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Research Article

Bifurcations of a Ratio-Dependent Holling-Tanner System with Refuge and Constant Harvesting

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The bifurcation properties of a predator prey system with refuge and constant harvesting are investigated. The number of the equilibria and the properties of the system will change due to refuge and harvesting, which leads to the occurrence of several kinds bifurcation phenomena, for example, the saddle-node bifurcation, Bogdanov-Takens bifurcation, Hopf bifurcation, backward bifurcation, separatrix connecting a saddle-node and a saddle bifurcation and heteroclinic bifurcation, and so forth. Our main results reveal much richer dynamics of the system compared to the system with no refuge and harvesting.

1. Introduction

The Holling-Tanner predator-prey system has attracted much attentions from both theoretical and mathematical biologists, especially, in [1] the authors considered the ratio-dependent system of the form

$$\begin{aligned} \dot{x} &= rx\left(1-\frac{x}{K}\right)-\frac{\alpha xy}{Ay+x}, \\ \dot{y} &= sy\left(1-\frac{by}{x}\right), \end{aligned} \tag{1}$$

where x and y stand for prey and predator population (or densities) at time t, respectively. The predator growth is of logistic type with growth rate r and carrying capacity K in the absence of predation; α and A stand for the predator capturing rate and half saturation constant, respectively; s is the intrinsic growth rate of predator; however, carrying capacity x/b (b is the conversion rate of prey into predators) is the function on the population size of prey. They studied the global properties and the existence and uniqueness of limit cycle for system (1).

Generally speaking, from the views of the optimal management and exploitation of bioeconomic resources, it is necessary and meaningful to consider the refuge or harvesting of populations in some bioeconomic models; one can see [2–11], and the references therein.

In this paper we will analyze the system (1) with refuge and harvesting of the form

$$\begin{aligned} \dot{x} &= rx\left(1-\frac{x}{K}\right)-\frac{\alpha y(x-\overline{m})}{Ay+x-\overline{m}}-\overline{h}, \\ \dot{y} &= sy\left(1-\frac{by}{x-\overline{m}}\right), \end{aligned} \tag{2}$$

where r, K, α, A, $\overline{m}$, $\overline{h}$, s, and b are positive constants. $\overline{m}$ is a constant number of prey using refuges, and $\overline{h}$ is the rate of prey harvesting.

For simplicity, we first rescale the system (2).

Let $X = x-\overline{m}$, $Y = y$; system (2) can be written as (still denote X, Y as x, y)

$$\begin{aligned} \dot{x} &= r(x+\overline{m})\left(1-\frac{x+\overline{m}}{K}\right)-\frac{\alpha xy}{Ay+x}-\overline{h}, \\ \dot{y} &= sy\left(1-\frac{by}{x}\right). \end{aligned} \tag{3}$$

Next, let $\tau = rt$, $X = x/K$, and $Y = \alpha y/rK$, then system (3) takes the form (still denote X, Y, and τ as x, y, t)

$$\begin{aligned} \dot{x} &= (x+m)(1-x-m) - \frac{xy}{ay+x} - h = P(x,y), \\ \dot{y} &= \delta y\left(\beta - \frac{y}{x}\right) = Q(x,y), \end{aligned} \tag{4}$$

where $m = \overline{m}/K$, $a = Ar/\alpha$, $\delta = s\overline{h}/\alpha$, $\beta = \alpha/br$, and $h = \overline{h}/r$.

From the view of biology, we are only interested in the dynamics of the system (4) in the first quadrant.

The organization of this paper is as follows. In Section 2, we discuss the existence and properties of the equilibria of system (4). In Section 3, all possible bifurcation phenomena of the model in terms of the five parameters are presented, and the numerical simulations about every bifurcation phenomena are exhibited.

2. Qualitative Analysis of Equilibria

To obtain the boundary equilibria the following equation can be obtained

$$x^2 + (2m-1)x + h - m(1-m) = 0. \tag{5}$$

Its discriminant is $\Delta_0 = 1 - 4h$.

Obviously, $\Delta_0 \geq 0$ if $0 < h \leq 1/4$ and $\Delta_0 < 0$ if $h > 1/4$.

Hence, (5) has two distinct positive solutions $x_{01} = (1 - 2m + \sqrt{1-4h})/2$, $x_{02} = (1 - 2m - \sqrt{1-4h})/2$ if $0 < m < 1/2$, $m(1-m) < h < 1/4$, a positive solution x_{01} if $0 < m < 1$, $0 < h < m(1-m)$, a double solution $\overline{x} = (1-2m)/2 > 0$ if $0 < m < 1/2$, $h = 1/4$, and a solution $x_{03} = 1 - 2m$ when $h = m(1-m)$ and $0 < m < 1/2$.

One can obtain the positive equilibrium of (4) by solving the equation

$$x^2 + \left(\frac{\beta}{a\beta+1} + 2m - 1\right)x + h + m(m-1) = 0. \tag{6}$$

We can derive that system (4) has two positive equilibria $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ if

$$\begin{aligned} &0 < m < \frac{1}{2}\left(1 - \frac{\beta}{a\beta+1}\right), \\ &m(1-m) < h < \frac{1}{4}\left(\frac{\beta}{a\beta+1} - 1\right)^2 + \frac{m\beta}{a\beta+1}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} x_i &= -\frac{1}{2}\left(\frac{\beta}{a\beta+1} + 2m - 1\right) \\ &\quad + (-1)^{i+1}\frac{\sqrt{\Delta}}{2}, \quad y_i = \beta x_i, \quad i = 1, 2, \\ \Delta &= \left(\frac{\beta}{a\beta+1} - 1\right)^2 + \frac{4m\beta}{a\beta+1} - 4h. \end{aligned} \tag{8}$$

Moreover, we can show that system (4) just exists one positive equilibrium P_1 if $0 < h < m(1-m)$ and $0 < m < 1$.

The positive equilibrium $P_3 = (x_3, y_3)$ ($P_* = (x_*, y_*)$) of system (4) exists if $0 < m < (1/2)(1 - \beta/(a\beta+1))$, and $h = m(1-m)(0 < m < (1/2)(1 - \beta/(a\beta+1))$, $h = (1/4)(\beta/(a\beta+1) - 1)^2 + m\beta/(a\beta+1))$, where $x_3 = 1 - 2m - \beta/(a\beta+1)$, $y_3 = \beta x_3$, $x_* = -(1/2)(\beta/(a\beta+1) + 2m - 1)$, and $y_* = \beta x_*$.

Summarizing the previous discussion, the number and location of equilibria of system (4) can be described by the following lemmas.

Lemma 1. *Let* $1/2 \leq m < 1$.

(i) *System* (4) *has no equilibria when* $h \geq m(1-m)$;

(ii) *System* (4) *exist two equilibria* $E_1 = (x_{01}, 0)$ *and* P_1 *when* $0 < h < m(1-m)$.

Lemma 2. *Let* $0 < (1/2)(1 - \beta/(a\beta+1)) \leq m < 1/2$.

(i) *System* (4) *has no equilibria when* $h > 1/4$.

(ii) *System* (4) *has a unique equilibrium* $\overline{E} = (\overline{x}, 0)$ *when* $h = 1/4$.

(iii) *System* (4) *has two equilibria* $E_1 = (x_{01}, 0)$ *and* $E_2 = (x_{02}, 0)$ *when* $m(1-m) < h < 1/4$.

(iv) *System* (4) *has an equilibrium* $E_3 = (x_{03}, 0)$ *when* $h = m(1-m)$.

(v) *System* (4) *has two equilibria* E_1 *and* P_1 *when* $0 < h < m(1-m)$.

Lemma 3. *Let* $0 < m < (1/2)(1 - \beta/(a\beta+1))$ *and* $\widetilde{h} = (1/4)(\beta/(a\beta+1) - 1)^2 + m\beta/(a\beta+1)$.

(i) *System* (4) *has no equilibria when* $h > 1/4$.

(ii) *System* (4) *has a unique equilibrium* $\overline{E} = (\overline{x}, 0)$ *when* $h = 1/4$.

(iii) *System* (4) *has two equilibria* E_1 *and* E_2 *when* $\widetilde{h} < h < 1/4$.

(iv) *System* (4) *has three equilibria* E_1, E_2, *and* P_* *when* $h = \widetilde{h}$.

(v) *System* (4) *has four equilibria* E_1, E_2, P_1, *and* P_2 *when* $m(1-m) < h < \widetilde{h}$.

(vi) *System* (4) *has two equilibria* E_1 *and* P_1 *when* $0 < h < m(1-m)$.

(vii) *System* (4) *has two equilibria* E_3 *and* P_3 *when* $h = m(1-m)$.

Next we discuss the dynamics of system (4) in the neighborhood of each feasible equilibria. Firstly, the Jacobian matrix of system (4) at E_1 is

$$M_{E_1} = \begin{bmatrix} -\sqrt{1-4h} & -1 \\ 0 & \delta\beta \end{bmatrix}. \tag{9}$$

It is easy to see that E_1, if exists, is a hyperbolic saddle.

Secondly, the Jacobian matrix of system (4) at E_2 is

$$M_{E_2} = \begin{bmatrix} \sqrt{1-4h} & -1 \\ 0 & \delta\beta \end{bmatrix}. \tag{10}$$

One can see that boundary equilibrium E_2, if exists, is an unstable hyperbolic node.

The Jacobian matrix of system (4) at E_3 is

$$M_{E_3} = \begin{bmatrix} 2m-1 & -1 \\ 0 & \delta\beta \end{bmatrix}. \tag{11}$$

Hence, E_3, if exists, is a saddle.

Similarly, we assume boundary equilibrium $\overline{E}$ exists, and the Jacobian matrix of system (4) at $\overline{E}$ is obtained as follows:

$$M_{\overline{E}} = \begin{bmatrix} 0 & -1 \\ 0 & \delta\beta \end{bmatrix}. \tag{12}$$

Hence, $\overline{E}$ is a saddle node.

The previous discussion can be summarized as follows.

Theorem 4. *If the equilibria E_1, E_2, and E_3 exist, then E_1 and E_3 are hyperbolic saddle, and E_2 is a hyperbolic unstable node. Moreover, E_1 and E_2 merge into a saddle node $\overline{E} = (\overline{x}, 0)$ when $h = 1/4$.*

Remark 5. Note that if $h = 1/4$, then $\dot{x} = -(x+m-1/2)^2 - xy/(x+ay) < 0$, if $h > 1/4$, then $\dot{x} \le 1/4 - h < 0$. Thus, the prey species may go extinct as time increases for some initial values when $h \ge 1/4$. That is, biological over harvesting occurs.

In the following, we will discuss the properties of interior equilibria of system (4).

2.1. The Properties of Interior Equilibria. The Jacobian matrix of system (4) at P_1 is

$$M_{P_1} = \begin{bmatrix} \dfrac{\beta}{(a\beta+1)^2} - \sqrt{\Delta} & -\dfrac{1}{(a\beta+1)^2} \\ \delta\beta^2 & -\delta\beta \end{bmatrix}. \tag{13}$$

The characteristic equation is $\lambda^2 + A_1\lambda + A_2 = 0$, where

$$A_1 = \sqrt{\Delta} - \frac{\beta}{(1+a\beta)^2} + \delta\beta, \qquad A_2 = \delta\beta\sqrt{\Delta} > 0. \tag{14}$$

Denote that

$$\begin{aligned} A_1^2 - 4A_2 = \beta^2\delta^2 - 2\left(\sqrt{\Delta} + \frac{\beta}{(1+a\beta)^2}\right)\beta\delta \\ + \left(\sqrt{\Delta} - \frac{\beta}{(1+a\beta)^2}\right)^2 := F(\delta). \end{aligned} \tag{15}$$

The discriminant of $F(\delta) = 0$ is $\Delta_1 = 16\beta^3\sqrt{\Delta}/(1+a\beta)^2 > 0$, then $F(\delta) = 0$ has two distinct solutions δ_1 and δ_2 denoted by

$$\delta_1 = \left(\sqrt{\frac{\sqrt{\Delta}}{\beta}} - \frac{1}{1+a\beta}\right)^2, \qquad \delta_2 = \left(\sqrt{\frac{\sqrt{\Delta}}{\beta}} + \frac{1}{1+a\beta}\right)^2. \tag{16}$$

If $\sqrt{\Delta} \ne \beta/(1+a\beta)^2$, it is easy to see that P_1 is a node if $0 < \delta < \delta_1$ or $\delta > \delta_2$, a degenerate node if $\delta = \delta_1$ or $\delta = \delta_2$, and a focus or a center type nonhyperbolic if $\delta_1 < \delta < \delta_2$.

If $\sqrt{\Delta} = \beta/(1+a\beta)^2$, P_1 is a node if $\delta > \delta_2$, and a degenerate node if $\delta = \delta_2$, and a focus or a center-type nonhyperbolic if $0 < \delta < \delta_2$.

To discuss the stability of P_1, we need to determine the sign of A_1. Define $\tilde{\delta} = 1/(1+a\beta)^2 - (1/\beta)\sqrt{\Delta}$, then $A_1 = \beta(\delta - \tilde{\delta})$.

Clearly, if $\beta/(1+a\beta)^2 \le \sqrt{\Delta}$ then $A_1 > 0$ for all δ; if $\beta/(1+a\beta)^2 > \sqrt{\Delta}$, then $A_1 > 0$ when $\delta > \tilde{\delta}$, $A_1 \le 0$ when $0 < \delta \le \tilde{\delta}$; by simple computation, one can obtain $F(\tilde{\delta}) = 4\sqrt{\Delta}(\sqrt{\Delta} - \beta/(a\beta+1)^2) < 0$, hence $\delta_1 < \tilde{\delta} < \delta_2$.

The Jacobian matrix of system (4) at P_2 is

$$M_{P_2} = \begin{bmatrix} \dfrac{\beta}{(a\beta+1)^2} + \sqrt{\Delta} & -\dfrac{1}{(a\beta+1)^2} \\ \delta\beta^2 & -\delta\beta \end{bmatrix}. \tag{17}$$

Its determinant is $\det M_{P_2} = -\delta\beta\sqrt{\Delta} < 0$.

Through the previous discussion, about the stability of P_1 and P_2, we have the following theorem.

Theorem 6. *Equilibrium P_2, if exists, must be a hyperbolic saddle. Equilibrium P_1, if exists, may be a node or a focus when $\beta/(1+a\beta)^2 \le \sqrt{\Delta}$. And when $\beta/(1+a\beta)^2 > \sqrt{\Delta}$, P_1 is a stable focus for $\delta_2 > \delta > \tilde{\delta}$, a stable degenerate node for $\delta = \delta_2$, a stable node for $\delta > \delta_2$, an unstable node for $0 < \delta < \delta_1$, an unstable degenerate node for $\delta = \delta_1$, an unstable focus for $\delta_1 < \delta < \tilde{\delta}$, and a weak focus or a center for $\delta = \tilde{\delta}$.*

The Jacobian matrix of system (4) at P_3 is

$$M_{P_3} = \begin{bmatrix} 2m-1+\dfrac{\beta(a\beta+2)}{(a\beta+1)^2} & -\dfrac{1}{(a\beta+1)^2} \\ \delta\beta^2 & -\delta\beta \end{bmatrix}. \tag{18}$$

then by the existence condition of P_3, $\det M_{P_3} = -\beta\delta(2m - 1 + \beta/(a\beta+1)) > 0$, $\operatorname{tr} M_{P_3} = 2m - 1 - \delta\beta + \beta/(1+a\beta) + \beta/(1+a\beta)^2$. Then by taking similar methods used in estimating the properties of P_1, we have the following theorem.

Theorem 7. *Let $h = m(1-m)$, $\overline{\delta} = (1/\beta)(2m - 1 + \beta/(1+a\beta) + \beta/(1+a\beta)^2)$. Then,*

(i) *assume $0 < m \le (1/2)(1 - \beta/(1+a\beta) - \beta/(1+a\beta)^2)$, then P_3 is stable;*

(ii) *assume $(1/2)(1 - \beta/(1+a\beta) - \beta/(1+a\beta)^2) < m < (1/2)(1 - \beta/(1+a\beta))$, then P_3 is stable if $\delta > \overline{\delta}$, is unstable if $0 < \delta < \overline{\delta}$, and is a weak focus or a center if $\delta = \overline{\delta}$.*

The Jacobian matrix of system (4) at P_* is

$$M_{P_*} = \begin{bmatrix} \dfrac{\beta}{(a\beta+1)^2} & -\dfrac{1}{(a\beta+1)^2} \\ \delta\beta^2 & -\delta\beta \end{bmatrix}. \tag{19}$$

One can see that $\det M_{P_*} = 0$, which indicates that P_* is a degenerate singularity and maybe has complicated properties, see the following theorem.

Theorem 8. *Let $0 < m < (1/2)(1 - \beta/(a\beta + 1))$, $h = \tilde{h}$. Then system (4) has three equilibria, where E_1 is a hyperbolic saddle, E_2 is a hyperbolic unstable node, and P_* is a degenerate singularity. More precisely,*

1° if $\delta \neq 1/(a\beta + 1)^2$, then P_* is a saddle node;

2° if $\delta = 1/(a\beta + 1)^2$, then P_* is a cusp of codimension 2.

Proof. In order to discuss the properties of system (4) in the neighborhood of the equilibrium $P_* = (x_*, y_*)$, we first take $\overline{x} = x - x_*$, $\overline{y} = y - y_*$, then P_* is translated to $(0, 0)$, and system (4) becomes (still denote $\overline{x}$, $\overline{y}$ as x, y)

$$\begin{aligned} \dot{x} &= \frac{\beta}{(a\beta+1)^2}x - \frac{1}{(a\beta+1)^2}y - g_1x^2 \\ &\quad + g_2xy - g_3y^2 + O\left(|x, y|^3\right), \\ \dot{y} &= \delta\beta^2x - \delta\beta y + g_4x^2 - g_5xy + g_6y^2 + O\left(|x, y|^3\right), \end{aligned} \tag{20}$$

where

$$\begin{aligned} g_1 &= \frac{2a\beta^2}{(a\beta+1)^2((2m-1)(a\beta+1) + \beta)} + 1, \\ g_2 &= \frac{4a\beta}{(a\beta+1)^2((2m-1)(a\beta+1)+\beta)}, \\ g_3 &= \frac{2a}{(a\beta+1)^2((2m-1)(a\beta+1)+\beta)}, \\ g_4 &= \frac{2\delta\beta^2(a\beta+1)}{(2m-1)(a\beta+1)+\beta}, \\ g_5 &= \frac{4\delta\beta(a\beta+1)}{(2m-1)(a\beta+1)+\beta}, \\ g_6 &= \frac{2\delta(a\beta+1)}{(2m-1)(a\beta+1)+\beta}. \end{aligned} \tag{21}$$

Clearly, if $\delta \neq 1/(a\beta + 1)^2$, then $\operatorname{tr}(M_{P_*}) \neq 0$. $P_* = (x_*, y_*)$ is a saddle-node. We finish the proof of the part 1°.

When $\delta = 1/(a\beta + 1)^2$, $\operatorname{tr} M_{P_*} = 0$, which implies that both eigenvalues of the matrix M_{P_*} are zero. We rewrite system (20) as

$$\begin{aligned} \dot{x} &= \frac{\beta}{(a\beta+1)^2}x - \frac{1}{(a\beta+1)^2}y - q_1x^2 \\ &\quad + q_2xy - q_3y^2 + O\left(|x, y|^3\right), \\ \dot{y} &= \frac{\beta^2}{(a\beta+1)^2}x - \frac{\beta}{(a\beta+1)^2}y + q_4x^2 \\ &\quad - q_5xy + q_6y^2 + O\left(|x, y|^3\right), \end{aligned} \tag{22}$$

where

$$\begin{aligned} q_1 &= g_1, \quad q_2 = g_2, \quad q_3 = g_3, \\ q_4 &= \frac{2\beta^2}{(a\beta+1)^2((2m-1)(a\beta+1)+\beta)}, \\ q_5 &= \frac{4\beta}{(a\beta+1)^2((2m-1)(a\beta+1)+\beta)}, \\ q_6 &= \frac{2}{(a\beta+1)^2((2m-1)(a\beta+1)+\beta)}. \end{aligned} \tag{23}$$

By introducing variable $\tau = (\beta/(a\beta + 1)^2)t$ into previous system and rewriting τ as t for simplicity, then we obtain that

$$\begin{aligned} \dot{x} &= x - \frac{1}{\beta}y - \omega_1x^2 + \omega_2xy - \omega_3y^2 + O\left(|x, y|^3\right), \\ \dot{y} &= \beta x - y + \omega_4x^2 - \omega_5xy + \omega_6y^2 + O\left(|x, y|^3\right), \end{aligned} \tag{24}$$

where

$$\begin{aligned} \omega_1 &= \frac{2a\beta}{(2m-1)(a\beta+1)+\beta} + \frac{(a\beta+1)^2}{\beta}, \\ \omega_2 &= \frac{4a}{(2m-1)(a\beta+1)+\beta}, \\ \omega_3 &= \frac{2a}{\beta((2m-1)(a\beta+1)+\beta)}, \\ \omega_4 &= \frac{2\beta}{(2m-1)(a\beta+1)+\beta}, \\ \omega_5 &= \frac{4}{(2m-1)(a\beta+1)+\beta}, \\ \omega_6 &= \frac{2}{\beta((2m-1)(a\beta+1)+\beta)}. \end{aligned} \tag{25}$$

We take transformation $X_0 = x$, $Y_0 = x - (1/\beta)y$ into (24), then system (24) is transformed to

$$\begin{aligned} \dot{X}_0 &= Y_0 + \eta_1X_0^2 + \eta_2X_0Y_0 - \omega_3\beta^2Y_0^2 + O\left(|X_0, Y_0|^3\right), \\ \dot{Y}_0 &= \eta_3X_0^2 + \eta_4X_0Y_0 - \eta_5Y_0^2 + O\left(|X_0, Y_0|^3\right), \end{aligned} \tag{26}$$

where

$$\begin{aligned} \eta_1 &= -\omega_3\beta^2 - \omega_1 + \omega_2\beta, \\ \eta_2 &= 2\omega_3\beta^2 - \omega_2\beta, \\ \eta_3 &= -\frac{\omega_1\beta - \omega_2\beta^2 + \omega_3\beta^3 + \omega_4 - \omega_5\beta + \omega_6\beta^2}{\beta}, \\ \eta_4 &= 2\omega_3\beta^2 - \omega_2\beta - \omega_5 + 2\omega_6\beta, \\ \eta_5 &= \beta(\omega_3\beta + \omega_6). \end{aligned} \tag{27}$$

In order to obtain the canonical normal forms of system (26), we will perform a series of C^∞ transformations of variables for system (26) in a small neighborhood of $(0,0)^{\mathrm{T}}$.

Firstly, performing the transformation by taking $X_1 = X_0$, $Y_1 = Y_0 - \omega_3\beta^2 Y_0^2$, then (26) becomes

$$\begin{aligned} \dot{X}_1 &= Y_1 + \eta_1 X_1^2 + \eta_2 X_1 Y_1 + O\left(\left|X_1, Y_1\right|^3\right), \\ \dot{Y}_1 &= \eta_3 X_1^2 + \eta_4 X_1 Y_1 - \eta_5 Y_1^2 + O\left(\left|X_1, Y_1\right|^3\right). \end{aligned} \tag{28}$$

Secondly, performing the transformation by taking $X_2 = X_1$, $Y_2 = Y_1 + \eta_5 X_1 Y_1$, then (28) becomes

$$\begin{aligned} \dot{X}_2 &= Y_2 + \eta_1 X_2^2 + \left(\eta_2 - \eta_5\right) X_2 Y_2 + O\left(\left|X_2, Y_2\right|^3\right), \\ \dot{Y}_2 &= \eta_3 X_2^2 + \eta_4 X_2 Y_2 + O\left(\left|X_2, Y_2\right|^3\right). \end{aligned} \tag{29}$$

We perform the final transformation of variables by

$$X = X_2 - \frac{\eta_2 - \eta_5}{2} X_2^2, \qquad Y = Y_2 + \eta_1 X_2^2 + O\left(\left|X_2, Y_2\right|^3\right). \tag{30}$$

Then, we obtain

$$\begin{aligned} \dot{X} &= Y, \\ \dot{Y} &= \eta_3 X^2 + \left(2\eta_1 + \eta_4\right) XY + O\left(|X, Y|^3\right). \end{aligned} \tag{31}$$

Note that

$$\eta_3 = -\frac{(a\beta+1)^2}{\beta} \neq 0, \qquad 2\eta_1 + \eta_4 = -\frac{2(a\beta+1)^2}{\beta} \neq 0, \tag{32}$$

which indicates that the origin $(0,0)$ of (31) is a cusp of codimension 2. We complete the proof. □

3. Bifurcation Analysis

From previous analysis, we can see the equilibria of system (4) may be hyperbolic or degenerate singularities under appropriate conditions, which indicate that some bifurcations may occur for system (4). It is interesting to investigate what kinds of bifurcations system (4) can undergo with the original parameters varying.

3.1. Hopf Bifurcation. Theorem 6 shows that P_1, if exists, is a weak focus or a center when

$$\begin{aligned} V = \Bigg\{ &(a, m, \beta, \delta, h) : 0 < m < \widetilde{m}, m(1-m) \\ &< h < \widetilde{h}, \frac{\beta}{(1+a\beta)^2} > \sqrt{\Delta}, \delta = \widetilde{\delta} \Bigg\}, \end{aligned} \tag{33}$$

where $\widetilde{m} = (1/2)(1 - \beta/(a\beta+1))$.

To determine the direction of Hopf bifurcation and stability of P_1 in this case, we need to compute the Liapunov coefficients of the equilibrium P_1. Let $\delta = \widetilde{\delta}$ and by the variable $u = x - x_1$, $v = y - y_1$. Then we rewrite system (4) (still denote u, v as x, y) as follows:

$$\begin{aligned} \dot{x} &= \widetilde{\delta}\beta x - \frac{1}{(a\beta+1)^2} y + f_1(x, y), \\ \dot{y} &= \widetilde{\delta}\beta^2 x - \widetilde{\delta}\beta y + f_2(x, y). \end{aligned} \tag{34}$$

We perform the transformations

$$\begin{aligned} X &= x, \qquad Y = \widetilde{\delta}\beta x - \frac{1}{(a\beta+1)^2} y, \\ u &= X, \qquad v = \frac{(a\beta+1) Y}{\beta\sqrt{\widetilde{\delta}\left(1 - \widetilde{\delta}(a\beta+1)^2\right)}} \end{aligned} \tag{35}$$

and rewrite u, v as x, y. Then the previous system can be transformed to

$$\begin{aligned} \dot{x} &= k_1 y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ &\quad + a_{30}x^3 + a_{21}x^2 y + a_{12}xy^2 + a_{03}y^3 + O\left(\left|(x, y)\right|^4\right), \\ \dot{y} &= -k_1 x + d_{11}xy + b_{20}x^2 + b_{11}xy + b_{02}y^2 \\ &\quad + b_{30}x^3 + b_{21}x^2 y + b_{12}xy^2 + b_{03}y^3 + O\left(\left|(x, y)\right|^4\right), \end{aligned} \tag{36}$$

where the expressions of a_{20}, a_{11}, a_{02}, a_{30}, a_{21}, a_{12}, a_{03}, b_{20}, b_{11}, b_{02}, b_{30}, b_{21}, b_{12}, and b_{03} depend on the parameters a, β, δ, h, and m, and $k_1 = \beta\sqrt{\widetilde{\delta}(1 - \widetilde{\delta}(a\beta+1)^2)}/(a\beta+1) > 0$.

Using the formula, the first Liapunov number is

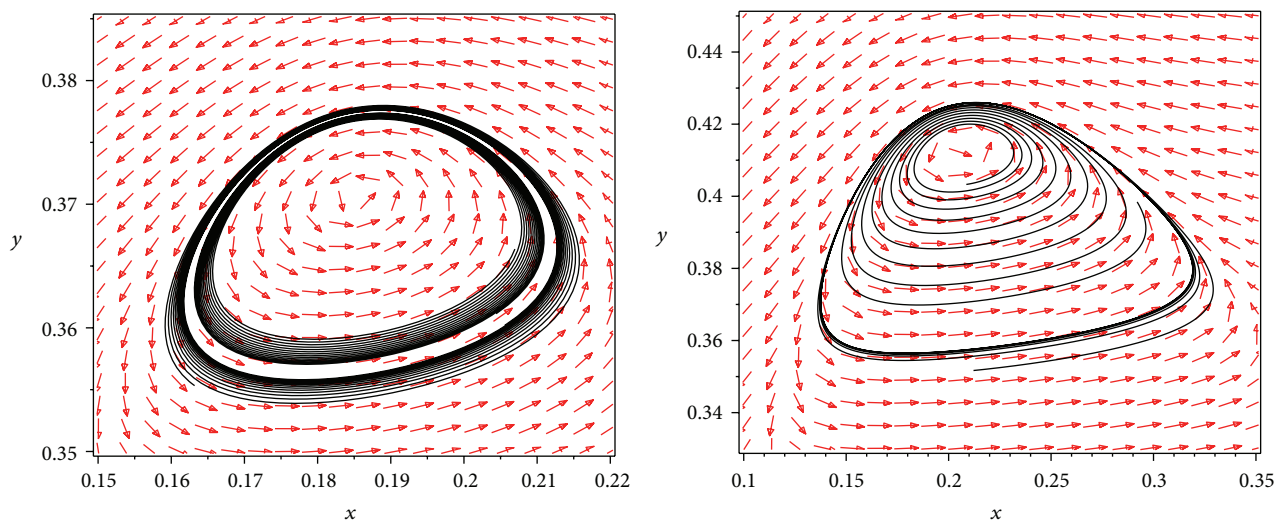
$$\begin{aligned} \sigma = -\frac{1}{8\beta k_1} \Bigg(&\frac{2(a\beta+1)^2}{\widetilde{\delta}(a\beta+1)^2 - 1} + \frac{2\beta(2a\beta+1)}{(a\beta+1)x_1} \\ &+ \frac{a\beta^3(2a\beta-1)\left[\widetilde{\delta}(a\beta+1)^2 - 1\right]}{x_1^2(a\beta+1)^4} \Bigg). \end{aligned} \tag{37}$$

Therefore, there exists a surface H_b (H_p) in the parameter space (a, m, β, δ, h) which satisfies

$$\begin{aligned} H_b &= \left\{(a, m, \beta, \delta, h) : \sigma > 0,\ (a, m, \beta, \delta, h) \in V\right\}, \\ \left(H_p\right. &= \left.\left\{(a, m, \beta, \delta, h) : \sigma < 0,\ (a, m, \beta, \delta, h) \in V\right\}\right). \end{aligned} \tag{38}$$

Hence, when the parameter (a, m, β, δ, h) is in H_b (H_p), the equilibrium P_1 of system (4) is a weak focus of multiplicity 1 and is unstable (stable) (see [8]). H_b (H_p) is called the subcritical (supercritical) Hopf bifurcation surface of system (4).

From Theorem 6, we know that P_1 is a stable focus for $\delta_2 > \delta > \widetilde{\delta}$ and $(a, m, \beta, \delta, h) \in V$, an unstable focus for $\delta_1 < \delta < \widetilde{\delta}$ and $(a, m, \beta, \delta, h) \in V$.



(a) (b)

Figure 1: (a) System (4) shows an unstable limit cycle when $a = 1.6$, $\beta = 2$, $m = 0.13095$, and $h = 0.128$. $\delta_2 \approx 0.2220 > \delta = 0.00249 > \tilde{\delta} \approx 0.00235$; (b) System (4) shows a stable limit cycle when $a = 1.2$, $\beta = 2$, $m = 0.08235$, and $h = 0.08396$. $\delta_1 \approx 0.0000391 < \delta = 0.002919 < \tilde{\delta} \approx 0.003639$.

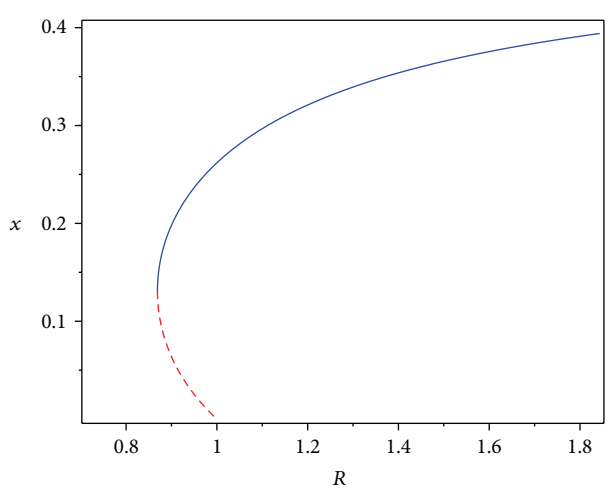


Figure 2: The figure of prey x at equilibria versus R when $a = 1.6$, $\beta = 2$, $m = 0.1309$, and $h = 0.13$, which displays a backward bifurcation at $R = 1$.

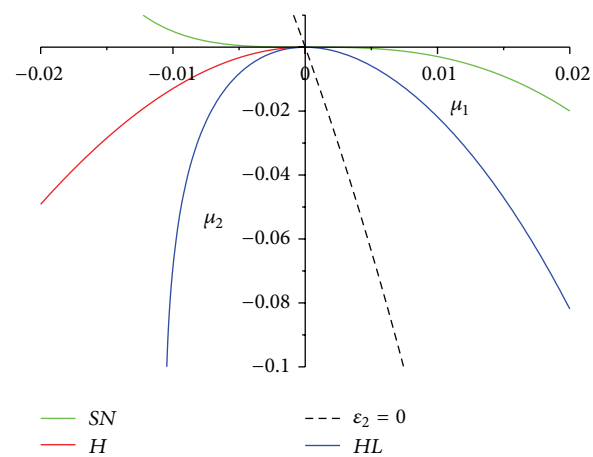


Figure 3: Bifurcation diagram of system (4) near P_* in the plane of μ_1 and μ_2.

Theorem 9. (i) *System* (4) *has at least one unstable limit cycle if* $(a, m, \beta, \delta, h) \in H_b$, $\delta_2 > \delta > \tilde{\delta}$, $|\delta - \tilde{\delta}| \ll 1$.

(ii) *System* (4) *has at least one stable limit cycle if* $(a, m, \beta, \delta, h) \in H_p$, $\delta_1 < \delta < \tilde{\delta}$, $|\delta - \tilde{\delta}| \ll 1$.

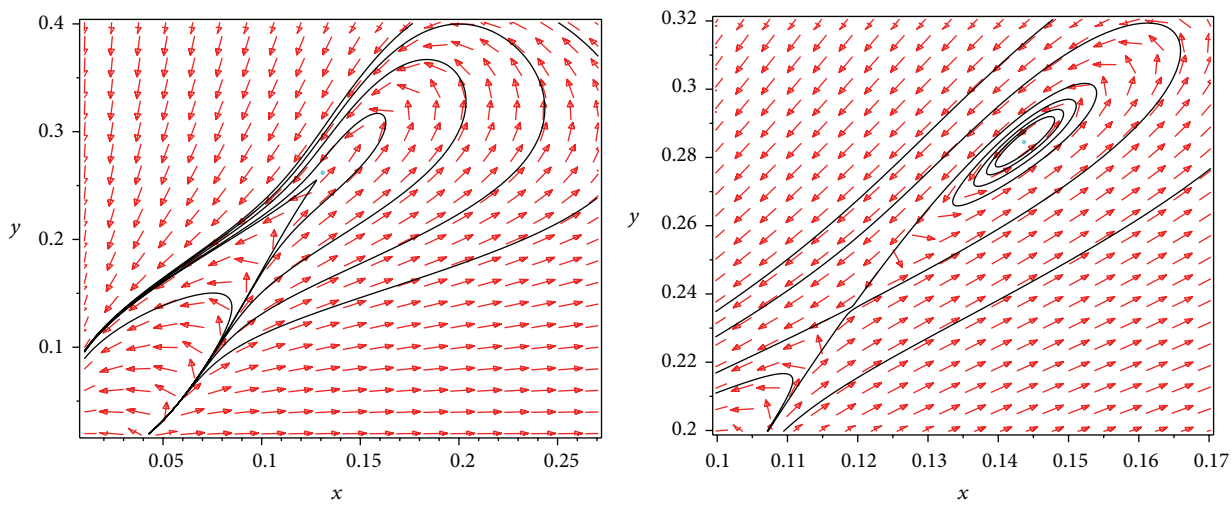
Remark 10. When $\sigma = 0$ system (4) maybe undergoes degenerate Hopf bifurcation for some parameter values; since the expression of σ is complicated, we do not discuss this case.

Note that by Theorem 7, if P_3 is a weak focus or a center, then we can obtain that its first Lyapunov number is

$$\sigma_1 = \frac{3a\beta}{8k_1(1 + a\beta)(2ma\beta - a\beta + \beta + 2m - 1)} < 0, \quad (39)$$

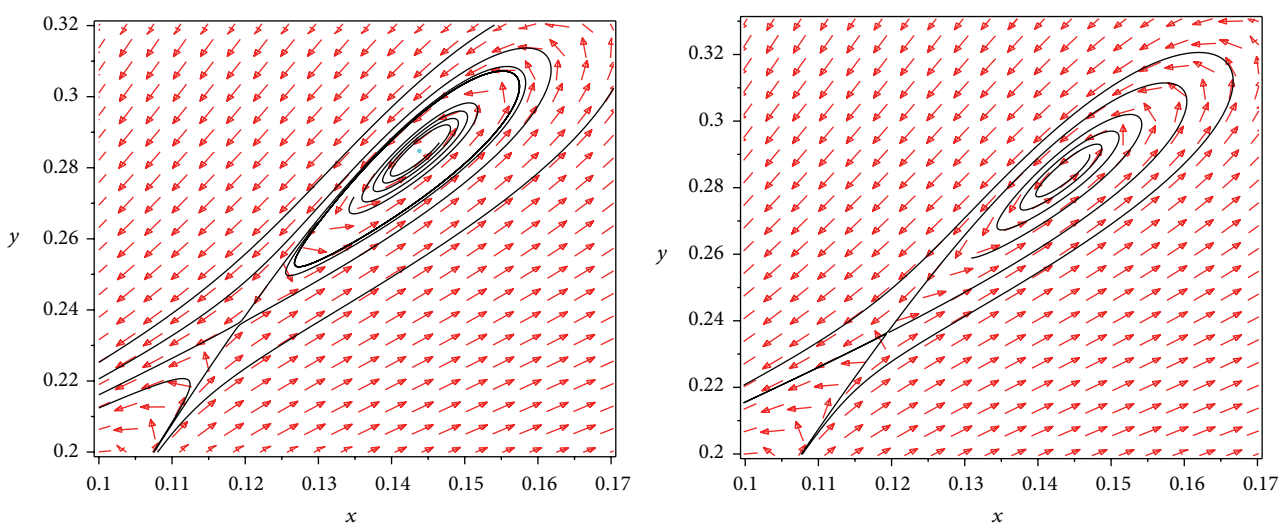
therefore, P_3 is a stable weak focus.

By numerical calculation, we give the parameter values $(a, \beta, m, h) = (1.6,\ 2.0,\ 0.13095,\ 0.128)$, then $\tilde{\delta} = 0.00235$, $\delta_2 = 0.2220$ and $k_1\sigma = 0.31085 > 0$, and the existence condition of subcritical Hopf bifurcation is satisfied. If we



(a) (b)

FIGURE 4: (a) System (4) shows a cusp of codimension 2 when $a = 1.6$, $\beta = 2$, $m = (1/4)(1 - \beta/(a\beta + 1))$, $h = (1/4)(\beta/(a\beta + 1) - 1)^2 + m\beta/(a\beta + 1)$, $\delta = 1/(a\beta + 1)^2$, and $\mu_1 = 0$, $\mu_2 = 0$; (b) the cusp of codimension 2 break into an unstable focus $\widehat{E}_1$ and a hyperbolic saddle $\widehat{E}_2$ when $\mu_1 = -0.0132$, $\mu_2 = -0.02$.



(a) (b)

FIGURE 5: (a) The cusp of codimension 2 breaks into a stable focus $\widehat{E}_1$ and a hyperbolic saddle $\widehat{E}_2$ when $\mu_1 = -0.0098$, $\mu_2 = -0.02$. The change of stability of the focus yields an unstable limit cycle. (b) The unstable limit cycle is broken when $\mu_1 \approx -0.0081768$, $\mu_2 = -0.02$, reachs the manifold of the saddle $\widehat{E}_2$, and leads to a homoclinic loop occur.

keep a, β, m, h fixed and choose $\delta = 0.00249$, then a unstable limit cycle can be shown in Figure 1(a).

When taking $(a, \beta, m, h) = (1.2,\ 2.0,\ 0.08235,\ 0.08396)$, then $\widetilde{\delta} = 0.003639$, $\delta_1 = 0.0000391$, and $k_1\sigma = -0.1741543 < 0$ which satisfy the existence condition of supercritical Hopf bifurcation. Furthermore, we choose $\delta = 0.002919$; according to Theorem 9, there exists a stable limit cycle, which can be shown in Figure 1(b).

3.2. Backward Bifurcation. Define $R = m(1 - m)/h$, $R_* = m(1 - m)/\widetilde{h}$.

Lemmas 2–3 and Theorems 6–8 illustrate that when the parameter h varies in the range of $(0, m(1-m)]$, system (4) just has only one positive equilibrium P_1 which is stable. However, when h varies in the range $(m(m-1), \widetilde{h})$, system (4) has two distinct positive equilibria P_1 and P_2, where P_1 is a stable node or focus and P_2 is a saddle. Furthermore, when $h = \widetilde{h}$, system (4) has unique positive equilibrium P_*. The previous discussion indicates the possibility of a backward bifurcation, which can be summarized as follows.

Theorem 11. *Let $0 < m < (1/2)(1-\beta/(a\beta+1))$, $\delta > \delta_2$. Then system (4) has a unique positive equilibrium P_* when $R = R_*$, has two distinct positive equilibria P_1 and P_2 when $R_* < R < 1$, where P_1 is a stable node and P_2 is a saddle, and has one positive equilibrium P_1 or P_3 when $R \geq 1$. Therefore, system (4) undergoes a backward bifurcation when $R = 1$.*

We give a numerical example in Figure 2 which displays that system (4) has a backward bifurcation at $R = 1$.

3.3. Saddle-Node Bifurcations. From Lemmas 2–3 and Theorem 4, we see that when $0 < m < 1/2$, $h = 1/4$, E_1 and E_2 degenerate into a saddle-node $\overline{E} = (\overline{x}, 0)$. This indicates that there is a saddle node bifurcation surface which takes the form

$$\mathrm{SN}_1 = \left\{(m, a, \beta, h, \delta) : 0 < m < \frac{1}{2},\ h = \frac{1}{4},\ \ \delta > 0,\ \beta > 0, a > 0\right\}. \tag{40}$$

Similarly, from Lemma 3 and the part 1° of Theorem 8, we know that when $0 < m < (1/2)(1-\beta/(a\beta+1))$ and $h = \widetilde{h}$, in $\mathbb{R}_2^+$, system (4) admits the double point $P_* = (x_*, y_*)$. And P_* is a saddle node if $\delta \neq 1/(a\beta+1)^2$.

One also can see that when the parameter h varies in the range of $(m(m-1), \widetilde{h})$, system (4) has two distinct positive equilibria P_1 and P_2. From Theorem 6, we know that P_1 may be a stable node, or a focus, and P_2 is a saddle. These imply that system (4) undergoes another saddle-node bifurcation of codimension 1. That is, there is a second saddle-node bifurcation surface SN_2 which is defined by

$$\mathrm{SN}_2 = \left\{(m, a, \beta, h, \delta) : 0 < m < \frac{1}{2}\left(1 - \frac{\beta}{a\beta+1}\right),\ h = \widetilde{h}, \delta \neq \frac{1}{(a\beta+1)^2}\right\}. \tag{41}$$

3.4. Bogdanov-Takens Bifurcation. From the part 2° of Theorem 8, we can see that system (4) exists a cusp of codimension 2, which implies that there may exist the Bogdanov-Takens bifurcation in system (4). Clearly, there exists a parameter space

$$\mathrm{BT} = \left\{(m, a, \beta, h, \delta) : 0 < m < \frac{1}{2} - \frac{\beta}{2(a\beta+1)}, \delta = \frac{1}{(a\beta+1)^2},\ h = \frac{1}{4}\left(\frac{\beta}{a\beta+1} - 1\right)^2 + \frac{m\beta}{a\beta+1}\right\}, \tag{42}$$

such that system (4) has a cusp of codimension 2 when $(m, a, \beta, h, \delta) \in \mathrm{BT}$.

To show that system (4) undergoes the Bogdanov-Takens bifurcation we choose δ and β as bifurcation, parameters. We need to find the universal unfolding of P_*.

Let $(m, a, \beta, h, \delta) \in \mathrm{BT}$, and consider the following unfold system

$$\begin{aligned} \dot{x} &= (x+m)(1-x-m) - \frac{yx}{ay+x} - h, \\ \dot{y} &= (\delta+\mu_1)\, y\left(\beta + \mu_2 - \frac{y}{x}\right), \end{aligned} \tag{43}$$

where μ_1 and μ_2 are small parameters and vary in the neighborhood of the origin.

Translating P_* to $(0,0)$ by the transformation $X = x - x_*$ and $Y = y - y_*$. Then system (43) is rewritten as

$$\begin{aligned} \dot{X} &= \alpha_1 X - \alpha_2 Y - \alpha_3 X^2 + \alpha_4 XY - \alpha_5 Y^2 + W_1(X, Y), \\ \dot{Y} &= l_1 + l_2 X + l_3 Y + l_4 X^2 + l_5 XY + l_6 Y^2 + W_2(X, Y), \end{aligned} \tag{44}$$

where W_1 and W_2 are smooth functions of X, Y at least of the third order. And

$$\alpha_1 = \frac{\beta}{(a\beta+1)^2}, \qquad \alpha_2 = \frac{1}{(a\beta+1)^2},$$

$$\alpha_3 = \frac{(a\beta+1)^2\left[(a\beta+1)(2m-1)+\beta\right] + 2a\beta^2}{(a\beta+1)^2\left[(a\beta+1)(2m-1)+\beta\right]},$$

$$\alpha_4 = \frac{4a\beta}{(a\beta+1)^2\left[(a\beta+1)(2m-1)+\beta\right]},$$

$$\alpha_5 = \frac{2a}{(a\beta+1)^2\left[(a\beta+1)(2m-1)+\beta\right]},$$

$$l_1 = -\frac{\left[1+\mu_1(a\beta+1)^2\right]\left[(a\beta+1)(2m-1)+\beta\right]\beta\mu_2}{2(a\beta+1)^3},$$

$$l_2 = \frac{\left(1+\mu_1(a\beta+1)^2\right)\beta^2}{(a\beta+1)^2},$$
$$l_3 = -\frac{\left(1+\mu_1(a\beta+1)^2\right)(\beta-\mu_2)}{(a\beta+1)^2},$$
$$l_4 = \frac{2\beta^2\left(1+\mu_1(a\beta+1)^2\right)}{(a\beta+1)\left[(a\beta+1)(2m-1)+\beta\right]}, \tag{45}$$
$$l_5 = -\frac{4\beta\left(1+\mu_1(a\beta+1)^2\right)}{(a\beta+1)\left[(a\beta+1)(2m-1)+\beta\right]},$$
$$l_6 = \frac{2\left(1+\mu_1(a\beta+1)^2\right)}{(a\beta+1)\left[(a\beta+1)(2m-1)+\beta\right]}.$$

Taking the change of variables $\overline{X} = X$, $\overline{Y} = \alpha_1 X - \alpha_2 Y - \alpha_3 X^2 + \alpha_4 XY - \alpha_5 Y^2 + W_1(X,Y)$ and rewriting $\overline{X}, \overline{Y}$ as X, Y, we obtain

$$\begin{aligned}\dot{X} &= Y,\\ \dot{Y} &= n_1 + n_2 X + n_3 Y + n_4 X^2 + n_5 XY + n_6 Y^2 + W_3(X,Y,\mu),\end{aligned} \tag{46}$$

where

$$\begin{aligned}
n_1 &= -\alpha_2 l_1 \longrightarrow 0,\\
n_2 &= \alpha_4 l_1 - \alpha_2 l_2 - l_3\alpha_1 - \frac{2\alpha_5\alpha_1 l_1}{\alpha_2} \longrightarrow 0,\\
n_3 &= l_3 + \alpha_1 + \frac{2\alpha_5 l_1}{\alpha_2} \longrightarrow 0,\\
n_4 &= -l_4\alpha_2 - l_5\alpha_1 + l_3\alpha_3 + \alpha_4 l_2\\
&\quad + \frac{-2\alpha_5\alpha_1 l_2 + 2\alpha_5 l_1\alpha_3 - l_6\alpha_1^2}{\alpha_2}\\
&\quad - \frac{2\alpha_5 l_1\alpha_4\alpha_1 + l_3\alpha_5\alpha_1^2}{\alpha_2^2} + 2\,\frac{\alpha_5^2 l_1\alpha_1^2}{\alpha_2^3} \longrightarrow -\frac{\beta}{(a\beta+1)^2},\\
n_5 &= l_5 - 2\alpha_3 + \frac{2\alpha_5 l_2 + \alpha_4\alpha_1 + 2\,l_6\alpha_1}{\alpha_2}\\
&\quad + \frac{2\alpha_5\alpha_1 l_3 + 2\alpha_5 l_1\alpha_4}{\alpha_2^2} - 4\,\frac{\alpha_5^2 l_1\alpha_1}{\alpha_2^3} \longrightarrow -2,\\
n_6 &= -\frac{l_6}{\alpha_2} - \frac{\alpha_4}{\alpha_2} - \frac{l_3\alpha_5}{\alpha_2^2}\\
&\quad + 2\frac{\alpha_5^2 l_1}{\alpha_2^3} \longrightarrow -\frac{2(a\beta+1)}{(a\beta+1)(2m-1)+\beta},
\end{aligned} \tag{47}$$

with $\mu_1 \to 0$, $\mu_2 \to 0$.

Taking $u = X + n_3/n_5$, substituting u in system (46), and rewriting u as X, we get that

$$\begin{aligned}\dot{X} &= Y,\\ \dot{Y} &= n_1 - \frac{n_2 n_3}{n_5} + \frac{n_4 n_3^2}{n_5^2} + \left(n_2 - \frac{2n_4 n_3}{n_5}\right)X\\ &\quad + n_4 X^2 + n_5 XY + n_6 Y^2 + W_4(X,Y,\mu),\end{aligned} \tag{48}$$

where W_4 is a smooth function of X, Y, and μ at least of order three. When $\mu_1 \to 0$, $\mu_2 \to 0$,

$$n_1 - \frac{n_2 n_3}{n_5} + \frac{n_4 n_3^2}{n_5^2} \longrightarrow 0, \qquad n_2 - \frac{2n_4 n_3}{n_5} \longrightarrow 0. \tag{49}$$

Next, let $s = t/(1-n_6 X)$, $x = X$, and $y = (1-n_6 X)Y$ into (48) and rewriting s, x, and y as t, X, and Y yields

$$\begin{aligned}\dot{X} &= Y,\\ \dot{Y} &= \varepsilon_1 + \varepsilon_2 X + n_5 XY + \varepsilon_3 X^2 + W_5(X,Y,\mu),\end{aligned} \tag{50}$$

where W_5 is a smooth function of X, Y, and μ at least of order three and

$$\begin{aligned}
\varepsilon_1 &= n_1 - \frac{n_2 n_3}{n_5} + \frac{n_4 n_3^2}{n_5^2} \longrightarrow 0,\\
\varepsilon_2 &= n_2 - 2\frac{n_4 n_3}{n_5} - 2n_6\varepsilon_1 \longrightarrow 0,\\
\varepsilon_3 &= n_4 - 2n_6\left(n_2 - 2\frac{n_4 n_3}{n_5}\right) + n_6^2\varepsilon_1 \longrightarrow -\frac{1}{(a\beta+1)^2} < 0,
\end{aligned} \tag{51}$$

when $\mu_1 \to 0$, $\mu_2 \to 0$. Let $x = (n_5^2/\varepsilon_3)X$, $y = (n_5^3/\varepsilon_3^2)Y$, $\nu = (\varepsilon_3/n_5)t$, and rewrite ν as t. Then system (50) becomes

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= \tau_1 + \tau_2 x + xy + x^2 + W_6(x,y,\mu),\end{aligned} \tag{52}$$

where W_6 is a smooth function of x, y, and μ at least of order three and $\tau_1 = n_5^4\varepsilon_1/\varepsilon_3^3$, $\tau_2 = n_5^2\varepsilon_2/\varepsilon_3^2$.

Then system (4) exists the following bifurcation curves in a small neighborhood of the origin in the (μ_1,μ_2) plane.

Theorem 12. *Let* $0 < m < 1/2 - \beta/2(a\beta+1)$, $\delta = 1/(a\beta+1)^2$, $h = (1/4)(\beta/(a\beta+1)-1)^2 + m\beta/(a\beta+1)$. *Then system* (43) *admits the following bifurcations:*

(i) *there exists a saddle node bifurcation curve* $SN = \{(\mu_1,\mu_2) : 4\varepsilon_3\varepsilon_1 = \varepsilon_2^2 + o(\|\mu\|)^2\}$;

(ii) *there is a Hopf bifurcation curve* $H = \{(\mu_1,\mu_2) : \varepsilon_1 = 0 + o(\|\mu\|)^2,\ \varepsilon_2 < 0\}$;

(iii) *there is a homoclinic bifurcation curve* $HL = \{(\mu_1,\mu_2) : 25\varepsilon_1\varepsilon_3 + 6\varepsilon_2^2 = 0 + o(\|\mu\|)^2\}$.

The biological interpretation for the Bogdanov-Takens bifurcation is that if the harvesting rate h and the prey refuge value m satisfy $0 < m < 1/2 - \beta/2(a\beta + 1)$, $h = (1/4)(\beta/(a\beta + 1) - 1)^2 + m\beta/(a\beta + 1)$, and $\delta = 1/(a\beta + 1)^2$, then the predator and prey coexist in the form of a positive equilibrium or a periodic orbit for different initial values, respectively. And there exist other values of parameters, such that the predator and prey coexist in the form of a positive equilibrium for all initial values lying inside the homoclinic loop, and the predator and prey coexist in the form of a periodic orbit with infinite period for all initial values on the homoclinic loop. By choosing $\beta = 2$, $a = 1.6$, $m = (1/4)(1 - \beta/(a\beta + 1))$, $h = (1/4)(\beta/(a\beta + 1) - 1)^2 + m\beta/(a\beta + 1)$, and $\delta = 1/(a\beta + 1)^2$, the numerical simulations for the Bogdanov-Takens bifurcation in Theorem 12 can be shown in Figures 3, 4 and 5.

3.5. Separatrix Connecting a Saddle-Node and a Saddle Bifurcation and Heteroclinic Bifurcation. From Theorem 8 and Lemma 3, when $0 < m < (1/2)(1 - \beta/(a\beta + 1))$, $h = \tilde{h}$, $\delta \neq 1/(a\beta + 1)^2$, there may exist a separatrix connecting the saddle-node P_* and the saddle E_1. When $0 < m < (1/2)(1 - \beta/(a\beta + 1))$, $m(1 - m) < h < \tilde{h}$, the saddle node P_* separates into the hyperbolic node P_1 and the hyperbolic saddle P_2, which implies that system (4) undergoes a separatrix connecting a saddle node and a saddle bifurcation. Furthermore, the heteroclinic bifurcation may occur if there exists a heteroclinic orbit connecting the separatrix of saddle E_1 and saddle P_2.

Acknowledgments

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