Research Article

Periodic Solutions of Duffing Equation with an Asymmetric Nonlinearity and a Deviating Argument

Zaihong Wang, Jin Li, and Tiantian Ma

School of Mathematical Sciences, Capital Normal University, Beijing 100048, China

Correspondence should be addressed to Zaihong Wang; zhwang@cnu.edu.cn

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We study the existence of periodic solutions of the second-order differential equation $x'' + ax^+ - bx^- + g(x(t - \tau)) = p(t)$, where a, b are two constants satisfying $1/\sqrt{a} + 1/\sqrt{b} = 2/n$, $n \in \mathbb{N}$, τ is a constant satisfying $0 \leq \tau < 2\pi$, $g, p : \mathbb{R} \rightarrow \mathbb{R}$ are continuous, and p is 2π -periodic. When the limits $\lim_{x \rightarrow \pm\infty} g(x) = g(\pm\infty)$ exist and are finite, we give some sufficient conditions for the existence of 2π -periodic solutions of the given equation.

1. Introduction

In this paper, we are concerned with the existence of periodic solutions of the second-order differential equation with an asymmetric nonlinearity and a deviating argument:

$$x'' + ax^+ - bx^- + g(x(t - \tau)) = p(t), \quad (1)$$

where a, b are two constants satisfying $1/\sqrt{a} + 1/\sqrt{b} = 2/n$, $n \in \mathbb{N}$, τ is a constant satisfying $0 \leq \tau < 2\pi$, $g, p : \mathbb{R} \rightarrow \mathbb{R}$ are continuous, and p is 2π -periodic.

In recent years, the periodic problem of the second-order differential equation with a deviating argument has been widely studied because of its background in applied sciences (see [1–6] and the references cited therein).

In case when $\tau = 0$ and $a = b = n^2$, (1) becomes

$$x'' + n^2 x + g(x) = p(t). \quad (2)$$

Assume that the limits

$$\lim_{x \rightarrow \pm\infty} g(x) = g(\pm\infty) \quad (g)$$

exist and are finite. Lazer and Leach [7] proved that (2) has one 2π -periodic solution provided that the function

$$\Psi(\theta) = 2[g(+\infty) - g(-\infty)] - \int_0^{2\pi} p(t) \sin n(t + \theta) dt \quad (3)$$

is of constant sign.

In case when $\tau = 0$ and a, b satisfy the equation $1/\sqrt{a} + 1/\sqrt{b} = 2/n$, $n \in \mathbb{N}$, (1) becomes

$$x'' + ax^+ - bx^- + g(x) = p(t). \quad (4)$$

Equation (4) was first introduced by Fučík [8]. Lately, the periodic problem of (4) was widely studied in the literature (see [9–13] and the references cited therein). To deal with the existence of periodic solutions of (4), Dancer [9] introduced a $2\pi/n$ -periodic function

$$\Phi(\theta) = 2n \left[\frac{g(+\infty)}{a} - \frac{g(-\infty)}{b} \right] - \int_0^{2\pi} p(t) c(t + \theta) dt, \quad (5)$$

where $c(t)$ is a $2\pi/n$ -periodic function defined by

$$c(t) = \begin{cases} \frac{1}{\sqrt{a}} \sin(\sqrt{a}t), & 0 \leq t \leq \frac{\pi}{\sqrt{a}}, \\ -\sqrt{\frac{1}{b}} \sin \left[\sqrt{b} \left(t - \frac{\pi}{\sqrt{a}} \right) \right], & \frac{\pi}{\sqrt{a}} \leq t \leq \frac{2\pi}{n}. \end{cases} \quad (6)$$

Obviously, $c(t)$ is a periodic solution of the equation $x'' + ax^+ - bx^- = 0$ satisfying the initial value $x(0) = 0, x'(0) = 1$. It was proved in [9] that (4) has at least one 2π -periodic solution provided that Φ has a constant sign in $[0, 2\pi/n]$.

In the present paper, we will deal with the periodic solutions of (1) under condition (g). Owing to the appearance

of the asymmetric nonlinearity $ax^+ - bx^-$, the methods in [4, 5] are no longer valid. To overcome this difficulty, we embed (1) into an operator equation with the form $Lx = N(x, \lambda)$ instead of $Lx = \lambda Nx$ as in [4, 5]. We first prove a continuation lemma and then apply this continuation lemma to prove the existence of periodic solution of (1).

Let us denote

$$\nu = \tau \left(\bmod \frac{2\pi}{n} \right). \quad (7)$$

Obviously, we have

$$0 \leq \nu < \frac{2\pi}{n}. \quad (8)$$

We obtain the following result.

Theorem 1. Assume that condition (g) holds and $0 \leq \nu \leq \min\{\pi/\sqrt{a}, \pi/\sqrt{b}\}$. Then (1) has at least one 2π -periodic solution provided that either

$$\begin{aligned} & ng(-\infty) \left(\frac{1 - \cos \sqrt{a}\nu}{a} - \frac{1 + \cos \sqrt{b}\nu}{b} \right) \\ & + ng(+\infty) \left(\frac{1 + \cos \sqrt{a}\nu}{a} - \frac{1 - \cos \sqrt{b}\nu}{b} \right) \\ & \neq \int_0^{2\pi} p(t) c(t + \theta) dt, \quad \forall \theta \in [0, 2\pi] \end{aligned} \quad (9)$$

or

$$\begin{aligned} & n(g(-\infty) - g(+\infty)) \left(\frac{\sin \sqrt{a}\nu}{\sqrt{a}} + \frac{\sin \sqrt{b}\nu}{\sqrt{b}} \right) \\ & \neq \int_0^{2\pi} p(t) s(t + \theta) dt, \quad \forall \theta \in [0, 2\pi] \end{aligned} \quad (10)$$

holds, where the function s is defined by $s(t) = c'(t)$, $t \in \mathbb{R}$.

Remark 2. In the case when $\min\{\pi/\sqrt{a}, \pi/\sqrt{b}\} \leq \nu < 2\pi/n$, we can obtain the similar sufficient conditions. For brevity, we omit the detailed description.

Remark 3. Obviously, if $\nu = 0$ or $\tau = 2k\pi/n$, $k = 0, 1, 2, \dots, n-1$, then the first inequality of Theorem 1 reduces to the condition as in [8]; namely,

$$\begin{aligned} & 2n \left[\frac{g(+\infty)}{a} - \frac{g(-\infty)}{b} \right] \neq \int_0^{2\pi} p(t) c(t + \theta) dt, \\ & \forall \theta \in [0, 2\pi]. \end{aligned} \quad (11)$$

Throughout this paper, we always use $\mathbb{R}$ to denote the real number set. For a multivariate function ζ depending on r , the notation $\zeta = o(1)$ always means that, for $r \rightarrow \infty$, $\zeta \rightarrow 0$ holds uniformly with respect to other variables, whereas $\zeta = O(1)$ (or $\zeta = O(r^{-1})$) always means that ζ (or $r \cdot \zeta$) is bounded for r large enough. For any continuous 2π -periodic function $\phi(t)$, we always set $\|\phi\|_\infty = \max_{0 \leq t \leq 2\pi} |\phi(t)|$.

2. Preliminary Lemmas

We now embed (1) into a family of equations with one parameter $\lambda \in [0, 1]$,

$$x'' + ax^+ - bx^- + (1 - \lambda)\psi(x') + \lambda g(x(t - \tau)) = \lambda p(t), \quad (12)$$

where $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies the sign condition as follows:

$$\psi(x)x > 0, \quad \forall x \in \mathbb{R}, x \neq 0. \quad (13)$$

Lemma 4. Suppose that there exist two positive constants M_1 and M_2 such that, for any 2π -periodic solution $x(t)$ of (12), the following conditions hold:

$$\|x\|_\infty < M_1, \quad \|x'\|_\infty < M_2. \quad (14)$$

Then (1) has at least one 2π -periodic solution.

Proof. We follow an argument in [14] to prove Lemma 4. At first, we introduce some notations. Let X and Y be two Banach spaces defined by

$$\begin{aligned} X &= \{x \in C^1(\mathbb{R}, \mathbb{R}) : x(t + 2\pi) = x(t), \forall t \in \mathbb{R}\}, \\ Y &= \{y \in C(\mathbb{R}, \mathbb{R}) : y(t + 2\pi) = y(t), \forall t \in \mathbb{R}\}, \end{aligned} \quad (15)$$

with the norms

$$\|x\|_X = \max\{\|x\|_\infty, \|x'\|_\infty\}, \quad \|y\|_Y = \|y\|_\infty. \quad (16)$$

Define a linear operator by

$$L : D(L) \subset X \rightarrow Y, \quad Lx = x'', \quad (17)$$

where $D(L) = \{x \in X : x'' \in C(\mathbb{R}, \mathbb{R})\}$, and a nonlinear operator $N : X \times [0, 1] \rightarrow Y$,

$$\begin{aligned} N(x, \lambda)(t) &= -(ax^+ - bx^-) - (1 - \lambda)\psi(x') \\ &\quad - \lambda g(x(t - \tau)) + \lambda p(t). \end{aligned} \quad (18)$$

It is easy to see that

$$\text{Ker } L = \mathbb{R}, \quad \text{Im } L = \left\{ y \in Y : \int_0^T y(t) dt = 0 \right\}. \quad (19)$$

It follows that L is a Fredholm mapping of index zero.

Let us define two continuous projectors $P : X \rightarrow \text{Ker } L$ and $Q : Y \rightarrow Y$ by setting

$$Px = x(0), \quad Qy = \frac{1}{T} \int_0^T y(t) dt. \quad (20)$$

Set $L_P = L|_{D(L) \cap \text{Ker } P} \rightarrow \text{Im } L$. Then L_P is an algebraic isomorphism, and we define $K_P : \text{Im } L \rightarrow D(L)$ by

$$K_P = L_P^{-1}. \quad (21)$$

Clearly, we have that, for any $y \in \text{Im } L$,

$$(K_P y)(t) = -\frac{t}{T} \int_0^T (t-s) y(s) ds + \int_0^t (t-s) y(s) ds. \quad (22)$$

For any open bounded set $\Omega \subset X$, we can prove by standard arguments that $K_P(I - Q)N$ and QN are relatively compact on the closure $\overline{\Omega}$. Therefore, N is L -compact on $\overline{\Omega}$.

It is noted that (12), together with the 2π -periodic boundary condition, is equivalent to the operator equation

$$Lx = N(x, \lambda). \quad (23)$$

Let $\Omega \subset X$ be the open bounded set defined by

$$\Omega = \{x \in X : \|x\|_\infty < M_1, \|x'\|_\infty < M_2\}. \quad (24)$$

From (14), we have

$$Lx \neq N(x, \lambda), \quad \text{for } x \in \partial\Omega \cap D(L), \lambda \in [0, 1]. \quad (25)$$

Since L is a Fredholm operator with index zero and N is L -compact on $\overline{\Omega} \times [0, 1]$, we get from the homotopic invariance of the coincidence degree that

$$D_L(L - N(\cdot, 1), \Omega) = D_L(L - N(\cdot, 0), \Omega). \quad (26)$$

Next, we will compute $D_L(L - N(\cdot, 0), \Omega)$. To this end, we introduce an auxiliary operator $S : \overline{\Omega} \times [0, 1] \rightarrow Y$ defined by

$$S(x, \mu) = -(ax^+ - bx^-) - \psi(x') - \mu x'. \quad (27)$$

Clearly, S is L -compact on $\overline{\Omega} \times [0, 1]$ and

$$S(x, 0) = N(x, 0), \quad \text{for } x \in \overline{\Omega}. \quad (28)$$

Now, we will prove that

$$Lx \neq S(x, \mu), \quad \text{for } x \in \partial\Omega \cap \text{dom } L, \mu \in [0, 1]. \quad (29)$$

Obviously, it follows from (25) and (28) that

$$Lx \neq S(x, 0), \quad \text{for } x \in \partial\Omega. \quad (30)$$

On the other hand, if $x \in \text{dom } L$ is a solution of $Lx = S(x, \mu)$, then x satisfies the equation as follows:

$$x'' + \mu x' + (ax^+ - bx^-) + \psi(x') = 0. \quad (31)$$

Multiplying both sides of (31) by x' and integrating over $[0, 2\pi]$, we get

$$\mu \int_0^{2\pi} x'^2(t) dt + \int_0^{2\pi} \psi(x') x' dt = 0. \quad (32)$$

If $\mu > 0$, then we infer from (13) and (32) that $x'(t) \equiv 0$ for every $t \in [0, 2\pi]$. Furthermore, $x(t) \equiv c$ for every $t \in [0, 2\pi]$, where c is a constant. Consequently, we have $x(t) \equiv 0$, and then $x \in \Omega$.

From the homotopic invariance of the coincidence degree, we have

$$D_L(L - S(\cdot, 0), \Omega) = D_L(L - S(\cdot, 1), \Omega). \quad (33)$$

In the following, we will compute $D_L(L - S(\cdot, 1), \Omega)$. To this end, we use the equality [15] as follows:

$$|D_L(L - S(\cdot, 1), \Omega)| = |d_B(-QS(\cdot, 1)|_{\text{Ker } L}, \Omega \cap \text{Ker } L, 0)|, \quad (34)$$

which holds provided that the following conditions are satisfied,

$$Lx \neq \lambda S(x, 1), \quad \forall x \in \partial\Omega \cap \text{dom } L, \lambda \in (0, 1], \quad (35)$$

$$QS(x, 1) \neq 0, \quad \forall x \in \partial\Omega \cap \text{Ker } L. \quad (36)$$

In what follows, we will prove that conditions (35) and (36) are satisfied. In fact, if $x \in \partial\Omega \cap \text{dom } L$ is a solution of $Lx = \lambda S(x, 1)$, then $x(t)$ satisfies the equation as follows:

$$x''(t) + \lambda x' + \lambda(ax^+ - bx^-) + \lambda\psi(x') = 0. \quad (37)$$

Using the same method as before, we can get $x \in \Omega$. This is a contradiction. To check condition (36), we notice that if $x \in \partial\Omega \cap \text{Ker } L$, then $x(t) = c'$ with $|c'| = M_1$. Hence, we have that, for $x \in \partial\Omega \cap \text{Ker } L$,

$$QS(x, 1) = \frac{1}{T} \int_0^T (-ax^+ + bx^-) dt = -ac' \quad \text{or} \quad -bc' \neq 0. \quad (38)$$

Finally, we can easily calculate the Brouwer degree $d_B(-QS(\cdot, 1)|_{\text{Ker } L}, \Omega \cap \text{Ker } L, 0)$ and obtain

$$d_B(-QS(\cdot, 1)|_{\text{Ker } L}, \Omega \cap \text{Ker } L, 0) = 1. \quad (39)$$

Therefore, we have

$$D_L(L - N(\cdot, 1), \Omega) \neq 0. \quad (40)$$

Consequently, the equation

$$Lx = N(x, 1) \quad (41)$$

has at least one 2π -periodic solution. Equivalently, (1) has at least one 2π -periodic solution. $\square$

Remark 5. In (12), if ψ satisfies the following condition,

$$x\psi(x) < 0, \quad \forall x \in \mathbf{R}, x \neq 0, \quad (42)$$

then the conclusion of Lemma 4 still holds. This claim can be proved by using the same method as the one used for proving Lemma 4. In fact, we only need to modify the term $-\mu x'$ in the auxiliary operator $S(x, \mu)$ to the term $\mu x'$.

3. Periodic Solutions of Duffing Equation with a Deviating Argument

At first, we choose a continuous function $\psi : R \rightarrow R$ satisfying

$$\lim_{x \rightarrow \pm\infty} \psi(x) = \psi(\pm\infty), \quad (43)$$

where $\psi(\pm\infty) \in R$ are constants. Moreover, ψ satisfies condition (13).

Considering the equivalent system of (12),

$$\begin{aligned} x' &= y, \\ y' &= -(ax^+ - bx^-) - \lambda g(x(t-\tau)) - (1-\lambda)\psi(x') \\ &\quad + \lambda p(t). \end{aligned} \quad (44)$$

Let $x(t)$ be any (possible) 2π -periodic solution of (12). Write $y(t) = x'(t)$. Then, $(x(t), y(t))$ is a 2π -periodic solution of system (44).

In what follows, we will introduce a transformation. To this end, let us denote by $c(t)$ a solution of equation $x'' + ax^+ - bx^- = 0$ satisfying the initial condition $c(0) = 0, c'(0) = 1$. Obviously, $c(t)$ is $2\pi/n$ -periodic. The derivative of $c(t)$ will be denoted by $s(t) = c'(t)$. It is easy to check that the following properties are satisfied:

- (1) $c(t + 2\pi/n) = c(t), s(t + 2\pi/n) = s(t)$.
- (2) $c'(t) = s(t), s'(t) = -(ac^+(t) - bc^-(t))$.
- (3) $s(t)^2 + ac^+(t)^2 + bc^-(t)^2 = 1, \forall t \in R$.

Let us define a mapping $\Phi : (\theta, \rho) \in S^1 \times (0, +\infty) \rightarrow (x, y) \in R^2 \setminus \{0\}$ as follows:

$$x = \rho^{1/2} c\left(\frac{\theta}{n}\right), \quad y = \rho^{1/2} s\left(\frac{\theta}{n}\right), \quad (45)$$

where $S^1 = R/2\pi Z$.

Under the transformation Φ , if $|x(t)| + |y(t)| \neq 0, \forall t \in [0, 2\pi]$, then the 2π -periodic solution $(x(t), y(t))$ of system (44) can be expressed in the form $(\rho(t), \theta(t))$ satisfying the equations as follows:

$$\begin{aligned} \frac{d\rho}{dt} &= -2\lambda\rho^{1/2} \\ &\quad \times \left(g\left(\rho^{1/2}(t-\tau)c\left(\frac{\theta(t-\tau)}{n}\right)\right)s\left(\frac{\theta}{n}\right) - p(t)s\left(\frac{\theta}{n}\right) \right) \\ &\quad - 2(1-\lambda)\rho^{1/2}\psi\left(\rho^{1/2}s\left(\frac{\theta}{n}\right)\right)s\left(\frac{\theta}{n}\right), \end{aligned}$$

$$\begin{aligned} \frac{d\theta}{dt} &= n + n\lambda\rho^{-1/2} \\ &\quad \times \left(g\left(\rho^{1/2}(t-\tau)c\left(\frac{\theta(t-\tau)}{n}\right)\right)c\left(\frac{\theta}{n}\right) - p(t)c\left(\frac{\theta}{n}\right) \right) \\ &\quad + n(1-\lambda)\rho^{-1/2}\psi\left(\rho^{1/2}s\left(\frac{\theta}{n}\right)\right)c\left(\frac{\theta}{n}\right). \end{aligned} \quad (46)$$

Let us denote $(\rho_0, \theta_0) = (\rho(0), \theta(0))$. From now on, we always assume that g is bounded. From the first equation of (46) we get that

$$\begin{aligned} \frac{d\rho^{1/2}}{dt} &= -\lambda g\left(\rho^{1/2}(t-\tau)c\left(\frac{\theta(t-\tau)}{n}\right)\right)s\left(\frac{\theta}{n}\right) \\ &\quad + \lambda p(t)s\left(\frac{\theta}{n}\right) - (1-\lambda)\psi\left(\rho^{1/2}s\left(\frac{\theta}{n}\right)\right)s\left(\frac{\theta}{n}\right). \end{aligned} \quad (47)$$

Therefore, we have

$$\rho(t)^{1/2} = \rho_0^{1/2} + O(1). \quad (48)$$

Furthermore, we get

$$\rho(t)^{-1/2} = \rho_0^{-1/2} + O(\rho_0^{-1}). \quad (49)$$

From the second equation of (46), we have

$$\frac{d\theta}{dt} = n + O(\rho_0^{-1/2}). \quad (50)$$

As a result,

$$\theta(t) = \theta_0 + nt + O(\rho_0^{-1/2}). \quad (51)$$

Substituting (51) in (47), we obtain that, for $t \in [0, 2\pi]$,

$$\begin{aligned} \frac{d\rho^{1/2}}{dt} &= -\lambda g\left(\rho_0^{1/2}c\left(t-\tau+\frac{\theta_0}{n}\right)+O(1)\right)s\left(t+\frac{\theta_0}{n}\right) \\ &\quad + \lambda p(t)s\left(t+\frac{\theta_0}{n}\right) - (1-\lambda)\psi\left(\rho_0^{1/2}s\left(\frac{\theta}{n}\right)\right)s\left(\frac{\theta}{n}\right) \\ &\quad + O(\rho_0^{-1/2}). \end{aligned} \quad (52)$$

Consequently,

$$\begin{aligned} \rho^{1/2}(2\pi) &= \rho_0^{1/2} - \lambda \int_0^{2\pi} g\left(\rho_0^{1/2}c\left(t-\tau+\frac{\theta_0}{n}\right)+O(1)\right)s\left(t+\frac{\theta_0}{n}\right)dt \\ &\quad - (1-\lambda) \int_0^{2\pi} \psi\left(\rho_0^{1/2}s\left(t+\frac{\theta_0}{n}\right)+O(1)\right)s\left(t+\frac{\theta_0}{n}\right)dt \\ &\quad + \lambda \int_0^{2\pi} p(t)s\left(t+\frac{\theta_0}{n}\right)dt + O(\rho_0^{-1/2}). \end{aligned} \quad (53)$$

Similarly, substituting (51) in the second equality of (44), we get that, for $t \in [0, 2\pi]$,

$$\begin{aligned} \frac{d\theta}{dt} &= n + n\lambda\rho_0^{-1/2}g\left(\rho_0^{1/2}c\left(t - \tau + \frac{\theta_0}{n}\right) + O(1)\right) \\ &\quad \times c\left(t + \frac{\theta_0}{n}\right) - n\lambda\rho_0^{-1/2}p(t)c\left(t + \frac{\theta_0}{n}\right) \\ &\quad + n(1 - \lambda)\rho_0^{-1/2}\psi\left(\rho_0^{1/2}s\left(t + \frac{\theta_0}{n}\right) + O(1)\right) \\ &\quad \times c\left(t + \frac{\theta_0}{n}\right) + O(\rho_0^{-1}). \end{aligned} \quad (54)$$

Therefore, we have

$$\begin{aligned} \theta(2\pi) &= \theta_0 + 2n\pi + n\lambda\rho_0^{-1/2} \\ &\quad \times \int_0^{2\pi} g\left(\rho_0^{1/2}c\left(t - \tau + \frac{\theta_0}{n}\right) + O(1)\right)c\left(t + \frac{\theta_0}{n}\right)dt \\ &\quad + n(1 - \lambda)\rho_0^{-1/2} \\ &\quad \times \int_0^{2\pi} \psi\left(\rho_0^{1/2}s\left(t + \frac{\theta_0}{n}\right) + O(1)\right)c\left(t + \frac{\theta_0}{n}\right)dt \\ &\quad - n\lambda\rho_0^{-1/2} \int_0^{2\pi} p(t)c\left(t + \frac{\theta_0}{n}\right)dt + O(\rho_0^{-1}). \end{aligned} \quad (55)$$

Write

$$\begin{aligned} \psi_1(\theta_0) &= \int_0^{2\pi} g\left(\rho_0^{1/2}c\left(t - \tau + \frac{\theta_0}{n}\right) + O(1)\right)s\left(t + \frac{\theta_0}{n}\right)dt, \\ \psi_2(\theta_0) &= \int_0^{2\pi} g\left(\rho_0^{1/2}c\left(t - \tau + \frac{\theta_0}{n}\right) + O(1)\right)c\left(t + \frac{\theta_0}{n}\right)dt, \\ \psi_3(\theta_0) &= \int_0^{2\pi} \psi\left(\rho_0^{1/2}s\left(t + \frac{\theta_0}{n}\right) + O(1)\right)s\left(t + \frac{\theta_0}{n}\right)dt, \\ \psi_4(\theta_0) &= \int_0^{2\pi} \psi\left(\rho_0^{1/2}s\left(t + \frac{\theta_0}{n}\right) + O(1)\right)c\left(t + \frac{\theta_0}{n}\right)dt. \end{aligned} \quad (56)$$

Recalling that $\nu = \tau \pmod{2\pi/n}$ and $0 \leq \nu < 2\pi/n$, we have the following estimates.

Lemma 6. Assume that condition (g) holds. Then, for $\rho_0 \rightarrow +\infty$,

$$\begin{aligned} \psi_1(\theta_0) &= \begin{cases} -n(g(+\infty) - g(-\infty))\left(\frac{\sin \sqrt{a}\nu}{\sqrt{a}} + \frac{\sin \sqrt{b}\nu}{\sqrt{b}}\right) \\ \quad + o(1), \\ \quad \text{for } \nu \leq \frac{\pi}{\sqrt{a}} \leq \frac{\pi}{\sqrt{b}}, \text{ or } \nu \leq \frac{\pi}{\sqrt{b}} \leq \frac{\pi}{\sqrt{a}}, \\ n(g(+\infty) - g(-\infty)) \\ \quad \times \left[\frac{\sin \sqrt{b}(\nu - \pi/\sqrt{a})}{\sqrt{b}} - \frac{\sin \sqrt{b}\nu}{\sqrt{b}}\right] + o(1), \\ \quad \text{for } \frac{\pi}{\sqrt{a}} < \nu \leq \frac{\pi}{\sqrt{b}}, \\ n(g(+\infty) - g(-\infty)) \\ \quad \times \left[\frac{\sin \sqrt{a}(\nu - \pi/\sqrt{b})}{\sqrt{a}} - \frac{\sin \sqrt{a}\nu}{\sqrt{a}}\right] + o(1), \\ \quad \text{for } \frac{\pi}{\sqrt{b}} < \nu \leq \frac{\pi}{\sqrt{a}}, \\ n(g(+\infty) - g(-\infty)) \\ \quad \times \left[\frac{\sin \sqrt{a}(\nu - \pi/\sqrt{b})}{\sqrt{a}} + \frac{\sin \sqrt{b}(\nu - \pi/\sqrt{a})}{\sqrt{b}}\right] \\ \quad + o(1), \\ \quad \text{for } \frac{\pi}{\sqrt{a}} \leq \frac{\pi}{\sqrt{b}} < \nu \text{ or } \frac{\pi}{\sqrt{b}} \leq \frac{\pi}{\sqrt{a}} < \nu. \end{cases} \end{aligned} \quad (57)$$

Proof. We only give the proof for the case $0 \leq \nu \leq \pi/\sqrt{a} \leq \pi/\sqrt{b} < 2\pi/n$. The other cases can be treated similarly. Since $s(t)$ is $2\pi/n$ -periodic, it follows from the expression of $\psi_1(\theta_0)$ that

$$\begin{aligned} \psi_1(\theta_0) &= \int_0^{2\pi} g\left(\rho_0^{1/2}c\left(t - \tau + \frac{\theta_0}{n}\right) + O(1)\right)s\left(t + \frac{\theta_0}{n}\right)dt \\ &= \int_0^{2\pi} g\left(\rho_0^{1/2}c(u) + O(1)\right)s(u + \tau)du \\ &= \int_0^{2\pi} g\left(\rho_0^{1/2}c(u) + O(1)\right)s(u + \nu)du. \end{aligned} \quad (58)$$

From the dominated convergent theorem, we have that, for $\rho_0 \rightarrow \infty$,

$$\begin{aligned} \psi_1(\theta_0) &= ng(+\infty) \int_0^{\pi/\sqrt{a}} s(u + \nu)du + ng(-\infty) \\ &\quad \times \int_{\pi/\sqrt{a}}^{2\pi/n} s(u + \nu)du + o(1) \\ &= ng(+\infty) \left[\int_0^{\pi/\sqrt{a}-\nu} s(u + \nu)du + \int_{\pi/\sqrt{a}-\nu}^{\pi/\sqrt{a}} s(u + \nu)du \right] \end{aligned}$$

$$\begin{aligned}
& +ng(-\infty) \left[\int_{\pi/\sqrt{a}}^{2\pi/n-\nu} s(u+\nu) du \right. \\
& \quad \left. + \int_{2\pi/n-\nu}^{2\pi/n} s(u+\nu) du \right] + o(1) \\
& = -ng(+\infty) \left(\frac{\sin \sqrt{a}\nu}{\sqrt{a}} + \frac{\sin \sqrt{b}\nu}{\sqrt{b}} \right) \\
& \quad + ng(-\infty) \left(\frac{\sin \sqrt{a}\nu}{\sqrt{a}} + \frac{\sin \sqrt{b}\nu}{\sqrt{b}} \right) + o(1) \\
& = -n(g(+\infty) - g(-\infty)) \left(\frac{\sin \sqrt{a}\nu}{\sqrt{a}} + \frac{\sin \sqrt{b}\nu}{\sqrt{b}} \right) + o(1). \tag{59}
\end{aligned}$$

Lemma 7. Assume that condition (g) holds. Then, for $\rho_0 \rightarrow +\infty$,

$$\begin{aligned}
& \psi_2(\theta_0) \\
& = \left\{ \begin{aligned}
& ng(+\infty) \left(\frac{1 + \cos \sqrt{a}\nu}{a} - \frac{1 - \cos \sqrt{b}\nu}{b} \right) \\
& \quad + ng(-\infty) \left(\frac{1 - \cos \sqrt{a}\nu}{a} - \frac{1 + \cos \sqrt{b}\nu}{b} \right) + o(1), \\
& \quad \text{for } \nu \leq \frac{\pi}{\sqrt{a}} \leq \frac{\pi}{\sqrt{b}}, \text{ or } \nu \leq \frac{\pi}{\sqrt{b}} \leq \frac{\pi}{\sqrt{a}}, \\
& ng(+\infty) \left[\frac{\cos \sqrt{b}\nu - \cos \sqrt{b}(\nu - \pi/\sqrt{a})}{b} \right] + ng(-\infty) \\
& \quad \times \left[\left(\frac{2}{a} - \frac{2}{b} \right) - \frac{\cos \sqrt{b}\nu - \cos \sqrt{b}(\nu - \pi/\sqrt{a})}{b} \right] \\
& \quad + o(1), \\
& \quad \text{for } \frac{\pi}{\sqrt{a}} < \nu \leq \frac{\pi}{\sqrt{b}}, \\
& ng(+\infty) \left[\left(\frac{2}{a} - \frac{2}{b} \right) - \frac{\cos \sqrt{a}(\nu - \pi/\sqrt{b}) - \cos \sqrt{a}\nu}{a} \right] \\
& \quad + ng(-\infty) \left[\frac{\cos \sqrt{a}(\nu - \pi/\sqrt{b}) - \cos \sqrt{a}\nu}{a} \right] \\
& \quad + o(1), \\
& \quad \text{for } \frac{\pi}{\sqrt{b}} < \nu \leq \frac{\pi}{\sqrt{a}}, \\
& ng(+\infty) \left[\frac{1 - \cos \sqrt{a}(\nu - \pi/\sqrt{b})}{a} \right. \\
& \quad \left. - \frac{1 + \cos \sqrt{b}(\nu - \pi/\sqrt{a})}{b} \right] \\
& \quad + ng(-\infty) \left[\frac{1 + \cos \sqrt{a}(\nu - \pi/\sqrt{b})}{a} \right. \\
& \quad \left. - \frac{1 - \cos \sqrt{b}(\nu - \pi/\sqrt{a})}{b} \right] \\
& \quad + o(1), \\
& \quad \text{for } \frac{\pi}{\sqrt{a}} \leq \frac{\pi}{\sqrt{b}} < \nu \text{ or } \frac{\pi}{\sqrt{b}} \leq \frac{\pi}{\sqrt{a}} < \nu.
\end{aligned} \right. \tag{60}
\end{aligned}$$

Proof. We also only give the proof for the case $0 \leq \nu \leq \pi/\sqrt{a} \leq \pi/\sqrt{b} < 2\pi/n$. The other cases can be treated similarly. Since $c(t)$ is $2\pi/n$ -periodic, it follows from the expression of $\psi_2(\theta_0)$ and the dominated convergent theorem that, for $\rho_0 \rightarrow \infty$,

$$\begin{aligned}
& \psi_2(\theta_0) \\
& = \int_0^{2\pi} g \left(\rho_0^{1/2} c \left(t - \tau + \frac{\theta_0}{n} \right) + O(1) \right) c \left(t + \frac{\theta_0}{n} \right) dt \\
& = \int_0^{2\pi} g \left(\rho_0^{1/2} c(u) + O(1) \right) c(u + \tau) du \\
& = \int_0^{2\pi} g \left(\rho_0^{1/2} c(u) + O(1) \right) c(u + \nu) du \\
& = ng(+\infty) \int_0^{\pi/\sqrt{a}} c(u + \nu) du \\
& \quad + ng(-\infty) \int_{\pi/\sqrt{a}}^{2\pi/n} c(u + \nu) du + o(1) \\
& = ng(+\infty) \\
& \quad \times \left[\int_0^{\pi/\sqrt{a}-\nu} c(u + \nu) du + \int_{\pi/\sqrt{a}-\nu}^{\pi/\sqrt{a}} c(u + \nu) du \right] \\
& \quad + ng(-\infty) \\
& \quad \times \left[\int_{\pi/\sqrt{a}}^{2\pi/n-\nu} c(u + \nu) du + \int_{2\pi/n-\nu}^{2\pi/n} c(u + \nu) du \right] + o(1) \\
& = ng(+\infty) \left(\frac{1 + \cos \sqrt{a}\nu}{a} - \frac{1 - \cos \sqrt{b}\nu}{b} \right) \\
& \quad + ng(-\infty) \left(\frac{1 - \cos \sqrt{a}\nu}{a} - \frac{1 + \cos \sqrt{b}\nu}{b} \right) + o(1). \tag{61}
\end{aligned}$$

Lemma 8. Assume that condition (43) holds. Then, for $\rho_0 \rightarrow +\infty$,

$$\begin{aligned}
& \psi_3(\theta_0) = 2 [\psi(+\infty) - \psi(-\infty)] + o(1), \\
& \psi_4(\theta_0) = n [\psi(+\infty) + \psi(-\infty)] \left(\frac{1}{a} - \frac{1}{b} \right) + o(1). \tag{62}
\end{aligned}$$

Proof. From the expression of $\psi_3(\theta_0)$ and the dominated convergent theorem we have that, for $\rho_0 \rightarrow \infty$,

$$\begin{aligned}
& \psi_3(\theta_0) \\
& = \int_0^{2\pi} \psi \left(\rho_0^{1/2} s \left(t + \frac{\theta_0}{n} \right) + O(1) \right) s \left(t + \frac{\theta_0}{n} \right) dt \\
& = \int_0^{2\pi} \psi \left(\rho_0^{1/2} s(u) + O(1) \right) s(u) du
\end{aligned}$$

$$\begin{aligned}
&= n\psi(+\infty) \left(\int_0^{\pi/2\sqrt{a}} s(u) du + \int_{\pi/\sqrt{a}+\pi/2\sqrt{b}}^{2\pi/n} s(u) du \right) \\
&\quad + n\psi(-\infty) \left(\int_{\pi/2\sqrt{a}}^{\pi/\sqrt{a}} s(u) du + \int_{\pi/\sqrt{a}}^{\pi/\sqrt{a}+\pi/2\sqrt{b}} s(u) du \right) \\
&\quad + o(1) \\
&= 2[\psi(+\infty) - \psi(-\infty)] + o(1).
\end{aligned} \tag{63}$$

Similarly, we have that, for $\rho_0 \rightarrow +\infty$,

$$\begin{aligned}
&\psi_4(\theta_0) \\
&= \int_0^{2\pi} \psi \left(\rho_0^{1/2} s \left(t + \frac{\theta_0}{n} \right) + O(1) \right) c \left(t + \frac{\theta_0}{n} \right) dt \\
&= \int_0^{2\pi} \psi \left(\rho_0^{1/2} s(u) + O(1) \right) c(u) du \\
&= n\psi(+\infty) \left(\int_0^{\pi/2\sqrt{a}} c(u) du + \int_{(\pi/\sqrt{a})+(\pi/2\sqrt{b})}^{2\pi/n} c(u) du \right) \\
&\quad + n\psi(-\infty) \left(\int_{\pi/2\sqrt{a}}^{\pi/\sqrt{a}} c(u) du + \int_{\pi/\sqrt{a}}^{(\pi/\sqrt{a})+(\pi/2\sqrt{b})} c(u) du \right) \\
&\quad + o(1) = n[\psi(+\infty) + \psi(-\infty)] \left(\frac{1}{a} - \frac{1}{b} \right) + o(1).
\end{aligned} \tag{64}$$

Proof of Theorem 1. We proceed to prove Theorem 1 in two different cases.

(1) Assume that the first inequality of Theorem 1 holds. Without loss of generality, we assume

$$\begin{aligned}
&ng(-\infty) \left(\frac{1 - \cos \sqrt{a}\gamma}{a} - \frac{1 + \cos \sqrt{b}\gamma}{b} \right) \\
&\quad + ng(+\infty) \left(\frac{1 + \cos \sqrt{a}\gamma}{a} - \frac{1 - \cos \sqrt{b}\gamma}{b} \right) \\
&> \int_0^{2\pi} p(t) c(t + \theta) dt, \quad \forall \theta \in [0, 2\pi].
\end{aligned} \tag{65}$$

Let us set

$$\begin{aligned}
\eta(\theta) &= ng(-\infty) \left(\frac{1 - \cos \sqrt{a}\gamma}{a} - \frac{1 + \cos \sqrt{b}\gamma}{b} \right) \\
&\quad + ng(+\infty) \left(\frac{1 + \cos \sqrt{a}\gamma}{a} - \frac{1 - \cos \sqrt{b}\gamma}{b} \right) \\
&\quad - \int_0^{2\pi} p(t) c(t + \theta) dt > 0, \quad \theta \in [0, 2\pi].
\end{aligned} \tag{66}$$

We now choose a function ψ satisfying (43) and (13). Moreover, $\psi(\pm\infty)$ satisfy

$$\mu = [\psi(+\infty) + \psi(-\infty)] \left(\frac{1}{a} - \frac{1}{b} \right) > 0. \tag{67}$$

Then we infer from Lemmas 7 and 8 that, for $\rho_0 \rightarrow \infty$,

$$\begin{aligned}
\theta(2\pi) &= \theta_0 + 2n\pi + n\rho_0^{-1/2} \left[\lambda\eta \left(\frac{\theta_0}{n} \right) + n(1 - \lambda)\mu \right] \\
&\quad + o(\rho_0^{-1/2}).
\end{aligned} \tag{68}$$

Since $\eta(\theta) > 0$, $\theta \in [0, 2\pi]$, and $\mu > 0$, there exists a constant $\gamma > 0$ such that, for $\theta \in [0, 2\pi]$ and $\lambda \in [0, 1]$,

$$\lambda\eta(\theta) + n(1 - \lambda)\mu \geq \gamma. \tag{69}$$

From (68) and (69) we have that, for $\rho_0 \rightarrow \infty$,

$$\theta(2\pi) = \theta_0 + 2n\pi + o(1), \quad \theta(2\pi) > \theta_0 + 2n\pi. \tag{70}$$

Consequently, there exists a constant $M > 0$ such that if $(\rho(t), \theta(t))$ is a 2π -periodic solution of system (46), then $\rho(t) \leq M$, $t \in [0, 2\pi]$. Furthermore, there exist constants $M_1 > 0$ and $M_2 > 0$ such that if $x(t)$ is a 2π -periodic solution of (12), then

$$\|x\|_\infty < M_1, \quad \|x'\|_\infty < M_2. \tag{71}$$

From Lemma 4, we know that (1) has at least one 2π -periodic solution.

(2) We assume that the second inequality of Theorem 1 holds. Without loss of generality, we assume

$$\begin{aligned}
&n(g(-\infty) - g(+\infty)) \left(\frac{\sin \sqrt{a}\gamma}{\sqrt{a}} + \frac{\sin \sqrt{b}\gamma}{\sqrt{b}} \right) \\
&> \int_0^{2\pi} p(t) s(t + \theta) dt, \quad \forall \theta \in [0, 2\pi].
\end{aligned} \tag{72}$$

Let us set

$$\begin{aligned}
\zeta(\theta) &= n(g(-\infty) - g(+\infty)) \left(\frac{\sin \sqrt{a}\gamma}{\sqrt{a}} + \frac{\sin \sqrt{b}\gamma}{\sqrt{b}} \right) \\
&\quad - \int_0^{2\pi} p(t) s(t + \theta) dt > 0, \quad \theta \in [0, 2\pi].
\end{aligned} \tag{73}$$

Similarly, we choose a continuous function ψ satisfying (43) and (13). Moreover, $\psi(\pm\infty)$ satisfy

$$\mu' = \psi(+\infty) - \psi(-\infty) > 0. \tag{74}$$

Then we infer from Lemmas 7 and 8 that, for $\rho_0 \rightarrow \infty$,

$$\rho^{1/2}(2\pi) = \rho_0^{1/2} - \left[\lambda\zeta \left(\frac{\theta_0}{n} \right) + 2(1 - \lambda)\mu' \right] + o(1). \tag{75}$$

Since $\zeta(\theta) > 0$, $\theta \in [0, 2\pi]$, and $\mu' > 0$, there exists a constant $\gamma' > 0$ such that, for $\theta \in [0, 2\pi]$ and $\lambda \in [0, 1]$ and $\rho_0 \rightarrow \infty$,

$$\lambda\zeta(\theta) + 2(1 - \lambda)\mu' \geq \gamma'. \tag{76}$$

From (75) and (76) we have that, for sufficiently large ρ_0 ,

$$\rho^{1/2}(2\pi) \leq \rho_0^{1/2} - \frac{\gamma'}{2}. \tag{77}$$

Consequently, there exists a constant $M' > 0$ such that if $(\rho(t), \theta(t))$ is a 2π -periodic solution of system (46), then $\rho(t) \leq M'$, $t \in [0, 2\pi]$. Furthermore, there exist constants $M'_1 > 0$ and $M'_2 > 0$ such that if $x(t)$ is a 2π periodic solution of (12), then

$$\|x\|_{\infty} < M'_1, \quad \|x'\|_{\infty} < M'_2. \quad (78)$$

From Lemma 4 we know that (1) has at least one 2π periodic solution. $\square$

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