l} \frac{d^n E_n}{dq^n} \frac{t^n}{n!}. \quad (50)$$

Therefore, by (7), (47), and (50), we obtain the following corollary.

**Corollary 4.** For $N \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, one has

$$\widetilde{E}_n^{(N)}(x) = \sum_{k=0}^{N-1} \frac{1}{k!} \binom{N-1}{k} q^k \sum_{l=0}^{n} \frac{n!}{l!} x^{n+l} \frac{d^n E_n}{dq^n} \frac{t^n}{n!}. \quad (51)$$

From (3) and (7), we get

$$\sum_{n=0}^{\infty} \sum_{k=0}^{N-1} \frac{E_n^{(N)}}{n!} \frac{t^n}{n!} = \left( \frac{2}{qe^2 + 1} \right) \times \cdots \times \left( \frac{2}{qe^2 + 1} \right) \overset{N\text{-times}}{=} \left( \sum_{l=0}^{N} \frac{E_l}{l!} \right) \times \cdots \times \left( \sum_{l=0}^{N} \frac{E_l}{l!} \right).$$

Therefore, by (49) and (52), we obtain the following corollary.

**Corollary 5.** For $N \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, one has

$$\sum_{n=0}^{\infty} \frac{E_n^{(N)}}{n!} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{E_n^{(N)}}{n!} \frac{t^n}{n!} = \frac{1}{2} \left( \frac{2}{qe^2 + 1} \right) \times \cdots \times \left( \frac{2}{qe^2 + 1} \right) \overset{N\text{-times}}{=} \left( \sum_{l=0}^{N} \frac{E_l}{l!} \right) \times \cdots \times \left( \sum_{l=0}^{N} \frac{E_l}{l!} \right).$$  

$$\sum_{n=0}^{\infty} \sum_{k=0}^{N-1} \frac{E_n^{(N)}}{n!} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{N} \frac{E_l}{l!} \right) \times \cdots \times \left( \sum_{l=0}^{N} \frac{E_l}{l!} \right) \frac{t^n}{n!}. \quad (52)$$

Therefore, by (49) and (52), we obtain the following corollary.

**Corollary 5.** For $N \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, one has

$$\sum_{n=0}^{\infty} \sum_{k=0}^{N-1} \frac{E_n^{(N)}}{n!} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{N} \frac{E_l}{l!} \right) \times \cdots \times \left( \sum_{l=0}^{N} \frac{E_l}{l!} \right) \frac{t^n}{n!}. \quad (53)$$

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**References**


