Research Article

An Operational Matrix Based on Legendre Polynomials for Solving Fuzzy Fractional-Order Differential Equations

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1. Introduction

The subject of fractional calculus has gained considerable popularity and importance during the past three decades. Some of the most recent works on this topic, that is, the theory of derivatives and integrals of fractional (noninteger) order, are such as the book of Podlubny [1], Baleanu et al. [2], Diethelm [3], Baleanu et al. [4], and Sabatier et al. [5]. Only in the last few years, the various applications of fractional calculus have been extended in the area of physics and engineering such as the modeling of nonlinear oscillation of earthquake [6], the fluid-dynamic models [7], continuum and statistical mechanics [8], and solid mechanics [9]. In a notably enormous number of recent works, one can find the robustness upon the usefulness of fractional calculus to derive particular solutions of different kinds of classical differential equations like Bessel differential equation of general order [10, 11]. Also, the most significant advantage of applying FDEs is their nonlocal property, which interprets that the next state of a system relies not only on its current phase but also on all of its past records of phases [12]. For example, with the fractional differentiability, the fluid dynamic traffic model can get rid of the shortage arising from the hypothesis of continuum traffic flow [12, 13].

On the other hand, the modeling of natural phenomena is stated using mathematical tools (mathematical arithmetic, mathematical logics, etc.). However, obtaining a deterministic model of such problems is not easy, even does not occur and always has some errors and vagueness. So, investigating a popular way to interpret such vagueness is important. Since 1965 with Zadeh’s well-known paper on introducing fuzzy sets, applications of fuzzy concept to the structure of any modeling has appeared more and more, instead of deterministic case. So the topic of fuzzy differential equations (FDEs) of integer order has been rapidly growing in recent years [14–20]. Additionally, the application of various techniques has been expanded by means of the interpolations and polynomials for approximating the fuzzy solutions of fuzzy integral equations vastly, like Bernstein polynomials [21, 22], Lagrange interpolation [23, 24], Chebyshev interpolation [25], Legendre wavelets [26], and Galerkin-type technique [27].

Recently, Agarwal et al. [28] proposed the concept of solutions for the fractional differential equations with uncertainty. They have considered the Riemann-Liouville’s differentiability with a fuzzy initial condition to solve FFDEs. In [29, 30], the authors considered the generalization of H-differentiability for the fractional case. Discovering a suitable
approximate or exact solution for FFDEs is a significant task which has been aroused simultaneously with the emerging of FFDEs, except for a few number of these equations, and we have hardship in finding their analytical solutions. Consequently, there have been limited efforts to develop new methods for gaining approximate solutions which reasonably estimate the exact solutions. Salahshour et al. [31] considered fuzzy laplace transforms for solving FFDEs under Riemann-Liouville H-differentiability. Also Mazandarani and Kamyad [32] generalized the fractional Euler method for solving FFDEs under Caputo-type derivative.

From another point of view, several methods have been exploited to solve fractional differential equations, and fractional partial differential equations, fractional integrodifferential equations such as Adomian’s decomposition method [7], He’s variational iteration method [33], homotopy perturbation method [34], and spectral methods [35, 36]. In this way, orthogonal functions have received considerable attention in dealing with the various kinds of fractional differential equations. The main characteristic behind the approach using this technique is that it reduces these problems to those of solving a system of algebraic equations thus greatly simplifying the problem. Saadatmandi and Dehghan [36] presented the shifted Legendre operational matrix for fractional derivatives and applied it with tau method for the numerical solution of fractional differential equations subject to initial conditions. Also in [37–39], the authors derived new formulas using shifted Chebyshev polynomials and shifted Jacobi polynomials of any degree, respectively and applied them together with tau and collocations spectral methods for solving multiterm linear and nonlinear fractional differential equations.

The essential target of this paper is to recommend a suitable way to approximate FFDEs using a shifted Legendre tau approach. This strategy demands a formula for fuzzy fractional-order Caputo derivatives of shifted Legendre polynomials of any degree which is provided and applied together with the tau method for solving FFDEs with initial conditions. Up till now, and to the best of our knowledge, few methods corresponding to those mentioned previously have been devoted to solve FFDEs and are traceless in the literature for FFDEs under Caputo differentiability. This partially motivates our interest in the operational matrix of fuzzy fractional derivative of shifted Legendre polynomials. Also another motivation is based on the reality that only a few terms of expansion of the shifted Legendre function is needed to reach to a high accuracy, therefore, it does not need to implement the method frequently for finding the approximate results in each particular point.

For finding the fuzzy solution, the shifted Legendre operational matrix is generalized for the fuzzy fractional derivative (0 < \( \alpha < 1 \)) which is based on the Legendre tau method for solving numerically FFDEs with the fuzzy initial conditions. It is worthy to note here that the method based on using the operational matrix of the Legendre orthogonal function for solving FFDEs is computer oriented.

The aim of this paper is to introduce the shifted Legendre operational matrix of fuzzy fractional derivative which is based on Legendre tau method for solving FFDEs under generalized differentiability. Also, we introduce a suitable way to estimate the nonlinear fuzzy fractional initial problems on the interval [0, 1], by spectral shifted Legendre collocation method based on Legendre operational matrix, to find the approximate fuzzy solution. Finally, the accuracy of the proposed algorithms is demonstrated by several test problems. We note that the two shifted Legendre and shifted Jacobi operational matrices have been introduced by Saadatmandi and Dehghan [36] and Doha et al. [39], respectively, in the crisp concept. We, therefore, motivated our interest in the shifted Legendre operational matrix in the fuzzy settings.

This paper is organized as follows: In Section 2, we begin by introducing some necessary definitions and mathematical preliminaries of the fuzzy calculus and fractional calculus. Some basic concepts, properties and theorems of fuzzy fractional calculus are presented in Section 3. Section 4 is devoted to the fuzzy Legendre functions and their properties. The shifted Legendre operational matrix of fuzzy fractional derivative for solving fuzzy fractional differential equation is obtained in Section 5. Section 6 illustrates the effectiveness of the proposed method through solving several examples which some of them are modelled based on the real phenomena. Finally, a conclusion is given in the last section.

2. Preliminaries

We give some definitions and introduce the necessary notation which will be used throughout the paper, see, for example, [40, 41]. Also for some definitions related to generalized fuzzy difference, one can find more in [42, 43].

We denote the set of all real numbers by \( \mathbb{R} \). A fuzzy number is a mapping \( \tilde{u} : \mathbb{R} \to [0,1] \) with the following properties:

(a) \( \tilde{u} \) is upper semicontinuous,
(b) \( \tilde{u} \) is fuzzy convex, that is, \( \tilde{u}(\lambda x + (1 - \lambda)y) \geq \min\{\tilde{u}(x), \tilde{u}(y)\} \) for all \( x, y \in \mathbb{R}, \lambda \in [0,1] \),
(c) \( \tilde{u} \) is normal, that is, \( \exists x_0 \in \mathbb{R} \) for which \( \tilde{u}(x_0) = 1 \),
(d) \( \text{supp} \tilde{u} = \{x \in \mathbb{R} \mid \tilde{u}(x) > 0\} \) is the support of the \( u \), and its closure \( \text{cl} (\text{supp} \tilde{u}) \) is compact.

Let \( E \) be the set of all fuzzy number on \( \mathbb{R} \). The \( \alpha \)-level set of a fuzzy number \( \tilde{u} \in E \), \( 0 \leq \alpha \leq 1 \), denoted by \( [\tilde{u}]_\alpha \), is defined as

\[
[\tilde{u}]_\alpha = \left\{ x \in \mathbb{R} \mid \tilde{u}(x) \geq \alpha \right\} \quad \text{if} \; 0 < \alpha \leq 1 \\
\text{cl} (\text{supp} \tilde{u}) \quad \text{if} \; \alpha = 0.
\]

It is clear that the \( r \)-level set of a fuzzy number is a closed and bounded interval \( [\tilde{u}(r), \tilde{u}(r)] \), where \( \tilde{u}(r) \) denotes the left-hand endpoint of \( [\tilde{u}]_r \), and \( \tilde{u}(r) \) denotes the right-hand endpoint of \( [\tilde{u}]_r \). Since each \( y \in \mathbb{R} \) can be regarded as a fuzzy number \( \tilde{y} \) defined by

\[
\tilde{y}(t) = \begin{cases} 
1 & \text{if } t = y, \\
0 & \text{if } t \neq y,
\end{cases}
\]

\( \mathbb{R} \) can be embedded in \( E \).
The addition and scalar multiplication of fuzzy number in \( E \) are defined as follows:

1. \( \tilde{u} \oplus \tilde{v} = (\tilde{u} + \tilde{v}, \tilde{u} + \tilde{v}) \),

2. \( (\lambda \odot \tilde{u}) = \begin{cases} (\lambda \tilde{u}(r), \lambda \tilde{u}(r)) & \lambda \geq 0, \\ (\lambda \tilde{u}(r), \lambda \tilde{u}(r)) & \lambda < 0. \end{cases} \) \( (3) \)

The metric structure is given by the Hausdorff distance \( D: E \times E \rightarrow \mathbb{R} \cup \{0\} \),

\[
D(\tilde{u}, \tilde{v}) = \sup_{r \in [0,1]} \max \left\{ \left| \tilde{u}(r) - \tilde{v}(r) \right|, \left| \tilde{u}(r) - \tilde{v}(r) \right| \right\}.
\] \( (4) \)

Definition 1. The property (iv) in the properties of the above metric space suggests the definition of a function \( \| \cdot \|_E : R \rightarrow E \) that \( \| u \|_E := D(u, 0) \), for all \( u \in E \) that has the properties of usual norms. In \([44]\), the properties of this function are presented as follows:

(i) \( \| u \| \geq 0 \), for all \( u \in E \) and \( \| u \| = 0 \) if and only if \( u = 0 \),

(ii) \( \| \lambda u \| = |\lambda| \| u \| \) and \( \| u + v \| \geq \| u \| + \| v \| \), for all \( u, v \in E \), for all \( \lambda \in \mathbb{R} \),

(iii) \( \| u \| - \| v \| \leq D(u, v) \) and \( D(u, v) \leq \| u \| + \| v \| \) for all \( u, v \in E \).

Definition 2 (see \([45]\)). Let \( f \) and \( g \) be the two fuzzy-number-valued functions on the interval \([a, b] \), that is, \( f, g : [a, b] \rightarrow E \). The uniform distance between fuzzy-number-valued functions is defined by

\[
D^u(f, g) = \sup_{x \in [a, b]} D(f(x), g(x)).
\] \( (5) \)

Remark 3 (see \([46]\)). Let \( f : [a, b] \rightarrow E \) be fuzzy continuous. Then from property (iv) of Hausdorff distance, we can define

\[
D(f(x), 0) = \sup_{r \in [0,1]} \max \left\{ \left| f^r(x) \right|, \left| f^r(x) \right| \right\}, \quad \forall x \in [a, b].
\] \( (6) \)

Definition 4 (see \([42]\)). Let \( K^n_e \) be the space of nonempty compact and convex sets of \( \mathbb{R}^n \). The generalized Hukuhara difference of two sets \( A, B \in K^n_e \) (gH-difference for short) is defined as follows:

\[
A \ominus_{gH} B = C \iff \begin{cases} (a) \: A = B + C \text{ or } \\ (b) \: B = A + (-1)C. \end{cases}
\] \( (7) \)

In case (a) of the above equation, the gH-difference is coincident with the H-difference. Thus the gH-difference is a generalization of the H-difference.

Definition 5 (see \([47]\)). Let \( x, y \in E \). If there exists \( z \in E \) such that \( x = y \oplus z \), and then \( z \) is called the H-difference of \( x \) and \( y \), and it is denoted by \( x \ominus y \).

In this paper, the sign “\( \ominus \)” always stands for H-difference and note that \( x \oplus y \neq x + (-y) \). Also throughout the paper is assumed that the Hukuhara difference and Hukuhara generalized differentiability existed.

Definition 6 (see \([42]\)). The generalized difference (g-difference for short) of two fuzzy numbers \( u, v \in E \) is given by its level sets as

\[
[u \ominus_{gH} v]_\alpha = \left\{ u \ominus_{gH} v : [u]_\beta - [v]_\beta \right\} \quad \forall \alpha \in [0, 1],
\] \( (8) \)

where the gH-difference \( u \ominus_{gH} v \) is with interval operands \([u]_\beta \) and \([v]_\beta \).

Proposition 7. The g-difference in Definition 6 is given by the expression

\[
[u \ominus_{gH} v]_\alpha = \left\{ \inf_{\beta \geq \alpha} \left\{ u_\beta - v_\beta, \overline{u}_\beta - \overline{v}_\beta \right\} \right\} \cup \left\{ \sup_{\beta \geq \alpha} \left\{ u_\beta - v_\beta, \overline{u}_\beta - \overline{v}_\beta \right\} \right\}.
\] \( (9) \)

Proof. See \([42]\). \( \square \)

The next proposition gives simplified notation for \( u \ominus_{gH} v \) and \( v \ominus_{gH} u \).

Proposition 8. For any two fuzzy numbers \( u, v \in E \) the two g-difference \( u \ominus_{gH} v \) and \( v \ominus_{gH} u \) exist and, for any \( \alpha \in [0, 1] \), one has \( u \ominus_{gH} v = -(v \ominus_{gH} u) \) with

\[
[u \ominus_{gH} v]_\alpha = [d_\alpha, \overline{d}_\alpha], \quad [v \ominus_{gH} u]_\alpha = [-\overline{d}_\alpha, -d_\alpha],
\] \( (10) \)

where

\[
d_\alpha = \inf(D_\alpha), \quad \overline{d}_\alpha = \sup(D_\alpha),
\] \( (11) \)

and the sets \( D_\alpha \) are

\[
D_\alpha = \left\{ u_\beta - v_\beta : \beta \geq \alpha \right\} \cup \left\{ \overline{u}_\beta - \overline{v}_\beta : \beta \geq \alpha \right\}.
\] \( (12) \)

Proof. See \([42]\). \( \square \)

The following proposition prove that the g-difference is well-defined.

Proposition 9 (see \([14]\)). For any fuzzy numbers \( u, v \in E \) the g-difference \( u \ominus_{gH} v \) exists and it is a fuzzy number.

Proof. See \([42]\). \( \square \)
The following property holds for g-derivative.

**Proposition 10.** Let \( u, v \in \mathbb{E} \) be two fuzzy numbers, and then

(i) \( u \oplus v = u \oplus g(\cdot)v \) whenever the expressions on the right exist, in particular, \( u \oplus u = 0 \),

(ii) \( (u + v) \oplus g(\cdot)v = u \),

(iii) \( 0 \oplus g(\cdot)v = v \oplus g(\cdot)u \),

(iv) \( u \oplus g(\cdot)v = v \oplus g(\cdot)u = w \) if and only if \( w = -w \).

Furthermore, \( w = 0 \) if and only if \( u = v \).

In this paper, we consider the following definition which was introduced by Bede and Gal in [14].

**Definition 11.** Let \( f : (a, b) \to \mathbb{E} \) and \( x_0 \in (a, b) \).

One says that \( f \) is strongly generalized differentiable at \( x_0 \), if there exists an element \( f'(x_0) \in \mathbb{E} \), such that

(i) for all \( h > 0 \) sufficiently small, \( \exists f(x_0 + h) \oplus f(x_0) \), \( \exists f(x_0) \oplus f(x_0 - h) \), and the limits (in the metric \( D \))

\[
\lim_{h \to 0^+} \frac{f(x_0 + h) \oplus f(x_0)}{h} = f'(x_0),
\]

(ii) for all \( h > 0 \) sufficiently small, \( \exists f(x_0) \oplus f(x_0 + h) \), \( \exists f(x_0 - h) \oplus f(x_0) \), and the limits (in the metric \( D \))

\[
\lim_{h \to 0^+} \frac{f(x_0) \oplus f(x_0 + h)}{h} = f'(x_0),
\]

(iii) for all \( h > 0 \) sufficiently small, \( \forall f(x_0) \oplus f(x_0 + h) \), \( \exists f(x_0 - h) \oplus f(x_0) \), and the limits (in the metric \( D \))

\[
\lim_{h \to 0^+} \frac{f(x_0 + h) \oplus f(x_0)}{h} = f'(x_0),
\]

(iv) for all \( h > 0 \) sufficiently small, \( \exists f(x_0) \oplus f(x_0 + h) \), \( \exists f(x_0) \oplus f(x_0 - h) \), and the limits (in the metric \( D \))

\[
\lim_{h \to 0^+} \frac{f(x_0) \oplus f(x_0 + h)}{h} = f'(x_0).
\]

**Remark 12.** Throughout this paper, we say that \( f \) is (1)-differentiable on \((a, b)\), if \( f \) is differentiable in the sense (i) of Definition 11 and also \( f \) is (2)-differentiable on \((a, b)\), if \( f \) is differentiable in the sense (ii) of Definition 11.

**Theorem 13** (see [17]). Let \( f : (a, b) \to \mathbb{E} \) be a function and denote \([F(t)]' = [f_r(t), g_r(t)]\), for each \( r \in [0, 1] \). Then

(i) if \( f \) is (1)-differentiable, then \( f_r(t) \) and \( g_r(t) \) are differentiable functions and

\[
[F'(t)]' = [f'_r(t), g'_r(t)],
\]

(ii) if \( f \) is (2)-differentiable, then \( f_r(t) \) and \( g_r(t) \) are differentiable functions and

\[
[F'(t)]' = [g'_r(t), f'_r(t)].
\]

**Definition 14** (see [42]). Let \( f : (a, b) \to \mathbb{E} \) and \( x_0 \in (a, b) \).

We say that \( f \) is g-differentiable at \( x_0 \), if there exists an element \( f'(x_0) \in \mathbb{E} \) such that

\[
f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.
\]

Next we review one of the main results from Bede [15] for fuzzy initial value problem (FIVP) under (1)-differentiability which Nieto et al. [48] generalized this results for FIVP under (2)-differentiability (let \( || \cdot || \) denote the usual Euclidean norm).

**Theorem 15** (see [15], characterization theorem). Let one consider the fuzzy initial value problem

\[
y'(t) = f(x, y(x)),
\]

\[
y(t_0) = y_0,
\]

where \( f : [x_0, x_0 + a] \times \mathbb{E} \to \mathbb{E} \) is such that

(i) \([f(x, y)]' = \frac{[f'(x, y, \tilde{y})]}{\tilde{f}'(x, y, \tilde{y})},\]

(ii) \( f' \) and \( \tilde{f}' \) are equicontinuous (i.e., for any \( e > 0 \) there is a \( \delta > 0 \) such that \( ||f'(x, y, z) - f'(x, y, z')|| < e \) and \( ||\tilde{f}'(x, y, z) - \tilde{f}'(x, y, z')|| < e \) for all \( r \in [0, 1] \), whenever \( (x, y, z), (x, y, z') \in [x_0, x_0 + a] \times \mathbb{R}^2 \) and \( ||x, y, z - (x, y, z')|| < \delta \) and uniformly bounded on any bounded set,

(iii) there exists an \( L > 0 \) such that \( ||f'(x_2, y_2, z_2) - f'(x_1, y_1, z_1)|| \leq L \max||y_2 - y_1, |z_2 - z_1|| for all \( r \in [0, 1] \), \( ||\tilde{f}'(x_2, y_2, z_2) - \tilde{f}'(x_1, y_1, z_1)|| \leq L \max||y_2 - y_1, |z_2 - z_1|| for all \( r \in [0, 1] \).

Then the FIVP (20) and system of ODEs

\[
\left( \frac{y'(x)}{\tilde{y}'(x)} \right)' = \frac{f'(x, y, \tilde{y})}{\tilde{f}'(x, y, \tilde{y})},
\]

\[
\left( \frac{y'(x)}{\tilde{y}'(x)} \right)' = \frac{f'(x, y, \tilde{y})}{\tilde{f}'(x, y, \tilde{y})},
\]

\[
\begin{align*}
y'(x_0) &= \left( \frac{y'}{\tilde{y}'} \right)_0, \\
\tilde{y}'(x_0) &= \left( \frac{\tilde{y}'}{\tilde{\tilde{y}}'} \right)_0,
\end{align*}
\]

are equivalent.
Corollary 16 (see [48]). If we consider FIVP (20) under (2)-
differentiability then the FIVP (20) and the following system of
ODEs are equivalent:

\[
\begin{align*}
(y'(x))' &= f'(x, y', y''), \\
(y'(t))' &= f'(x, y', y''), \\
y'(x_0) &= \left(\begin{array}{c}
y_0' \\
y_0''
\end{array}\right), \\
y'(x_0) &= \left(\begin{array}{c}
y_0' \\
y_0''
\end{array}\right).
\end{align*}
\]

(22)

Theorem 17 (see [49]). Let \( f(x) \) be a fuzzy-valued function on \([a, \infty)\) and it is represented by \( (f(x; r), \overline{f}(x; r)) \). For any
fixed \( r \in [0, 1] \), assume that \( (f(x; r), \overline{f}(x; r)) \) are Riemann-
integrable on \([a, b] \) for every \( b > a \), and assume that there are two positive \( M(r) \) and \( M(r) \) such that:

\[
\int_a^b |f(x; r)| \, dx \leq M(r), \quad \int_a^b \overline{f}(x; r) \, dx \leq \overline{M}(r)
\]

for every \( b > a \).

(23)

Then \( f(x) \) is improper fuzzy Riemann-integrable on \([a, \infty)\)
and the improper fuzzy Riemann-integral is a fuzzy number.

Furthermore, we have

\[
\int_a^\infty f(x) \, dx = \left[ \int_a^\infty f(x; r) \, dx, \int_a^\infty \overline{f}(x; r) \, dx \right].
\]

(24)

Definition 18 (see [45]). \( f(x) : [a, b] \to \mathbb{E} \). We say that \( f \)
Fuzzy-Riemann integrable to \( I \in \mathbb{E} \), if for any \( \varepsilon > 0 \), there
exists \( \delta > 0 \) such that for any division \( P = \{[u, v]; \xi\} \) of \([a, b] \)
with the norms \( \Delta(P) < \delta \), we have

\[
D \left( \sum_\varepsilon^* (u - v) \odot f(\xi), I \right) < \varepsilon,
\]

(25)

where \( \sum^* \) means addition with respect to \( \oplus \) in \( \mathbb{E} \),

\[
I := (FR) \int_a^b f(x) \, dx.
\]

(26)

We also call an \( f \) as above, \( (FR) \)-integrable.

Definition 19 (see [50]). Consider the \( n \times n \) linear system of equations

\[
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2, \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n.
\]

(27)

The matrix form of the above equations is

\[
AX = Y,
\]

(28)

where the coefficient matrix \( A = (a_{ij}) \), \( 1 \leq i, j \leq n \) is a crisp
\( n \times n \) matrix and \( y_j \in \mathbb{E} \), \( 1 \leq i \leq n \). This system is called a fuzzy
linear system (FLS).

Definition 20 (see [50]). A fuzzy number vector \( (x_1, x_2, \ldots, x_n) \)
given by \( x_i = (x_i(r), \overline{x}_i(r)) \), \( 1 \leq i \leq n, 0 \leq r \leq 1 \)
is called a solution of the fuzzy linear system (27) if

\[
\sum_{j=1}^n a_{ij}x_j = y_i, \\
\sum_{j=1}^n a_{ij}\overline{x}_j = \overline{y}_i.
\]

(29)

If for a particular \( k, a_{kj} > 0 \), \( 1 \leq j \leq n \), we simply get

\[
\sum_{j=1}^n a_{kj}x_j = y_k, \\
\sum_{j=1}^n a_{kj}\overline{x}_j = \overline{y}_k.
\]

(30)

To solve fuzzy linear systems, one can refer to [51–53].

Now we define some notations which are used for the
fuzzy fractional calculus throughout the paper.

(i) \( L_p^E(a, b), 1 \leq p < \infty \) is the set of all fuzzy-valued
measurable functions \( f \) on \([a, b] \) where \( \|f\|_p = (\int_a^b (d(f(t), 0))^p dt)^{1/p} \).

(ii) \( C^E[a, b] \) is a space of fuzzy-valued functions which are
continuous on \([a, b] \).

(iii) \( AC^E[a, b] \) denotes the set of all fuzzy-valued functions
which are absolutely continuous.

One can easily find this definition in the crisp sense in
[1, 54].

Definition 21 (see [54]). The Riemann-Liouville fractional
integral operator of order \( \alpha, n - 1 < \alpha \leq n \), of a function \( f \in C[a, b] \) is defined as

\[
(I^\alpha_a f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \quad x > a,
\]

(31)

\[
(I^\alpha_b f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt \quad x < b.
\]

Properties of the operator \( I^\alpha \) can be found in [1, 54, 55].

we refer to only the following

For \( f \in C[a, b] \), \( \alpha, \beta \geq 0 \) and \( y > -1 \),

(1) \( I^{\alpha+\beta} f(x) = I^\beta I^\alpha f(x) \),

(2) \( I^{\alpha+\beta} f(x) = I^\beta I^\alpha f(x) \),

(3) \( I^\alpha x^y = \frac{\Gamma(y+1)}{\Gamma(\alpha+y+1)} x^{\alpha+y} \).
Definition 22 (see [54]). The Riemann-Liouville fractional derivatives of order $0 < \alpha < 1$ of a function $f \in C[a, b]$ are expressed by
\[
(D_0^\alpha f)(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f(t) dt}{(x-t)^\alpha}, \quad (x > a),
\]
\[
(D_0^\alpha f)(x) = -\frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f(t) dt}{(x-t)^\alpha}, \quad (x < b).
\]

Definition 23 (see [55]). The fractional Caputo derivatives $C^\alpha D_0^b f(x)$ and $C^\alpha D_0^a f(x)$ on $[a, b]$ for $0 < \alpha < 1$ are defined via the above Riemann-Liouville fractional derivatives by
\[
(C^\alpha D_0^a f)(x) = (D_0^a f)(x) - \frac{f(a)}{\Gamma(1-\alpha)}(x-a)^{1-\alpha},
\]
\[
(C^\alpha D_0^b f)(x) = (D_0^b f)(x) - \frac{f(b)}{\Gamma(1-\alpha)}(b-x)^{1-\alpha}.
\]

Lemma 26. Let $0 < \alpha < 1$ and $f \in C[a, b] \cap L_p[a, b]$. Then the Caputo fractional derivatives are bounded for any $x \in [a, b]$ and $(1 \leq p < \infty)$ as
\[
|\langle C^\alpha D_0^a f \rangle(x) \rangle| \leq \frac{\|f'(x)\|_p}{|\Gamma(1-\alpha)|} (x-a)^{1-\alpha} = M_a^\alpha,
\]
\[
|\langle C^\alpha D_0^b f \rangle(x) \rangle| \leq \frac{\|f'(x)\|_p}{|\Gamma(1-\alpha)|} (b-x)^{1-\alpha} = M_b^\alpha.
\]

3. Fuzzy Caputo Fractional Derivatives

In this section, some definitions and theorems related to the fuzzy Caputo fractional derivatives are presented which are an extension of the fractional derivative in the crisp sense. The generalization differentiability should be considered to expand the concept of Caputo fractional derivatives for the fuzzy space. For more details, see [14, 30].

Definition 27. Let $f : L^E \cap C^E$ be a fuzzy set-value function and $\Phi(x) = (1/\Gamma(1-\alpha)) \int_a^x (f(t)dt/(x-t)^\alpha)$, and then $f$ is said to be g-Caputo fuzzy fractional differentiable at $x$, when
\[
(\gamma D_0^a f)(x) = \lim_{h \to 0} \frac{\Phi(x+h) \oplus_g \Phi(x)}{h},
\]
where
\[
(\gamma D_0^a f)(x; r) = \left[ \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f'(t; r) dt}{(x-t)^\alpha} \right],
\]
\[
(\gamma D_0^a f)(x; r) = \left[ \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f'(r; t) dt}{(x-t)^\alpha} \right].
\]

Remark 28. A fuzzy-valued function $f$ is $C[1-\alpha]$-differentiable, if it is differentiable as in Definition 27, Case (i), and it is $C[2-\alpha]$-differentiable, if it is differentiable as in Definition 27, case (ii).

Theorem 29. Let $0 < \alpha < 1$ and $f \in AC^E[a, b]$, then Caputo fuzzy fractional derivative exists almost everywhere on $(a, b)$ and for all $0 \leq r \leq 1$ we have
\[
(C^\alpha D_0^a f)(x; r) = \left[ \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f'(t; r) dt}{(x-t)^\alpha} \right],
\]
when $f$ is $(1)$-differentiable, and
\[
(C^\alpha D_0^a f)(x; r) = \left[ \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f'(r; t) dt}{(x-t)^\alpha} \right].
\]
when \( f \) is \((2)\)-differentiable, in which \((I_\alpha^a f)(x) = (1/\Gamma(\alpha)) \int_0^x f(t)(x-t)^{1-\alpha}\) for \( x > a \).

Proof. It is straightforward by applying Definitions 20 and 24. \(\Box\)

**Theorem 30.** Let one assume that \( f \in C^\infty[a, b] \), and then one has the following:

\[
(I_a^\alpha C D_a^\alpha f)(x) = f(x) \oplus f(a), \quad 0 < \alpha < 1,
\]

when \( f \) is \([1-\alpha]\)-differentiable and

\[
(I_a^\alpha C D_a^\alpha f)(x) = f(x) \ominus f(a), \quad 0 < \alpha < 1,
\]

when \( f \) is \([2-\alpha]\)-differentiable.

Proof. See [30]. \(\Box\)

**Lemma 31.** Let \( 0 < \alpha < 1 \) and \( f \in AC^\infty[a, b] \), then the fuzzy Caputo derivative can be expressed by means of the fuzzy Riemann-Liouville integral as follows:

\[
(C D_a^\alpha f)(x) = (I_a^\alpha C D_f)(x) = \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}},
\]

when \( f \) is \((1)\)-differentiable, and

\[
(C D_a^\alpha f)(x) = (I_a^\alpha C D_f)(x) = \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}},
\]

when \( f \) is \((2)\)-differentiable.

Now we consider the generalization of Taylor's formula for the fuzzy Caputo fractional derivative which was introduced in [32, 45]. It should be mentioned that this theorem is the extension of Taylor's formula for the Caputo fractional derivative in the crisp context [58].

**Theorem 32.** Let \( \tilde{f}(x) \in AC^\infty(0, b) \) and suppose that \( C D^k \tilde{f}(x) \in C^\infty(0, b) \) for \( k = 0, 1, \ldots, n + 1 \) where \( 0 < \alpha < 1, 0 \leq x_0 \leq x \leq x \in [0, b] \). Then we have

\[
\tilde{f}^{(n)}(x) = \sum_{i=0}^n \frac{x^i}{\Gamma(\alpha + i + 1)} C D_a^\alpha \tilde{f}^{(i)}(x) + \frac{\alpha}{\Gamma(\alpha + x)} x^{(n+1)} \tilde{f}^{(n+1)}(x),
\]

\[
\tilde{f}^{(n)}(x) = \sum_{i=0}^n \frac{x^i}{\Gamma(\alpha + i + 1)} C D_a^\alpha \tilde{f}^{(i)}(x) + \frac{\alpha}{\Gamma(\alpha + x)} x^{(n+1)} \tilde{f}^{(n+1)}(x),
\]

also (50) and the following system of FDEs is equivalent when \( y(x) \) is \([2-\alpha]\)-differentiable

\[
(c D_{x}^\alpha y)(x; r) = \tilde{f}^{*}(x, y', y''),
\]

\[
(c D_{x}^\alpha \tilde{y})(x; r) = \tilde{f}^{\ast}(x, y', y''),
\]

Proof. See [32]. \(\Box\)

Now, characterization theorem (Theorem 15), which was introduced by Bede in [15] and established by Pederson and Sambandham in [59] for hybrid fuzzy differential equations, is extended for fuzzy Caputo-type fractional differential equations. To this end, we first consider the FFDEs under Caputo's H-differentiability for \( 0 < \alpha < 1 \) as follows:

\[
(I_a^\alpha D_a^\alpha) y(x) = f(a, y(x)), \quad y(a) \in \mathcal{E}.
\]

\[
(I_a^\alpha D_a^\alpha) y(x) = f(a, y(x)), \quad y(a) \in \mathcal{E}.
\]

**Theorem 33** (characterization theorem). Let one consider the fuzzy fractional differential equation under Caputo's H-differentiability (50) where \( f : [x_0, x_0 + a] \times \mathcal{E} \rightarrow \mathcal{E} \) and such that:

(i) \( [f(x, y)]' = \tilde{f}^{*}(x, y, \tilde{y}), \tilde{f}^{\ast}(x, y', \tilde{y}'') \),

(ii) \( f^{*} \) and \( \tilde{f} \) are equicontinuous (i.e., for any \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( |f^{*}(x, y, z) - f^{*}(x_1, y_1, z_1)| < \varepsilon \)

and \( |\tilde{f}(x, y, z) - \tilde{f}(x_1, y_1, z_1)| < \varepsilon \) for all \( r \in [0, 1] \), whenever \( (x, y, z), (x_1, y_1, z_1) \in [x_0, x_0 + a] \times \mathcal{E} \) and \( \|(x, y, z) - (x_1, y_1, z_1)\| < \delta \) and uniformly bounded on any bounded set,

(iii) there exists an \( L > 0 \) such that \( |f^{*}(x_2, y_2, z_2) - f^{*}(x_1, y_1, z_1)| \leq L \max\|y_2 - y_1\|, \|z_2 - z_1\| \) for all \( r \in [0, 1], |\tilde{f}(x_2, y_2, z_2) - \tilde{f}(x_1, y_1, z_1)| \leq L \max\|y_2 - y_1\|, \|z_2 - z_1\| \) for all \( r \in [0, 1] \).

Then, (50) and the following system of FDEs is equivalent when \( y(x) \) is \([2-\alpha]\)-differentiable

\[
(c D_{x}^\alpha y)(x; r) = \tilde{f}^{*}(x, y', y''),
\]

\[
(c D_{x}^\alpha \tilde{y})(x; r) = \tilde{f}^{\ast}(x, y', y''),
\]

also (50) and the following system of FDEs is equivalent when \( y(x) \) is \([2-\alpha]\)-differentiable

\[
(c D_{x}^\alpha y)(x; r) = \tilde{f}^{*}(x, y', y''),
\]

\[
(c D_{x}^\alpha \tilde{y})(x; r) = \tilde{f}^{\ast}(x, y', y''),
\]

Proof. In the papers [15, 59], the authors proved for fuzzy ordinary differential equations and hybrid fuzzy differential equations. The result for FFDEs is obtained analogously by using Theorem 2 in [15] and Theorem 3.1 in [59]. \(\Box\)
4. Proposed Method for Solving FFDEs

Saadatmandi and Dehghan [36] introduced the shifted Legendre operational matrix for derivative with fractional order using a spectral method which has been followed by Doha et al. [37–39]. They presented the shifted Chebyshev polynomials and Jacobi polynomials for solving fractional differential equations by tau method. In this section, we try to approximate fuzzy solution using shifted Legendre polynomials under H-differentiability as follows.

4.1. Properties of Shifted Legendre Polynomials. The Legendre polynomials, denoted by \( P_n(z) \), are orthogonal with Legendre weight function:

\[
\int_{-1}^{1} P_n(z) P_m(z) \, dz = \frac{2}{2n+1} \delta_{nm},
\]

where \( \delta_{nm} \) is the Kronecker function and can be specified with the help of following recurrence formula:

\[
P_0(z) = 1, \quad P_1(z) = z,
\]

\[
P_{i+1}(z) = \frac{2i+1}{i+1} z P_i(z) - \frac{i}{i+1} P_{i-1}(z), \quad i = 1, 2, \ldots
\]

In order to use these polynomials on the interval \([0,1]\), we define the so-called shifted Legendre polynomials by introducing the change of variable

\[
z = 2x - 1.
\]

Let the shifted Legendre polynomials be denoted by \( L_n(x) \). The shifted Legendre polynomials are orthogonal with respect to the weight function \( w_s(x) = 1 \) in the interval \((0,1)\) with the following orthogonality property:

\[
\int_{0}^{1} L_n(x) L_m(x) \, dx = \frac{1}{2n+1} \delta_{nm}.
\]

The shifted Legendre polynomials are generated from the following three-term recurrence relation:

\[
L_{i+1}(t) = \frac{(2i+1)(2x-1)}{i+1} L_i(t) - \frac{i}{i+1} L_{i-1}(t), \quad i = 1, 2, \ldots
\]

\[
L_0(t) = 1, \quad L_1(t) = 2x - 1.
\]

The analytic form of the shifted Legendre polynomial \( L_n(x) \) of degree \( n \) is given by

\[
L_n(x) = \sum_{i=0}^{n} \frac{(-1)^{n+i} (n+i)!}{(n-i)! i!} x^i = \sum_{i=0}^{n} \epsilon_{i,n} x^i,
\]

in which

\[
\epsilon_{i,n} = (-1)^{n+i} \frac{(n+i)!}{(n-i)! (i!)^2},
\]

where

\[
L_n(0) = (-1)^n, \quad L_n(1) = 1.
\]

A function \( u(x) \) of independent variable defined for \( 0 \leq x \leq 1 \) may be expanded in terms of shifted Legendre polynomials as

\[
u(x) = \sum_{i=0}^{m} b_i L_i(x) = \Lambda^T \Phi(x),
\]

where the shifted Legendre coefficients matrix \( B \) and the shifted Legendre vector \( \Phi(x) \) are given by

\[
\Lambda^T = [\lambda_1, \lambda_2, \ldots, \lambda_m],
\]

\[
\Phi(x) = [L_0(x), L_1(x), \ldots, L_m(x)]^T.
\]

Also, the derivative of \( \Phi(x) \) can be expressed by

\[
\frac{d\Phi(x)}{dx} = D \Phi(x),
\]

where \( D \) is the \( (m+1) \times (m+1) \) operational matrix of derivative given by

\[
D = (d_{ij}) = \begin{cases} 2(2j+1), & \text{for } j = i - k, \\ k = 1, 3, \ldots, m, & \text{if } m \text{ odd}, \\ k = 1, 3, \ldots, m-1, & \text{if } m \text{ even}, \\ 0, & \text{otherwise}. \end{cases}
\]

The graph of some shifted Legendre polynomials \((3 \leq m \leq 8)\) shown in Figure 1 to depict their behaviors.

Now, we use the shifted Legendre functions to approximate a fuzzy function.

4.2. The Approximation of Fuzzy Function. In this section, we propose a shifted Legendre approximation for the fuzzy-valued functions. To this end, we use Legendre's nodes and fuzzy shifted Legendre polynomials to calculate the fuzzy best approximation. For more details, see [61–67].
Definition 34. For \( y \in L_p^E(0, 1) \cap C^E(0, 1) \) and Legendre polynomial \( L_n(x) \) a real valued function over \((0, 1)\), the fuzzy function is approximated by
\[
y(x) = \sum_{j=0}^{\infty} c_j \odot L_j(x), \quad x \in (0, 1),
\]
where the fuzzy coefficients \( c_j \) are obtained by
\[
c_j = (2j + 1) \odot \int_0^1 y(x) \odot L_j(x) \, dx,
\]
in which \( L_j(x) \) is the same as in (57), and \( \sum^* \) means addition with respect to \( \odot \) in \( E \).

Remark 35. In actuality, only the first \((m + 1)\)-terms shifted Legendre polynomials are considered. So we have
\[
y(x) = \sum_{j=0}^{\infty} c_j \odot L_j(x) = \Phi_m(x),
\]
hence
\[
y'(x) = \Phi'_m(x) = \sum_{j=0}^{\infty} c_j \odot L_j(x),
\]
that the fuzzy shifted Legendre coefficient vector \( C^T_m \) and shifted Legendre function vector \( \Phi_{m+1}(x) \) are defined as
\[
C^T_m = [c_0, c_2, \ldots, c_m], \quad \Phi_m(x) = [L_0(x), L_1(x), \ldots, L_m(x)].
\]

Definition 36 (see [68]). A fuzzy-valued polynomial \( \tilde{p}^* \in \prod_N^* \) is the best approximation to fuzzy function \( f \) on \( \chi = \{x_0, x_1, x_2, \ldots, x_N\} \), if
\[
\max_{i=0,1,2,\ldots,N} D(\tilde{p}^*(x_i), f_i) = \min_{\tilde{p} \in \prod_N^*} \left\{ \max_{i=0,1,2,\ldots,N} D(\tilde{p}(x_i), f_i) \right\},
\]
in which \( \prod_N^* \) is the set of all fuzzy valued polynomials.

The problem is referred to as the best shifted Legendre approximation, as we use Legendre’s nodes.

Theorem 37. The best approximation of a fuzzy function based on the Legendre nodes exists and is unique.

Proof. The proof is an instantaneous outcome of Theorem 4.2.1 in [68].

Now, we want to show that the fuzzy approximation converges of Legendre functions to function \( y(x) \).

Lemma 38. Suppose that \( y(x) \in AC^E(0, 1) \cap L_p^E(0, 1) \) and \( \mathcal{D}^\alpha y(x) \in C^E(0, 1) \), \( 0 < \alpha < 1 \), \( 0 \leq x_0 \leq x \), and \( x \in (0, 1) \), and also assume that \( y(x) \) is \( C^{1 - \alpha} \)-differentiable. Therefore using Theorem 32, we have
\[
\left| y'(x) - \sum_{j=0}^{\infty} \frac{x^{j+\alpha}}{\Gamma(j + \alpha + 1)} \mathcal{D}^\alpha y'(0^+) \right| \leq \frac{M_x}{\Gamma(n_\alpha + \alpha + 1)},
\]
\[
\left| y''(x) - \sum_{j=0}^{\infty} \frac{x^{j+\alpha}}{\Gamma(j + \alpha + 1)} \mathcal{D}^\alpha y''(0^+) \right| \leq \frac{M_x}{\Gamma(n_\alpha + \alpha + 1)},
\]
in which \( |D^{(m+1)\alpha} y(x_0)| < M_x \) and \( |D^{(m+1)\alpha} \mathcal{F}(x_0)| < M_x \).

Proof. From Theorem 32, we have
\[
\left| y'(x) - \sum_{j=0}^{\infty} \frac{x^{j+\alpha}}{\Gamma(j + \alpha + 1)} \mathcal{D}^\alpha y'(0^+) \right| \leq \frac{M_{\mathcal{D}^\alpha}}{\Gamma(n_\alpha + \alpha + 1)} \frac{M_x}{\Gamma(n_\alpha + \alpha + 1)},
\]
and the following relation can be obtained:
\[
\left| y''(x) - \sum_{j=0}^{\infty} \frac{x^{j+\alpha}}{\Gamma(j + \alpha + 1)} \mathcal{D}^\alpha y''(0^+) \right| \leq \frac{M_{\mathcal{D}^\alpha}}{\Gamma(n_\alpha + \alpha + 1)} \frac{M_x}{\Gamma(n_\alpha + \alpha + 1)},
\]
in which \( M_{\mathcal{D}^\alpha} = |D^{(m+1)\alpha} y(x_0)| \) and \( M_x = |D^{(m+1)\alpha} \mathcal{F}(x_0)| \).
\textbf{Theorem 40.} Let $cD^\alpha y(x) \in C^E(0,1] \cap L^E_2(0,1]$, and $0 < \alpha < 1$. Also consider a sequence of finite dimensional fuzzy space $X_E \subset X$, $m \geq 1$, in which $X_E$ have dimension $d_{m+1}$. Additionally, $X_E$ have a basis $\{L_i(x)\}_{i=0}^m$. If one assumes that $\tilde{y}_m(x) = cT^\Phi$ is the best fuzzy approximation for fuzzy function $y(x)$ from $\{L_i(x)\}_{i=0}^m$, then the error estimation is as follows:

$$\lim_{m \to \infty} D_\infty (y(x), \tilde{y}_m(x)) = 0.$$  \hfill (75)

\textbf{Proof.} Let $f: R \to \mathbb{E}$ be a fuzzy valued function such that $[f(x)]^\Gamma = [f'(x), f''(x)]$. Also consider the fuzzy Taylor’s formula in Theorem 32, $f'(x) = \sum_{i=0}^m (x^{ia}/\Gamma(ia + 1)) \xi^{ia}f''(0^+)$ and $f''(x) = \sum_{i=0}^m (x^{ia}/\Gamma(ia + 1)) cD^\alpha f''(0^+)$. From Lemma 38 we have

$$\left| y'(x) - \tilde{y}'_m(x) \right| \leq M' \frac{x^{(m+1)\alpha}}{\Gamma(m \alpha + 1 + 1)},$$

$$\left| y''(x) - \tilde{y}''_m(x) \right| \leq M' \frac{x^{(m+1)\alpha}}{\Gamma(m \alpha + 1 + 1)}.$$  \hfill (76)

From the assumption, $cT^\Phi$ is the best fuzzy approximation to $y$ from $\{L_i(x)\}_{i=0}^m$ and $f \in X_E$, $x \in (0,1)$. So one has

$$\left| y'(x) - \tilde{y}'_m(x) \right| \leq \left| y''(x) - f''(x) \right| \leq M' \frac{x^{(m+1)\alpha}}{\Gamma(m \alpha + 1 + 1)},$$

$$\left| y''(x) - \tilde{y}''_m(x) \right| \leq \left| y''(x) - f''(x) \right| \leq M' \frac{x^{(m+1)\alpha}}{\Gamma(m \alpha + 1 + 1)}.$$  \hfill (77)

and thus, taking into account Theorem 1 in [69] and above relations, we have

$$\left\| y'(x) - \tilde{y}'_m(x) \right\|^2 \leq \left\| y''(x) - f''(x) \right\|^2 \leq \frac{M'^2}{\Gamma(m \alpha + 1 + 1)^2} \int_0^1 x^{2(m+1)\alpha} \, dx,$$

$$\left\| y''(x) - \tilde{y}''_m(x) \right\|^2 \leq \left\| y''(x) - f''(x) \right\|^2 \leq \frac{M'^2}{\Gamma(m \alpha + 1 + 1)^2} \int_0^1 x^{2(m+1)\alpha} \, dx.$$  \hfill (78)

From Remark 39 and Lemma 38, we have

$$\left\| y'(x) - \tilde{y}'_m(x) \right\| \leq \frac{M_x}{\Gamma(m \alpha + 1 + 1)} \sqrt{\frac{1}{(2(m + 1) \alpha + 1)^2}},$$

$$\left\| y''(x) - \tilde{y}''_m(x) \right\| \leq \frac{M_x}{\Gamma(m \alpha + 1 + 1)} \sqrt{\frac{1}{(2(m + 1) \alpha + 1)^2}}.$$  \hfill (79)

and if $m \to \infty$, we get $\left\| y'(x) - \tilde{y}'_m(x) \right\| \to 0$, $\left\| y''(x) - \tilde{y}''_m(x) \right\| \to 0$. Therefore, from Remark 39 and the definition of Hausdorff distance in Section 2, it can be implied that

$$\lim_{m \to \infty} D_\infty (y(x), \tilde{y}_m(x)) = 0,$$  \hfill (80)

which completes the proof. \hfill \Box

\textbf{Remark 41.} The same result can be obtained for $y(x)$ under $[2 - \alpha]$-differentiability.

4.3. Operational Matrix of Caputo Fractional Derivative

\textbf{Lemma 42.} The fuzzy Caputo fractional derivative of order $0 < \alpha < 1$ over the shifted Legendre functions can be gained in the form of

$$cD^\alpha L_i(x) = \sum_{k=0}^i \tilde{c}_{k,i} \Gamma(\alpha + k + 1) x^{k - \alpha},$$  \hfill (81)

where $\tilde{c}_{i,j} = 0$ when $i < [\alpha]$ and for $i \geq [\alpha]$, and so one has $\tilde{c}_{i,j} = c_{k,j}$.

\textbf{Proof.} Employing the analytic form of shifted Legendre polynomials explained in Section 4.1 and Definition 25, we have:

$$D^\alpha L_i(x) = \sum_{k=0}^i \tilde{c}_{k,i} D^\alpha \otimes x^{k - \alpha}. $$  \hfill (82)

Now, by exploiting Definition 24, the lemma can be proved. \hfill \Box

\textbf{Lemma 43.} Let $0 < \alpha < 1$, and the integral of the product of the fuzzy Caputo fractional derivative with order $\alpha$ over the shifted Legendre functions can be obtained by

$$\int_0^1 D^\alpha L_i(x) L_j(x) \, dx = \sum_{k=0}^i \sum_{l=0}^j \tilde{c}_{k,j} c_{k,l} \Gamma(k + 1) \Gamma(k - \alpha + 1).$$  \hfill (83)

\textbf{Proof.} Using Lemma 42 and the analytic form of shifted Legendre polynomials explained in Section 4.1, we can acquire:

$$\int_0^1 D^\alpha L_i(x) L_j(x) \, dx$$

$$= \sum_{k=0}^i \sum_{l=0}^j \tilde{c}_{k,j} c_{k,l} \Gamma(k + 1) \Gamma(k - \alpha + 1) \int_0^1 x^{(k+l+1)-\alpha} \, dx.$$  \hfill (84)
Abstract and Applied Analysis

The operational matrix of different orthogonal functions for solving differential equations was introduced in the crisp concept [36, 37, 39]. Here, the Legendre operational matrix (LOM) in [36] is applied to the FFDEs using Caputo-type derivative.

The Caputo fractional derivatives operator of order \(0 < \alpha < 1\) of the vector \(\Phi\) defined in (69) can be stated by

\[
D^\alpha \Phi(x) = D^{(\alpha)} \Phi(x),
\]

where \(D^{(\alpha)}\) is the \((m + 1)\)-square operational matrix of fractional Caputo’s derivative of Legendre functions. Regarding the following theorem, the LOM elements \(D^{(\alpha)}_{ij}\) are determined under Caputo fractional derivative. This theorem is generalizing the operational matrix of derivatives of shifted Legendre given in Section 4.1 to the fractional calculus.

**Theorem 44.** Let \(\Phi\) be Legendre functions vector. \(D^{(\alpha)}_{ij}\) is the \((m + 1)\)-square operational matrix of fractional Caputo’s derivative of order \(0 < \alpha < 1\). Then the elements of \(D^{(\alpha)}_{ij}\) are achieved as

\[
D^{(\alpha)}_{ij} = \sum_{k=\lfloor \alpha \rfloor}^i \theta_{i,j,k},
\]

in which \(\theta_{i,j,k}\) are acquired by

\[
\theta_{i,j,k} = (2j + 1) \times \sum_{l=0}^j \frac{(-1)^{i+j+k+l}(i+k)!(l+j)!}{(i-k)!k!(k-\alpha+1)(j-l)!(l!)^2(k+l-\alpha+1)}.
\]

Consider that in \(D^{(\alpha)}\), the first \([\alpha]\) rows, are all zero.

**Proof.** Employing the relation (85) and the orthogonal properties of shifted Legendre functions (56), we have

\[
D^{(\alpha)} = \langle D^\alpha \Phi(x), \Phi(x)^T \rangle E^{-1},
\]

in which \(\langle D^\alpha \Phi(x), \Phi(x)^T \rangle\) and \(E^{-1}\) are \((m + 1)\)-square matrices defined as

\[
\langle D^\alpha \Phi(x), \Phi(x)^T \rangle = \left\{ \int_0^1 D^\alpha L_i(x) L_j(x) dx \right\}_{i,j=0}^{m},
\]

\[
E^{-1} = \text{diag}(2i+1)_{i=0}^{m-1}.
\]

Hence, applying Lemma 43 and inserting the above matrices in the product \(D^{(\alpha)}\), the theorem be proved. \(\square\)

**Remark 45.** If \(\alpha = n \in \mathbb{N}\), then Theorem 44 gives the mentioned result as in Section 4.1.

The following property of the product of two shifted Legendre polynomials fuzzy vectors will also be applied which is the extension of the crisp case introduced in [36],

\[
\Phi^T \Lambda = \Lambda \Phi,
\]

where \(\Lambda\) is an \((m + 1)\)-square product operational matrix for the fuzzy vector \(\Lambda = [\Lambda, \Lambda]\). Using the above equation and by the orthogonal property equation (56), the elements \(\Lambda_{ij}\) can be computed from

\[
\Lambda_{ij} = (2j + 1) \sum_{k=0}^m \lambda_k \otimes g_{ijk},
\]

where \(\sum^*\) denotes the fuzzy summation and \(\otimes\) indicates fuzzy multiplication. Moreover, \(g_{ijk}\) are obtained by

\[
g_{ijk} = \int_0^1 L_i(x) L_j(x) L_k(x) dx,
\]

that in the simpler form, it is given by

\[
g_{ijk} = \begin{cases} 
\frac{d_{i+j+k+l}}{(2i+2j-2l+1)} & k = i + j - 2l; \\
0 & k \neq i + j - 2l;
end{cases}
\]

in which \(d_l = (2l)!/2^l l!^2\).

**Remark 46.** As a matter of fact, from (93), it is obvious that the elements of the product of shifted Legendre operational matrix are independent from the fuzzy vector.

### 5. Application of the LOM for Solving FFDEs

In this section, Legendre operational matrix of fractional Caputo’s derivative is considered to present its significance for solving fuzzy fractional differential equation which is alike to the concept of the Caputo-type derivative in crisp case [36]. The existence and uniqueness of the solutions under fuzzy Caputo’s derivative to this problem are discussed in [30].

#### 5.1. General Linear FFDEs

Let us consider the general linear fuzzy fractional differential equation

\[
\left( \partial^\alpha D^\alpha_{0,n} y \right)(x) = a_1 \otimes y(x) \oplus a_2 \otimes f(x), \quad 0 < \alpha \leq 1,
\]

\[
y(0) = y_0 \in \mathbb{E},
\]

in which \(a_j\) for \(j = 1, 2\) are fuzzy constant coefficients, \(y(x) : I \subseteq \mathbb{R} \rightarrow L^E \cap C^E\) is a continuous fuzzy-valued function, and \(\partial^\alpha D^\alpha_{0,n}\) represents the fuzzy Caputo fractional derivative of order \(\alpha\).

**Remark 47.** In this section, considering more simplicity, it is assumed that for the fuzzy set-valued function \(y(x) : I \subseteq \mathbb{R} \rightarrow L^E \cap C^E\), we present the functions \(y^L, y^R : I \subseteq \mathbb{R}, r \in [0, 1]\) by \(y^L(x) = (y(x))^L, y^R(x) = (y(x))^R\), for all \(x \in I\), for all \(r \in [0, 1]\), which are in the same previous manner, called the left and right \(r\)-cut functions of \(y(x)\).
Let \(r \in [0, 1]\) and \(0 < \alpha < 1\). Then from (94), we have that

\[
[a_1 \odot y(x)]^r = \left[ (a_1 \odot y(x))^r, (a_1 \odot y(x))^r \right] = \left[ (a_1^r y(x))_+, (a_1^r y(x))_+ \right] = a_1^{r \alpha} y(x)^r,
\]

where \(a_1^{r \alpha} y(x)^r = (a_1 y(x))^r\).

Therefore,

\[
\left( \mathbb{C} D_0^r y \right)_+^r(x) = (a_1)_+^r y(x)_+^r + (a_2)_+^r f(x)_+^r.
\]

Hence, we obtain

\[
\left[ \mathbb{C} D_0^r y(x) \right]^r = \left[ \mathbb{C} D_0^r y \right]_+^r(x), \mathbb{C} D_0^r y \right]_+^r(x) = \left[ (a_1)_+^r y(x)_+^r + (a_2)_+^r f(x)_+^r \right] = [a_1 \odot y(x)]^r + [a_2 \odot f(x)]^r, r \in [0, 1],
\]

We can rewrite (94) in the operator form

\[
(a_1^{r \beta} I \otimes g \mathbb{C} D^a) y^r = a_2^{r \beta} f^r,
\]

in which the fuzzy operator \(\mathbb{C} D^a = 1^{1-\alpha} D\) is supposed to be compact on a fuzzy Banach space \(X_E \to X_E\) and \(\otimes g\) is the notation for \(g\)-difference.

For solving fuzzy fractional differential equation (94), we attempt to find a fuzzy function \(y_m^{(r)} \in X_E\); therefore, let \((\mathbb{C} D_0^r y)(x), y(x)\) and \(f(x)\) be approximated using Definition 34 as:

\[
y(x) = \bar{y}_m(x) = \sum_{j=0}^m c_j \odot L_j(x) = C_m^T \odot \Phi_m,
\]

that

\[
[y_m(x)]^r = \sum_{j=0}^m c_j^{(r)} \odot [L_j(x)]^r, x \in I \subset R,
\]

\[
f(x) = \bar{f}_m(x) = \sum_{j=0}^m f_j \odot L_j(x) = F_m^T \odot \Phi_m,
\]

where \(F_m = [f_0, f_1, \ldots, f_m]^T\) is obtained as

\[
f_j = (2j + 1) \int_0^1 f(x) \odot L_j(x) \, dx.
\]

Also, using relations (85) and (100), we obtain

\[
\mathbb{C} D^a y(x) = C^T \odot D^a \Phi_m(x) = C^T \odot D^a \Phi_m(x).
\]

Substituting (100)–(103) in problem (94), the coefficients \(c_j^{(r)}\) are specified by imposing the equation to be almost fuzzy exact in the Legendre operational matrix form. Now, we establish fuzzy residual for the approximation of (94), when \([y(x)]^r \equiv [y_m(x)]^r\).

**Theorem 48.** Let \(y^r \in C^E[0, 1]\) and \(0 < \alpha \leq 1\), and then

\[
\left[ \mathbb{C} D^a \bar{y}_m(x) \right]^{(r)} = \left[ R_m(x) \oplus a_2 \odot \bar{f}_m(x) \oplus a_1 \odot \bar{y}_m(x) \right]^{(r)}.
\]

**Proof.** Let \(r \in [0, 1]\). We have \([\mathbb{C} D^a \bar{y}_m(x)]^{(r)} = [\mathbb{C} D^a \bar{y}_m(x), \mathbb{C} D^a \bar{y}_m(x)]^{(r)} = [\bar{y}_m(x)^r, \bar{y}_m(x)^r]^{(r)}\) and regarding to what was conversed previously,

\[
\left[ \mathbb{C} D^a \bar{y}_m(x) \right]^{(r)} = R_m^{(r)}(x) - (a_1)_+^r \bar{y}_m(x)_+^r - (a_2)_+^r \bar{f}_m(x)_+^r,
\]

so

\[
\left[ \mathbb{C} D^a \bar{y}_m(x) \right]^{(r)} = \left[ \left[ \mathbb{C} D^a \bar{y}_m^{(r)}(x) \right] \left[ \mathbb{C} D^a \bar{y}_m^{(r)}(x) \right] \right]^{(r)} = \left[ R_m^{(r)}(x), R_m^{(r)}(x) \right] - \left[ (a_1)_+^r \bar{y}_m^{(r)}(x)_+^r, (a_1)_+^r \bar{y}_m^{(r)}(x)_+^r \right] - \left[ (a_2)_+^r \bar{f}_m^{(r)}(x)_+^r, (a_2)_+^r \bar{f}_m^{(r)}(x)_+^r \right] = R_m^{(r)}(x) \oplus \left[ (a_1)_+^r \bar{y}_m^{(r)}(x) \right] \oplus \left[ (a_2)_+^r \bar{f}_m^{(r)}(x) \right].
\]

Using \(g\)-difference, we have

\[
R_m^{(r)}(x) = \left[ \left( \mathbb{C} D^a \bar{y}_m(x) \right) \right]^{(r)} \oplus \left[ (a_1)_+^r \bar{y}_m^{(r)}(x) \right] \oplus \left[ (a_2)_+^r \bar{f}_m^{(r)}(x) \right],
\]

or in the sense of fuzzy operator,

\[
R_m^{(r)}(x) = \left( a_1^{(r)} I \otimes g \mathbb{C} D^a \right) y^{(r)} \oplus \left( a_2^{(r)} f^{(r)} \right).
\]

It is anticipated that the deriving fuzzy function \([y_m(x)]^r\) will be a suitable approximation of the exact solution \([y(x)]^r\). To this end, let \(X_E = L^2_E(0, 1)\), and let \(\langle \cdot, \cdot \rangle_E\) indicate the fuzzy inner product for \(X_E\). It is demanded that \(R_m^{(r)}\) satisfy

\[
\langle R_m^{(r)}, L_{iE} \rangle = 0, \quad i = 0, 1, \ldots, m - 1, \quad r \in [0, 1],
\]

where \(L_{iE} = L^2_E(0, 1)\).
in which \( \langle R_{m}^{(r)} L_{i} \rangle_E = [(FR) \int_{0}^{1} R_{m}(x) \odot L_{i}(x) dx]^{(r)} \). The left side is the shifted legendre coefficients associated with \( L_{i} \). If \( \{ L_{i} \}_{i=0}^{m} \) are the main members of shifted Legendre family \( \Phi = \{ L_{i} \}_{i=0}^{m} \) which is complete in \( X_{E} \), then (109) needs the main terms to be zero in the Fourier extension of \( R_{m}^{(r)} \) with respect to \( \Phi \) which is called tau method in the crisp context and it is in a similar manner with the meaning of fuzzy (for more details, see [71–73]).

To discover \( E_{\alpha} \), implementing (109) to (108), regarding the (100)–(103), we generate \( m \) fuzzy linear equations. These fuzzy linear equations can be acquired as the following system of equations:

\[
\sum_{j=0}^{m-1} c_{j}^{(r)} \odot \left\{ (D^{(a)} L_{i}) - (a_{i}^{(r)} L_{i}) \right\} = \sum_{j=0}^{m-1} f_{j}^{(r)} \odot \left\{ (a_{i}^{(r)} L_{i}) \right\}, \quad i = 0, 1, \ldots, m-1
\]

or

\[
\sum_{j=0}^{m-1} c_{j}^{(r)} \odot \left\{ (FR) \int_{0}^{1} D^{(a)} L_{j}(x) L_{i}(x) dx \right\} = \sum_{j=0}^{m-1} f_{j}^{(r)} \odot (FR) \int_{0}^{1} a_{i}^{(r)} L_{j}(x) L_{i}(x) dx dt
\]

Afterwards, substitution of (100) in the initial condition of (94) yields

\[
y(0) = \sum_{j=0}^{m} c_{j}^{(r)} \odot L_{j}(0) = y_{0},
\]

that this equation be coupled with the previous fuzzy linear equations and constructed \( m+1 \) fuzzy linear equations. Clearly, after solving this fuzzy system, the coefficients \( c_{j}^{(r)} \) will be gained.

5.1.1. Error Analysis. The aim of this section is to acquire the error bound for the Legendre approximation using shifted Legendre polynomials. We consider the best shifted Legendre approximation of a smooth fuzzy function under Caputo derivative for \( 0 < \alpha \leq 1 \) to reach the result. It should be considered that the results of this section are the extension of this concept in the crisp context (see more in [69, 74–76]). Initially, we state the following lemma which can offer an upper bound for approximating the error of Caputo fractional derivative. So we define the error vector \( E_{\alpha} \) as

\[
E_{\alpha} = D^{\alpha} \Phi - D^{a_{i}} \Phi = [E_{0,\alpha}, E_{1,\alpha}, \ldots, E_{m,\alpha}]^{T},
\]

where

\[
E_{i,\alpha} = D^{\alpha} L_{i}(x) - \sum_{j=0}^{m} D^{(a)} L_{j}(x), \quad i = 0, 1, \ldots, m.
\]

Now, we can propose the next lemma by using Theorem 1 in [69] to depict the error bound of Caputo fractional derivative operator for the shifted Legendre polynomials.

**Lemma 49.** Let the error function of Caputo fractional derivative operator for Legendre polynomials \( E_{i,\alpha} : [x_{0}, 1] \rightarrow \mathbb{R} \) be \( m+1 \) times continuously differentiable for \( 0 < x_{0} \leq x \), \( x \in (0,1] \). Also \( E_{i,\alpha} \in C_{m+1}[x_{0}, 1] \) and \( \alpha < m+1 \) and then the error bound is presented as follows:

\[
\| E_{i,\alpha} \|_{\infty} \leq \frac{|\Gamma(i+1)|}{|\Gamma(\alpha)|} \sqrt{\frac{1}{\Gamma(2(m+1)\alpha+1)}} x_{0}^{-\alpha}.
\]

**Proof.** Firstly, we acquire a bound for \( ^{c} D^{a} x^{i} \), regarding Definition 24, as

\[
^{c} D^{a} x^{i} = \frac{\Gamma(i+1)}{\Gamma(i+1-\alpha)} x^{i-\alpha} \leq \frac{|\Gamma(m+2)|}{|\Gamma(1-\alpha)|} x_{0}^{-\alpha}, \quad i = 0, 1, \ldots, m.
\]

Then using Lemma 1 in [69], we have:

\[
D^{a} L_{i}(x) = \sum_{k=0}^{i} e_{k,\alpha} x^{k} \leq \frac{|\Gamma(i+1)|}{|\Gamma(1-\alpha)|} x_{0}^{-\alpha} \sum_{k=0}^{i} e_{k,\alpha} \quad \text{in which}
\]

\[
e_{k,\alpha} = (-1)^{k+i} \frac{(k+i)!}{(i-k)!(k)!}.
\]

Now, utilizing Theorem 1 in [69], the lemma can be proved. \( \square \)

Therefore, the maximum norm of error vector \( E_{\alpha} \) is attained as

\[
\| E_{i,\alpha} \|_{\infty} \leq \frac{|\Gamma(m+2)|}{|\Gamma(1-\alpha)|} \sqrt{\frac{1}{\Gamma(2(m+1)\alpha+1)}} x_{0}^{-\alpha}.
\]
Now we extend these results for the fuzzy case. Hence, the error of fuzzy Caputo fractional derivative is defined as

\[
D^\alpha \left( D^\alpha \Phi (x) , D^{(\alpha)} \Phi (x) \right) \\
= \sup_{x \in [0,1]} D \left( D^\alpha \Phi (x) , D^{(\alpha)} \Phi (x) \right) \\
= \sup_{x \in [0,1]} \sup_{r \in [0,1]} \max \left\{ \left| D^\alpha \Phi (x)^r_+ (x) - D^{(\alpha)} \Phi (x)^r_+ (x) \right| \right\} \\
= \sup_{r \in [0,1]} \max \left\{ \left| D^\alpha \Phi (x)^r_+ (x) - D^{(\alpha)} \Phi (x)^r_+ (x) \right| \right\}
\]

where \(0 < x_0 \leq x, x \in [0,1]\).

**Lemma 50.** Consider the error function of the fuzzy Caputo fractional derivative operator for shifted Legendre polynomials which is continuously differentiable and \(0 < \alpha < 1\). Then the error bound is given as follows:

\[
D^\alpha \left( D^\alpha \Phi (x) , D^{(\alpha)} \Phi (x) \right) \\
\leq \left| \frac{\Gamma (m + 1)}{\Gamma (1 - \alpha)} \right| \left| \frac{\Gamma (m + 1)}{\Gamma (1 - \alpha)} \right| \Gamma (m + \alpha + \alpha + 1) \sqrt{\frac{1}{(2 (m + 1) \alpha + 1)^2} x_0^{-\alpha}},
\]

that \(M_\alpha = \max \{M_{\alpha,-}, M_{\alpha,+}\} \).

**Proof.** It is straightforward from the definition of \((E_\alpha)\), **Definition 1**, and **Remark 3**.

For an error assessment of the approximation solution of (94), we assume that \(y_m (x)\) and \(y(x)\) reveal the fuzzy approximate and exact solutions of the fuzzy fractional differential equations, respectively. Then we rewrite (94) as

\[
(c^D D^\alpha y_m^{(r)} (x) - (a_1)_h^{(r)} y_m^{(r)} (x) + H_m^{(r)} (x) + (a_2)_h^{(r)} f_m^{(r)} (x) + f_m^{(r)} (x),
\]

that \(H_m^{(r)} (x)\) is the fuzzy perturbation function that depends only on \(y_m^{(r)} (x)\). By subtracting (94) from above equation and using Lemmas 38 and 50, **Theorem 40**, and **Definition 2**, one can obtain

\[
\left\| H_m^{(r)} (x) \right\|_E \leq \sup_{r \in [0,1]} \max \left\{ \left| D^\alpha \Phi (x)^r_+ (x) - D^{(\alpha)} \Phi (x)^r_+ (x) \right| \right\}
\]

\[
\leq \left( \frac{\Gamma (m + 1)}{\Gamma (1 - \alpha)} \right) \left( \frac{\Gamma (m + \alpha + \alpha + 1)}{\Gamma (m + \alpha + \alpha + 1)} \right) \sqrt{\frac{1}{(2 (m + 1) \alpha + 1)^2} x_0^{-\alpha}}
\]

\[
+ \left( \frac{\Gamma (m + 1)}{\Gamma (1 - \alpha)} \right) \left( \frac{\Gamma (m + \alpha + \alpha + 1)}{\Gamma (m + \alpha + \alpha + 1)} \right) \sqrt{\frac{1}{(2 (m + 1) \alpha + 1)^2} x_0^{-\alpha}},
\]

\[
\leq C \left( 1 + x_0^\alpha \right),
\]

at which \(C = \left( \frac{\Gamma (m + 1)}{\Gamma (1 - \alpha)} \frac{\Gamma (m + \alpha + \alpha + 1)}{\Gamma (m + \alpha + \alpha + 1)} \sqrt{\frac{1}{(2 (m + 1) \alpha + 1)^2}} \right), \) for \(0 < x_0 \leq x, x \in (0, 1).\) Therefore, \(H_m^{(r)} (x)\) is bounded.

5.2. **Nonlinear FFDEs.** Consider the nonlinear FFDE

\[
(c^D D^\alpha y (x) = F (x, y(x), \varphi (x)), \quad 0 < \alpha < 1,
\]

\[
y (0) = \left[ y_0, \varphi_0 \right],
\]

where \(F\) can be nonlinear in general, \(y(x) : \mathbb{L} \rightarrow \mathbb{C}^E\) is a continuous fuzzy-valued function, and \(c^D D^\alpha y\) indicates the fuzzy Caputo fractional derivative of order \(\alpha\).

In order to use LOM for this problem, we first approximate \(y(x)\) and \(c^D D^\alpha y (x)\) as (100) and (103), respectively. By replacing these equations in (128), we have

\[
C^T \otimes D^{(\alpha)} \Phi (x) = F \left( x, C^T \Phi (x) , \varphi (x) \right).
\]

We intend to find the fuzzy coefficients \(C_m\). Also by substituting (100) in the initial condition of nonlinear FFDE (128), we have

\[
y (0) = C^T \Phi (0) = y_0,
\]

\[
\varphi (0) = C^T \Phi (0) = \varphi_0.
\]
To find the approximate fuzzy solution \( \tilde{y}_m(x) \), we first collocate (129) at \( (m) \) points. For appropriate collocation points we use the first \( (m) \) shifted Legendre polynomials roots. These equations together with (130) generate \((m + 1)\) nonlinear fuzzy equations which can be solved using Newton’s iterative method presented in [77].

Remark 51. The solvability of system (129) and (130) is a complicated problem and we cannot prove the existence and uniqueness of such a fuzzy solution. But in our accomplishment, we have solved this system, regarding the method in [77], using MATLAB functions. In the assumed example, these functions have prospered to gain an accurate fuzzy approximate solution of the system, even starting with a zero initial guess.

### 6. Numerical Examples

In this section, to demonstrate the effectiveness of the proposed method in the present paper, two different test examples are carried out. Also, the obtained numerical solutions be compared with exact solutions.

**Example 52.** Let us consider the following FFDE:

\[
\left( \frac{\alpha}{\Gamma(\alpha+1)} \right) \left( \frac{d^\alpha}{dt^\alpha} y \right)(x) = \lambda \circ y(x), \quad 0 < \alpha \leq 1, \\
y(0; r) = y_0(r), \quad 0 < r \leq 1, \\
\left( \frac{\alpha}{\Gamma(\alpha+1)} \right) \left( \frac{d^\alpha}{dt^\alpha} \tilde{y} \right)(x; r) = \tilde{y}(x), \quad 0 < \alpha \leq 1, \\
\tilde{y}(0; r) = \tilde{y}_0(r), \quad 0 < r \leq 1,
\]

in which \( y(x) : L^E \cap C^E \) is a continuous fuzzy-valued function and \( \frac{\alpha}{\Gamma(\alpha+1)} \) denotes the fuzzy Caputo fractional derivative of order \( \alpha \). We solve this example according to two following cases for \( \lambda = 1, -1 \).

**Case 1.** Suppose that \( \lambda = 1 \), and then using \( \frac{\alpha}{\Gamma(\alpha+1)} \) differentiability and Theorem 33, we have the following:

\[
\left( \frac{\alpha}{\Gamma(\alpha+1)} \right) \left( \frac{d^\alpha}{dt^\alpha} y \right)(x; r) = y(x), \quad 0 < \alpha \leq 1, \\
y(0; r) = y_0(r), \quad 0 < r \leq 1, \\
\left( \frac{\alpha}{\Gamma(\alpha+1)} \right) \left( \frac{d^\alpha}{dt^\alpha} \tilde{y} \right)(x; r) = \tilde{y}(x), \quad 0 < \alpha \leq 1, \\
\tilde{y}(0; r) = \tilde{y}_0(r), \quad 0 < r \leq 1,
\]

in which \( y(0; r) = [0.5 + 0.5r, 1.5 - 0.5r] \).

The exact solution of FFDE is as follows:

\[
Y(x; r) = (0.5 + 0.5r) E_{0.1} \left[ x^0 \right], \quad 0 < \alpha \leq 1, \\
\tilde{Y}(x; r) = (1.5 - 0.5r) E_{0.1} \left[ x^0 \right], \quad 0 < r \leq 1,
\]

where \( E_{0.1} \) is the classical Mittag-Leffler function

\[
E_{0.1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha (k + 1))}.
\]

Note that for \( \alpha = 1 \) this representation is still valid since \( \Gamma(1) = 1 \) and \( E_{1,1}(z) = e^z \).

**Remark 53.** If \( \alpha = 1 \), then the fuzzy fractional differential equations (132) are converted to fuzzy integer-order differential equations. So the exact solution of problem under (1)-differentiability using Theorem 15 is as follows:

\[
\tilde{Y}(x; r) = (0.5 + 0.5r) e^r, \quad 0 < r \leq 1.
\]

By utilizing the technique explained in Section 5, (131) with \( \lambda = 1 \) becomes:

\[
m \sum_{j=0}^{m} \tilde{c}_j \left[ d^\alpha_{i,j}(r) - I \right] L_j(x; r) = 0 \quad i = 0, \ldots, m,
\]

or we can rewrite it in the matrix form as

\[
C^T_{m,-} \left[ D^\alpha(r) - I \right] \Phi(x) = 0,
\]

in which \( \{c_j\} = \{c_{i,j}, i = 0, 1, \ldots, m \} \) and for \( j = 0, 1, \ldots, m \). As it was described in Section 5, we produce \( m \) fuzzy algebraic equations multiplied above fuzzy residual system by \( L_j(x) \) for \( i = 0, 1, \ldots, m \) and solve the problem using orthogonal property, so we obtain

\[
C^T_{m,-} \left[ D^\alpha(r) - I \right] = 0,
\]

Now, using the initial condition (131), we have

\[
Y(0; r) = C^T_{m,-} \Phi_m = (0.5 + 0.5r), \\
\tilde{Y}(0; r) = C^T_{m,+} \Phi_m = (1.5 - 0.5r).
\]

The above three equations generate a set of \( m + 1 \) fuzzy linear algebraic equations. As a result, for \( \alpha = 0.85 \) and \( m = 3 \), one can gain

\[
D^{(0.85)} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0.8696 & -0.4794 & 6.7831 & 1.3797 \\
0 & 0.3901 & -0.1755 & 0.1095 \\
-0.3901 & 4.5267 & 0.8696 & -0.4078 \\
-0.1755 & -0.4078 & 0.3901 & -0.1755 \\
1.3797 & -0.4794 & 6.7831 & 1.3797
\end{bmatrix}
\]

the unknown coefficients \( c_j \) can be gained by substituting matrix \( D^{(0.85)} \) in the previous mentioned systems.

**Case 2.** Suppose that \( \lambda = -1 \), then using \( \frac{\alpha}{\Gamma(\alpha+1)} \) differentiability and Theorem 33, the FDEs are obtained as same as (132). Also, the exact solution is given by

\[
Y(x; r) = (0.5 + 0.5r) E_{0.1} \left[ -x^0 \right], \quad 0 < \alpha \leq 1, \\
\tilde{Y}(x; r) = (1.5 - 0.5r) E_{0.1} \left[ -x^0 \right], \quad 0 < r \leq 1.
\]
By applying the technique described in Sections 4 and 5, namely \( m = 2 \), we may write the approximate solution for Cases 1 and 2 in the forms

\[
\begin{align*}
\tilde{y}^1_2(x) &= \zeta_0 L_0(x) + \zeta_1 L_1(x) + \zeta_2 L_2(x), \\
\overline{y}^2_2(x) &= \overline{\zeta}_0 \overline{L}_0(x) + \overline{\zeta}_1 \overline{L}_1(x) + \overline{\zeta}_2 \overline{L}_2(x).
\end{align*}
\] (142)

So regarding the (100)–(110) and (142), the unknown parameters, \( \zeta_j, \overline{\zeta}_j, \ j = 0, 1, 2 \), are achieved for both cases.

**Remark 54.** If \( \alpha = 1 \), then the fuzzy fractional differential equation (131) with \( \lambda = -1 \) is converted to fuzzy integer-order differential equation. So the exact solution of problem under (2)-differentiability using Corollary 16 is as follows:

\[
\begin{align*}
Y(x; r) &= (0.5 + 0.5r)e^{-x}, \\
\overline{Y}(x; r) &= (1.5 - 0.5r)e^{-x}, \quad 0 < r \leq 1.
\end{align*}
\] (143)

The approximate solution, exact solution and absolute errors are depicted for Case 1 of Example 52 in Table 1 for \( x = 1 \) with \( \alpha = .85 \). It can be seen that a few terms of the shifted Legendre functions are required to achieve a suitable approximation which demonstrate the applicability of the proposed method for this problem. Also the absolute errors for \( m = 2, 4, 8 \) with \( \alpha = .85 \) are plotted in Figure 2 which show the decreasing of the error with the increasing of the number of Legendre functions. The fuzzy approximate solution is shown in Figure 3 for \( \alpha = .85, m = 4 \).

**Remark 55.** Figure 2 depicts the absolute errors for \( \tilde{y}(x; r) \) of Example 52, Case 1. In the same way, if we consider \( \tilde{y}(x; r) \), then analogously to the demonstration of Figure 2, we can obtain the absolute errors.

In the Table 2, the fuzzy approximate solution for Case 2 of Example 52 is considered. The fractional Caputo derivative is \( \alpha = .75 \) and the number of Legendre functions are \( m = 9 \). The result are computed for \( x = 1 \) with different \( r \)-cut. Again we can see that Table 2 demonstrates the validity of the method for this kind of problems. Furthermore, the absolute errors of Example 52, Case 2, are plotted in Figure 4 with \( m = 5, 9, 11 \) and the fuzzy approximate solution is shown in Figure 5 for \( \alpha = .75, m = 9 \).

**Remark 56.** Figure 4 shows the absolute errors for \( y(x; r) \) of Example 52, Case 2. In the same way, if we consider \( y(x; r) \),
Table 1: The result of the proposed method for Case 1 of Example 52 with $\alpha = 0.85$ and $m = 4$.

<table>
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<th>Error</th>
<th>$\frac{y'}{4}$</th>
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<td>2.3441</td>
<td>3.682e-4</td>
<td>3.9063</td>
<td>3.9069</td>
<td>6.137e-4</td>
</tr>
<tr>
<td>0.6</td>
<td>2.5000</td>
<td>2.5004</td>
<td>3.928e-4</td>
<td>3.7500</td>
<td>3.7506</td>
<td>5.891e-4</td>
</tr>
<tr>
<td>0.7</td>
<td>2.6563</td>
<td>2.6567</td>
<td>4.173e-4</td>
<td>3.5938</td>
<td>3.5943</td>
<td>5.646e-4</td>
</tr>
<tr>
<td>0.8</td>
<td>2.8125</td>
<td>2.8129</td>
<td>4.419e-4</td>
<td>3.4375</td>
<td>3.4380</td>
<td>5.400e-4</td>
</tr>
<tr>
<td>0.9</td>
<td>2.9688</td>
<td>2.9692</td>
<td>4.664e-4</td>
<td>3.2813</td>
<td>3.2818</td>
<td>5.155e-4</td>
</tr>
<tr>
<td>1</td>
<td>3.1250</td>
<td>3.1255</td>
<td>4.909e-4</td>
<td>3.1250</td>
<td>3.1255</td>
<td>4.909e-4</td>
</tr>
</tbody>
</table>

Table 2: The result of the proposed method for Case 2 of Example 52 with $\alpha = 0.75$ and $m = 9$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\frac{y'}{5}$</th>
<th>$\frac{Y}{r}(r)$</th>
<th>Error</th>
<th>$\frac{y'}{5}$</th>
<th>$\frac{Y}{r}(r)$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1962</td>
<td>0.1966</td>
<td>3.149e-4</td>
<td>0.5887</td>
<td>0.5897</td>
<td>9.446e-4</td>
</tr>
<tr>
<td>0.1</td>
<td>0.2159</td>
<td>0.2162</td>
<td>3.464e-4</td>
<td>0.5691</td>
<td>0.5700</td>
<td>9.131e-4</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2355</td>
<td>0.2359</td>
<td>3.778e-4</td>
<td>0.5495</td>
<td>0.5504</td>
<td>8.816e-4</td>
</tr>
<tr>
<td>0.3</td>
<td>0.2551</td>
<td>0.2555</td>
<td>4.093e-4</td>
<td>0.5298</td>
<td>0.5307</td>
<td>8.501e-4</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2747</td>
<td>0.2752</td>
<td>4.408e-4</td>
<td>0.5102</td>
<td>0.5110</td>
<td>8.186e-4</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2944</td>
<td>0.2948</td>
<td>4.723e-4</td>
<td>0.4906</td>
<td>0.4914</td>
<td>7.872e-4</td>
</tr>
<tr>
<td>0.6</td>
<td>0.3140</td>
<td>0.3145</td>
<td>5.038e-4</td>
<td>0.4710</td>
<td>0.4717</td>
<td>7.557e-4</td>
</tr>
<tr>
<td>0.7</td>
<td>0.3336</td>
<td>0.3341</td>
<td>5.353e-4</td>
<td>0.4514</td>
<td>0.4521</td>
<td>7.242e-4</td>
</tr>
<tr>
<td>0.8</td>
<td>0.3532</td>
<td>0.3538</td>
<td>5.668e-4</td>
<td>0.4317</td>
<td>0.4324</td>
<td>6.927e-4</td>
</tr>
<tr>
<td>0.9</td>
<td>0.3729</td>
<td>0.3735</td>
<td>5.982e-4</td>
<td>0.4121</td>
<td>0.4128</td>
<td>6.612e-4</td>
</tr>
<tr>
<td>1</td>
<td>0.3925</td>
<td>0.3931</td>
<td>6.297e-4</td>
<td>0.3925</td>
<td>0.3931</td>
<td>6.297e-4</td>
</tr>
</tbody>
</table>

Figure 5: The fuzzy approximate solution of Example 52, Case 2. $\alpha = 0.75, m = 9$.

Here, suppose that $\lambda = -1$, then using the $\alpha$-differentiability and Theorem 33 we have the following:

\[
\left( {}^{\alpha}D_{0}^{\alpha} \frac{y}{r}(x,r) \right) = -\frac{y}{r}(x)+x+1, \quad 0 < \alpha \leq 1, \\
\frac{y}{r}(0;r) = \frac{y_{0}}{r}(r), \quad 0 < r \leq 1, \\
\left( {}^{\alpha}D_{0}^{\alpha} \overline{y}(x,r) \right) = -\overline{y}(x)+x+1, \quad 0 < \alpha \leq 1, \\
\overline{y}(0;r) = \overline{y_{0}}(r), \quad 0 < r \leq 1,
\]

(145)

where $\overline{y}(0;r) = [0.5 + 0.5r, 1.5 - 0.5r]$. Also, regarding (145), the exact solution of FFDE (144) is determined as

\[
\frac{y}{r}(x,r) = (0.5 + 0.5r) E_{\alpha,1} [x^{\alpha}] \\
+ \int_{0}^{x} (x-t)^{\alpha-1} E_{\alpha,\alpha} \left[-(x-t)^{\alpha}\right] (x+1) dt, \\
0 < \alpha \leq 1,
\]

(146)

Example 57. Let us consider the following FFDE:

\[
\left( {}^{\alpha}D_{0}^{\alpha} y \right) (x) = \lambda \odot y (x) \oplus (x+1), \quad 0 < \alpha \leq 1, \\
y(0;r) = \left[ \overline{y_{0}}(r), \underline{y_{0}}(r) \right], \quad 0 < r \leq 1.
\]

(144)
Applying the shifted Legendre method with LOM technique explained in Sections 4 and 5, we have
\[
\sum_{j=0}^{m} c_j^r \left[ d_j(\alpha) + I \right] L_j(x; r) = \sum_{j=0}^{m} f_j L_j(x; r) \quad i = 0, \ldots, m,
\]
\[
\sum_{j=0}^{m} \xi_j \left[ d_j(\alpha) + I \right] L_j(x; r) = \sum_{j=0}^{m} f_j L_j(x; r) \quad i = 0, \ldots, m,
\]
(147)

or it can be written in the matrix form as
\[
C_{m-}^T \left[ D(\alpha) + I \right] \Phi(x) = F_{m-}^T \Phi(x),
\]
\[
C_{m+}^T \left[ D(\alpha) + I \right] \Phi(x) = F_{m+}^T \Phi(x),
\]
(148)
in which \( c_j^r \) = \([c_j^r, c_j^{r+}]\) and \( (f_j^r) = \[f_j^r, f_j^{r+}\] \) for \( j = 0, 1, \ldots, m \). As was illustrated in Section 5, we make \( m \) fuzzy algebraic equations by the result of the inner product of fuzzy residual with \( L_j(x) \), \( i = 0, 1, \ldots, m - 1 \), so we gain
\[
C_{m-}^T \left[ D(\alpha) + I \right] = F_{m-}^T,
\]
\[
C_{m+}^T \left[ D(\alpha) + I \right] = F_{m+}^T.
\]
(149)

Also for the initial condition (144), we have
\[
\tilde{y}(0; r) = C_{m-}^T \Phi_m = (0.5 + 0.5r),
\]
\[
\tilde{\Phi}(0; r) = C_{m+}^T \Phi_m = (1.5 - 0.5r).
\]
(150)
The above equations produce a set of \( m + 1 \) fuzzy linear algebraic equations. As a result, namely, with \( m = 2 \), the approximate fuzzy solution should be written in the form
\[
\tilde{y}(x) = \sum_{j=0}^{2} \xi_j L_j(x) = \left[ \xi_0 \quad \xi_1 \quad \xi_2 \right] \Phi(x),
\]
(151)
Finally, the corresponding fuzzy approximate solution from (151) can be acquired.

For case \( \alpha = 0.95 \) and \( m = 6 \), \( x = 1 \) with \( r = 0.1 \), we obtain the fuzzy approximate solution in a series expansion as
\[
\tilde{y}(x, 0.1) = 1.4500 - 0.5496x + 1.3466x^2 - 0.4478x^3 + 2.4940x^4 - 1.9084x^5 + 0.5500 + 0.5533x - 0.1915x^2 + 1.5046x^3 - 1.525594 \times 10^{-6}x^4 + 1.544174 \times 10^{-6}x^5 - 1.544174 \times 10^{-6}x^6.
\]
(152)

Remark 58. If \( \alpha = 1 \), then the fuzzy fractional differential equation (144) with \( \lambda = -1 \) is converted to fuzzy integer-order differential equation. Therefore, the exact solution of problem under (2)-differentiability using Corollary 16 is as follows:
\[
\tilde{y}(x; r) = x + (0.5 + 0.5r)e^{-x},
\]
(153)
In order to evaluate the advantages and the accuracy of using the presented method for the fuzzy fractional differential equations, Example 57 is considered. The results are illustrated in Table 3 are the approximate solutions are compared with exact solutions and the absolute errors are derived for \( \alpha = 0.95 \) with \( m = 6 \) at \( x = 1 \). It is obvious that the fuzzy approximate solutions are in high agreement with the fuzzy exact solutions. Moreover, different numbers of Legendre polynomials are applied to obtain the absolute errors for this problem which can be seen in Figure 6. This graph shows that the method has a good convergence rate. Also, the fuzzy approximate solution is shown for \( \alpha = 0.95 \) in Figure 7. The absolute error of different order of fractional differentiability can be considered for \( x = 1 \) in Figure 8 and the approximate solutions are shown in Figure 9 that shows that this approach can apply to solve the problem effectively with different fuzzy fractional order of derivatives with suitable errors. It can be seen that the \( \alpha \) approaches an integer order, and the error has a tendency to decrease, as expected.
Remark 59. Figure 6 illustrates the absolute errors for \( \bar{y}(x; r) \) of Example 57. In the same way, if we consider for \( \hat{y}(x; r) \), then analogously to the demonstration of Figure 6, we can obtain the absolute errors.

Example 60. For our third example, consider the inhomogeneous linear equation in [78] with fuzzy initial values, and so we have:

\[
\left( C D_0^{\alpha} y \right)(x; r) + y(x) = \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + x^2 - x,
\]

\[
y(0; r) = [-1 + r, 1 - r], \quad 0 < \alpha \leq 1, \quad 0 \leq x \leq 1,
\]

in which \( y(x) : L^E[0, 1] \cap C^E[0, 1] \) is a continuous fuzzy-valued function and \( C D_0^{\alpha} \) denotes the fuzzy Caputo fractional derivative of order \( \alpha \).

Now, using \([1 - \alpha]\)-differentiability and Theorem 33, we have the following:

\[
\left( C D_0^{\alpha} \bar{y} \right)(x; r) + \bar{y}(x; r) = \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + x^2 - x,
\]

\[
\bar{y}(0; r) = -1 + r, \quad 0 < \alpha \leq 1, \quad 0 \leq x \leq 1,
\]

\[
\left( C D_0^{\alpha} \hat{y} \right)(x; r) + \hat{y}(x; r) = \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + x^2 - x,
\]

\[
\hat{y}(0; r) = 1 - r, \quad 0 < \alpha \leq 1, \quad 0 \leq x \leq 1.
\]
Solving (155) leads to determining the solution of FFDE (154) as follows:

\[ y(x;r) = (-1 + r) E_{\alpha,1}[-x^\alpha] + \int_0^x (x-t)^{\alpha-1} E_{\alpha,\alpha}[-(x-t)^\alpha] \times \left( \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{x^{1-\alpha}}{\Gamma(3-\alpha)} + x^2 - x \right) dt, \]

\[ 0 \leq r \leq 1, \]

\[ \bar{y}(x;r) = (1 - r) E_{\alpha,1} [-x^\alpha] + \int_0^x (x-t)^{\alpha-1} E_{\alpha,\alpha} [-\bar{y}(x-t)^\alpha] \times \left( \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{x^{1-\alpha}}{\Gamma(3-\alpha)} + x^2 - x \right) dt, \]

\[ 0 \leq r \leq 1. \]

By exploiting the method proposed in Section 5, the equations are acquired by

\[ \sum_{j=0}^{m} c_j^T [d^{(a)} + I] L_j (x;r) = \sum_{j=0}^{m} f_j L_j (x;r) \quad i = 0, \ldots, m, \]

\[ \sum_{j=0}^{m} c_j^T [d^{(a)} + I] L_j (x;r) = \sum_{j=0}^{m} f_j L_j (x;r) \quad i = 0, \ldots, m, \]

or we can rewrite it in the matrix form as

\[ C_{m,-}^T [D^{(a)} + I] \Phi(x) = F_{m,-}^T \Phi(x), \]

\[ C_{m,+}^T [D^{(a)} + I] \Phi(x) = F_{m,+}^T \Phi(x), \]

in which \((c_j^T) = [c_{j,-}^T, c_{j,+}^T]\) and \((f_j^T) = [f_{j,-}^T, f_{j,+}^T]\) for \(j = 0, 1, \ldots, m\). As it was described in Section 5, we create \(m\) fuzzy algebraic equations multiplied in above system by \(L_j(x)\) for \(i = 0, 1, \ldots, m-1\) and implemented in the inner product using orthogonal property, and so we gain:

\[ C_{m,-}^T [D^{(a)} + I] = F_{m,-}^T, \]

\[ C_{m,+}^T [D^{(a)} + I] = F_{m,+}^T. \]

Also for the initial condition (154), we have

\[ y(0;r) = C_{m,-}^T \Phi_m = -1 + r, \]

\[ \bar{y}(0;r) = C_{m,+}^T \Phi_m = 1 - r. \]

Equations (159) and (160) produce a set of \(m+1\) fuzzy linear algebraic equations. As a result, for \(\alpha = 0.75r = 0.1\) and \(m = 3\) the unknown coefficients \(c_j\) can be achieved as \(C_{3,-}^T = [-0.6914, 0.3147, 0.1144, 0.0083]\) and \(C_{3,+}^T = [0.3725, -0.1418, 0.2954, -0.0903]\). Also with these assumptions, one has

\[ D^{(0.75)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1.7652 & 0.5884 & -0.2263 & 0.1305 \\ -0.5884 & 3.6662 & 1.2114 & -0.4979 \\ 1.5389 & -0.6230 & 5.1081 & 1.8265 \end{bmatrix}. \]

The approximate and exact solution of FFDEs (154) is illustrated in Table 4 at \(x = 1\) with the absolute errors. In this table, three shifted Legendre polynomials are considered to derive the approximate solution. It is clear that with the lower number of shifted Legendre functions, using proposed method, one can reach a reassemblable approximate solution. Also Figure 10 depicts the absolute error with \(m = 5\) and various values of \(\alpha\). This figure exhibits the applicability and validity of the proposed technique for this example. Furthermore, the fuzzy approximate solution for \(\alpha = 0.75\) with \(m = 5\) is shown in Figure 11 at \(x = 1\). Additionally, in Figure 12, the fuzzy approximate solution is compared with different values of \(\alpha\). It can be seen that as \(\alpha\) tends to 1, the solution of the fuzzy fractional differential equations tends to that of the fuzzy integer-order differential equations. Finally, Figure 13 displays the absolute error for \(\alpha = 0.75\) with different values of \(m\) which is obvious that with increasing the number of shifted Legendre functions, the absolute error of the problem has been decreased gently.

**Remark 61.** Figures 10 and 13 illustrate the absolute errors for \(\bar{y}(x;r)\) of Example 60. In the same way, if we consider \(\bar{y}(x;r)\), then analogously to the demonstration of Figures 10 and 13, we can obtain the absolute errors.

Now, we consider two applicable examples which are new under uncertainty represented by fuzzy-valued functions. Firstly, a brief history is given in the deterministic case of Example 62. For more details, see [1, 79].

There are two kinds of electrical circuits which are related to the fractional calculus. Circuits of the first type are supposed to consist of capacitors and resistors, which are described by conventional (integer-order) models. A circuit expressing fractional order behavior is called *fractance*. Circuits of the second type may consist of resistors, capacitors, and fractances. The term fractance was suggested initially by Mehaute and Crepy [80]. Now, in order to scrutinize the
Table 4: The result of the proposed method for Example 60 with $\alpha = 0.95$ and $m = 3$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$y_3(r)$</th>
<th>$\overline{Y}(r)$</th>
<th>Error</th>
<th>$\overline{y}$</th>
<th>$\overline{Y}(r)$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-0.371581</td>
<td>-0.371573</td>
<td>7.857218e-5</td>
<td>0.371581</td>
<td>0.371573</td>
<td>7.857218e-5</td>
</tr>
<tr>
<td>0.1</td>
<td>-0.334423</td>
<td>-0.334416</td>
<td>7.071496e-5</td>
<td>0.334423</td>
<td>0.334416</td>
<td>7.071496e-5</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.297265</td>
<td>-0.297258</td>
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<td>0.297265</td>
<td>0.297258</td>
<td>6.285775e-5</td>
</tr>
<tr>
<td>0.3</td>
<td>-0.260107</td>
<td>-0.260101</td>
<td>5.500053e-5</td>
<td>0.260107</td>
<td>0.260101</td>
<td>5.500053e-5</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.222948</td>
<td>-0.222944</td>
<td>4.714331e-5</td>
<td>0.222948</td>
<td>0.222944</td>
<td>4.714331e-5</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.185790</td>
<td>-0.185786</td>
<td>3.928609e-5</td>
<td>0.185790</td>
<td>0.185786</td>
<td>3.928609e-5</td>
</tr>
<tr>
<td>0.6</td>
<td>-0.148632</td>
<td>-0.148629</td>
<td>3.142887e-5</td>
<td>0.148632</td>
<td>0.148629</td>
<td>3.142887e-5</td>
</tr>
<tr>
<td>0.7</td>
<td>-0.111474</td>
<td>-0.111472</td>
<td>2.357165e-5</td>
<td>0.111474</td>
<td>0.111472</td>
<td>2.357165e-5</td>
</tr>
<tr>
<td>0.8</td>
<td>-0.074316</td>
<td>-0.074314</td>
<td>1.571443e-5</td>
<td>0.074316</td>
<td>0.074314</td>
<td>1.571443e-5</td>
</tr>
<tr>
<td>0.9</td>
<td>-0.037158</td>
<td>-0.037157</td>
<td>9.520160e-14</td>
<td>0.037158</td>
<td>0.037157</td>
<td>9.520160e-14</td>
</tr>
<tr>
<td>1</td>
<td>0.000000</td>
<td>0.000000</td>
<td>9.520160e-14</td>
<td>0.000000</td>
<td>0.000000</td>
<td>9.520160e-14</td>
</tr>
</tbody>
</table>

Figure 11: The fuzzy approximate solution of Example 60, $\alpha = 0.75, m = 5$.

Figure 12: The fuzzy approximate solution for different $\alpha$:(1: star, 0.95: square, 0.85: $\circ$, 0.75: $\times$, 0.65: pentagram) of Example 60, $m = 5$.

Figure 13: The absolute error for different $m$ of Example 60, $\alpha = 0.75$.

mentioned problem in a real case, we use the fuzzy initial value $y(0; r)$ and the concept of Caputo’s H-differentiability for fractional derivative of $y(x)$. So we have the following fuzzy fractional oscillation differential equation.

Example 62. Consider an electrical circuit (LR circuit) with an AC source. The current equation of this circuit can be written as follows:

$$\left( \frac{\alpha}{L} D_0^\alpha y \right)(x) = -y(x) + v(x), \quad 0 \leq x \leq 1,$$

$$y(0; r) = [0.96 + 0.04r, 1.01 + 0.01r],$$

in which $R$ is the circuit resistance, and $L$ is a coefficient, corresponding to the solenoid and $0 \leq r \leq 1$. Suppose that $v(x) = \sin(x)$, $R = 1$ ohm and $L = 1$H, so (162) can be rewritten as

$$\left( \frac{\alpha}{L} D_0^\alpha y \right)(x) = -y(x) + \sin(x), \quad 0 \leq x \leq 1,$$

$$y(0; r) = [0.96 + 0.04r, 1.01 + 0.01r].$$
Table 5: The result of the proposed method for Example 62 with $\alpha = 0.85$ and $m = 8$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$y'_b$</th>
<th>$\bar{Y}(r)$</th>
<th>Error</th>
<th>$\bar{y}'_b$</th>
<th>$\bar{Y}(r)$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.7208</td>
<td>0.7206</td>
<td>1.437e - 4</td>
<td>0.7398</td>
<td>0.7397</td>
<td>1.511e - 4</td>
</tr>
<tr>
<td>0.1</td>
<td>0.7223</td>
<td>0.7221</td>
<td>1.443e - 4</td>
<td>0.7394</td>
<td>0.7393</td>
<td>1.509e - 4</td>
</tr>
<tr>
<td>0.2</td>
<td>0.7238</td>
<td>0.7237</td>
<td>1.4492e - 4</td>
<td>0.7391</td>
<td>0.7389</td>
<td>1.5083e - 4</td>
</tr>
<tr>
<td>0.3</td>
<td>0.7253</td>
<td>0.7252</td>
<td>1.4551e - 4</td>
<td>0.7387</td>
<td>0.7385</td>
<td>1.5068e - 4</td>
</tr>
<tr>
<td>0.4</td>
<td>0.7269</td>
<td>0.7267</td>
<td>1.4610e - 4</td>
<td>0.7383</td>
<td>0.7381</td>
<td>1.5053e - 4</td>
</tr>
<tr>
<td>0.5</td>
<td>0.7284</td>
<td>0.7282</td>
<td>1.4669e - 4</td>
<td>0.7379</td>
<td>0.7378</td>
<td>1.5039e - 4</td>
</tr>
<tr>
<td>0.6</td>
<td>0.7299</td>
<td>0.7298</td>
<td>1.4729e - 4</td>
<td>0.7375</td>
<td>0.7374</td>
<td>1.5024e - 4</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7314</td>
<td>0.7313</td>
<td>1.4788e - 4</td>
<td>0.7372</td>
<td>0.7370</td>
<td>1.5009e - 4</td>
</tr>
<tr>
<td>0.8</td>
<td>0.7330</td>
<td>0.7328</td>
<td>1.4847e - 4</td>
<td>0.7368</td>
<td>0.7366</td>
<td>1.4994e - 4</td>
</tr>
<tr>
<td>0.9</td>
<td>0.7345</td>
<td>0.7343</td>
<td>1.4906e - 4</td>
<td>0.7364</td>
<td>0.7362</td>
<td>1.4980e - 4</td>
</tr>
<tr>
<td>1</td>
<td>0.7360</td>
<td>0.7359</td>
<td>1.4965e - 4</td>
<td>0.7360</td>
<td>0.7359</td>
<td>1.4965e - 4</td>
</tr>
</tbody>
</table>

Now, using $[2 - \alpha]$-differentiability and Theorem 33, we have the following:

\[
\left( D^\alpha_0 y \right)(x; r) + y(x; r) = \sin(x),
\]
\[
y(0; r) = (0.96 + 0.04 r),
\]
\[
0 < \alpha \leq 1, 0 \leq x \leq 1.
\]

The exact solution of (163) under (2)-differentiability for integer order is given by

\[
\sum_1 (t; r) = \frac{1}{2} (\sin(t) - \cos(t)) + e^{-t} (1.46 + 0.04 r),
\]
\[
\sum_1 (t; r) = \frac{1}{2} (\sin(t) - \cos(t)) + e^{-t} (1.51 - 0.01 r).
\]

By applying the technique described in Section 5, we approximate fuzzy solution as

\[
D^\alpha y(x) = C^T \otimes D(\alpha) \Phi_m(x),
\]
\[
y(x) = \bar{y}_m(x) = C^T \otimes \Phi_m,
\]
\[
\sin(x) = \bar{f}_m(x) = F_m^T \otimes \Phi_m,
\]

where vector $F_m^T$ is obtained as (102). With $m = 3$, $\alpha = 0.75$, and $r = 0.1$, we have

\[
D^{(0.75)} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
1.7652 & 0.5884 & -0.2263 & 0.1305 \\
-0.5884 & 3.6662 & 1.2114 & -0.4979 \\
1.5389 & -0.6230 & 5.1081 & 1.8265
\end{bmatrix},
\]
\[
F_4^T = \begin{bmatrix}
0.4597 & 0.4279 & -0.0392 & -0.0072
\end{bmatrix},
\]

and subsequently, we generate $m + 1$ fuzzy linear equations using (110) and also from substituting (100) in the initial conditions of (164), we can obtain the unknown fuzzy coefficients $c_j$ as

\[
c_0 = 0.7222, \quad c_1 = -0.0497, \quad c_2 = 0.1272, \quad c_3 = -0.0650,
\]
\[
c_0 = 0.7488, \quad c_1 = -0.0611, \quad c_2 = 0.1317, \quad c_3 = -0.0674,
\]

and finally, the fuzzy approximate solution is given by

\[
\bar{y}_4(x; 0.1) = 0.9640 - 1.6420 x + 2.7118 x^2 - 1.2990 x^3,
\]
\[
\bar{y}_4(x; 0.1) = 1.0090 - 1.7215 x + 2.8128 x^2 - 1.3483 x^3.
\]

In Table 5, the numerical results of (162) are depicted at $x = 1$ for $\alpha = 0.85$ and $m = 8$. The absolute error confirms that the proposed method approximates the fuzzy solution with a suitable accuracy. Figure 14 shows the absolute error of the proposed method with a different value of $\alpha$ with $m = 10$ which is clear as $\alpha$ tends to 1, and the error is decreasing dramatically, also with the increasing of the number of Legendre functions, the absolute error is decreasing that is, shown in Figure 16. Finally the fuzzy approximate solution of (162) is demonstrated in Figure 15 in the interval $0 \leq x \leq 1$.

Now we consider the second application of FFDES in the real world. But before we consider the problem of fuzzy model, We glimpse the vision of a non-fuzzy case.

A pharmacodynamic model is usually separated in two parts: a link model and a transduction model. The link model depicts the distribution of drug from an observed compartment (e.g., plasma) into a biophase (the effect compartment). Drug in the biophase induces a pharmacodynamic response (the effect), which is represented by a transduction model. For more details, see [81, 82].

The model describing the transfer of drug to the effect compartment, from the mechanistic interpretation, it can be considered as the pharmacokinetics of the drug. However, practically, and because drug condensation in the effect
In pharmacodynamics, modeling of the compartment system is not uncommon. It is often not clear if the model is correctly linked to the site of action, or if it exhibits some pharmacodynamic delay or a combination of the two. It is begun by representing drug concentration in the effect compartment by the (Caputo) fractional differential equation as follows:

$$\left(D^{\alpha}_{0+} C_e\right)(t) = k_{\text{e0}} (-C_e(t) + f(t)),$$  \hspace{1cm} (170)

in which \(f(t)\) describes the drug input into the central compartment and \(k_{\text{e0}}\) is the elimination rate constant from the effect compartment. Also, the action of drug is described by a nonlinear memory-less function of \(C_e\).

Now, we consider the action of drug delivery by a fuzzy function under the fuzzy Caputo’s H-differentiability which is modelled by fuzzy fractional differential equation.

**Example 63.** Consider the following fuzzy pharmacodynamic model

$$\left(D^{\alpha}_{0+} C_e\right)(t) = k_{\text{e0}} (-C_e(t) + f(t)), \quad 0 < \alpha < 1, \quad 0 \leq t \leq 1,$$

$$C_e(0; r) = (0.5 + 0.5r, 1.5 - 0.5r), \quad (171)$$

in which \(f(t)\) describes the drug input into the central compartment and \(k_{\text{e0}}\) is the elimination rate constant from the effect compartment. Also, the action of drug is described by a nonlinear memory-less function of \(C_e\).

The exact solution of (171) with \(\alpha = 1\) under generalized differentiability is given by

$$Y(t; r) = e^{-t}\left(-\frac{20}{9} e^{t/2} - \frac{5}{18} e^{-t} + 3 + 0.5r\right),$$

$$\bar{Y}(t; r) = e^{-t}\left(-\frac{20}{9} e^{t/2} - \frac{5}{18} e^{-t} + 4 - 0.5r\right).$$  \hspace{1cm} (172)

We apply the method presented in Section 5 and solve this problem for \(m = 3\), \(\alpha = 0.95\), and \(r = 0.2\). Hence, we have

$$D^{(0.95)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1.9566 & 0.1432 & -0.0743 & 0.0501 \\ -0.1432 & 5.4849 & 1.3447 & -0.1823 \\ 1.8823 & -0.2015 & 8.8258 & 0.5752 \end{bmatrix}, \quad (173)$$

and thereafter the fuzzy unknown coefficients for the fuzzy approximation of (171) can be acquired easily by replacing the above results in (110) and solving this algebraic fuzzy linear equations system. So we have

$$\xi_0 = 0.1744, \quad \xi_1 = -0.4368,$$

$$\xi_2 = 0.0573, \quad \xi_3 = 0.0185,$$

$$\bar{\xi}_0 = 0.6417, \quad \bar{\xi}_1 = -0.6635,$$

$$\bar{\xi}_2 = 0.1043, \quad \bar{\xi}_3 = 0.0095,$$  \hspace{1cm} (174)
and ultimately the fuzzy approximate solution is given by

\[
y_4(x, 0.1) = 0.6 - 0.9392x - 2.2482x^2 + 0.3823x^3,
\]

or we can rewrite it in the matrix form as

\[
\begin{align*}
\sum_{j=0}^{m} c_{j}^{r}d_{i,j}L_j(x; r) - 3 (3 - r) \sum_{j=0}^{m} (c_{j}^{r}L_j(x; r))^2 &= 0 \\
\sum_{j=0}^{m} c_{j}^{r}d_{i,j}L_j(x; r) - 3 (1 + r) \sum_{j=0}^{m} (c_{j}^{r}L_j(x; r))^2 &= 0
\end{align*}
\]

for \(i = 0, \ldots, m\), \(j = 0, \ldots, m\), \(r = 0.1\), and \(r = 0.95\).

\[
\begin{array}{cccccccc}
\hline
r & y_1^2(r) & Y(r) & \text{Error} & y_1^2(r) & Y(r) & \text{Error} \\
\hline
0 & -0.243121 & -0.243123 & 1.232190e-6 & 0.128454 & 0.128450 & 3.891957e-6 \\
0.1 & -0.224543 & -0.224544 & 1.365176e-6 & 0.109875 & 0.109871 & 3.758969e-6 \\
0.2 & -0.205964 & -0.205965 & 1.498166e-6 & 0.091296 & 0.091293 & 3.625981e-6 \\
0.3 & -0.187385 & -0.187387 & 1.631153e-6 & 0.072717 & 0.072714 & 3.492989e-6 \\
0.4 & -0.168806 & -0.168808 & 1.764141e-6 & 0.054139 & 0.054135 & 3.360011e-6 \\
0.5 & -0.150227 & -0.150229 & 1.897131e-6 & 0.035560 & 0.035557 & 3.227013e-6 \\
0.6 & -0.131648 & -0.131651 & 2.030121e-6 & 0.016981 & 0.016978 & 3.094025e-6 \\
0.7 & -0.113070 & -0.113072 & 2.163107e-6 & 0.001597 & 0.001600 & 2.961033e-6 \\
0.8 & -0.094491 & -0.094493 & 2.296095e-6 & -0.020176 & -0.020178 & 2.828050e-6 \\
0.9 & -0.075912 & -0.075914 & 2.429084e-6 & -0.038754 & -0.038757 & 2.695061e-6 \\
1 & -0.057333 & -0.057336 & 2.562075e-6 & -0.057333 & -0.057336 & 2.562075e-6 \\
\hline
\end{array}
\]
equations by collecting (162) in the first $m$ root of $L_{m+1}(x)$. Also using the initial condition (162), we have
\[
\begin{align*}
\bar{y}(0; r) &= C^T_{m, n} \Phi_m(0) = (0.5 \sqrt{r}) , \\
\bar{y}(0; r) &= C^T_{m, n} \Phi_m(0) = (0.2 \sqrt{1 - r} + 0.5) .
\end{align*}
\] (179)

These roots can be put in the (178) and deriving three nonlinear fuzzy equation, then these equations be coupled with (179). Finally with solving the fuzzy nonlinear equations system, the unknown coefficients $\bar{c}_i$ are obtained.

Now, as an explanation, assume that $m = 3$, then the first three roots of $L_4(x)$ are as follows:
\[
x_0 = 0.0694, \quad x_1 = 0.9306, \quad x_2 = 0.3300 , \quad (180)
\]
which for $\alpha = 0.95$ and $r = 0.1$ are
\[
\begin{align*}
\bar{c}_0 &= 0.2593 , \quad \bar{c}_1 = 0.1106 , \\
\bar{c}_2 &= -0.0032 , \quad \bar{c}_3 = 0.0054 , \\
\bar{c}_0 &= 0.1171 , \quad \bar{c}_1 = -0.0203 , \\
\bar{c}_2 &= 0.0045 , \quad \bar{c}_3 = 0.0155 .
\end{align*}
\] (181)

Finally, using the above results and (100), the approximate solution is computed in each particular point easily.

The approximate solution for Example 64 with different fractional Caputo order is derived in Table 7. The solution has been estimated using 3 shifted Legendre functions. The results in this table are at $x = 1$. It is clear that the method is applicable and valid for different fuzzy fractional order $\alpha$. Furthermore, by changing the number of shifted Legendre functions, the approximate solution for $\alpha = 0.75$ at $x = 1$ is revealed in Figure 20 which shows that with only a
few number of shifted Legendre polynomials, desired results are available. Ultimately, the fuzzy approximate solution is described in Figure 21 with $\alpha = 0.75$ and $m = 3$.

Remark 65. It is consequential to note that, for a crisp differential equation of integer or fractional order, the problem

\[
\left( D^\alpha_0 y \right) (x) + F(x, y(x)) = 0, \quad x \in [0, X],
\]

(182)
is the same as

\[
\left( D^\alpha_0 y \right) (x) = -F(x, y(x)), \quad x \in [0, X],
\]

(183)
in which if $F$ is nonlinear operator, and in the case of space $\mathbb{E}$, both problems are not equivalent [83].

7. Conclusion

In this paper, a cluster of orthogonal functions named shifted Legendre function is used to solve fuzzy fractional differential equations under Caputo type. The benefit of the shifted Legendre operational matrices method over other existing orthogonal polynomials is its simplicity of execution besides some other advantages. A general formulation of the proposed method is provided in details. Also a complete error analysis is considered in Section 5. In this paper, we tried to answer some questions:

(i) how to derive a shifted Legendre operational matrix method over the fuzzy fractional integration and differentiation

(ii) how to solve the fuzzy fractional order differential equations via the shifted Legendre operational matrices of the fuzzy fractional derivative.

Moreover, it should be mentioned that another advantage of this technique is that it decreases these problems to the degree of solving a system of fuzzy algebraic equations thus extremely making the problems easier. Several examples was carried out to depict the effectiveness and the absence of complexity of the proposed method. The achieved solutions have a satisfactory results obtained with only a small number of Legendre polynomials.

For future research, we will consider this method for solving FFDEs with order $1 < \alpha < 2$. Also we will apply it under Riemann-Liouville’s H-differentiability. Apart from this, the other orthogonal functions like Jacobi polynomials will be extended for solving FFDEs.

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References


