Research Article

Some Spaces of Double Sequences Obtained through Invariant Mean and Related Concepts

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1. Preliminaries, Background, and Notation

In 1900, Pringsheim [1] presented the following notion of convergence for double sequences. A double sequence \( x = (x_{jk}) \) is said to converge to the limit \( L \) in Pringsheim’s sense (shortly, \( p \)-convergent to \( L \)) if for every \( \varepsilon > 0 \) there exists an integer \( N \) such that \( |x_{jk} - L| < \varepsilon \) whenever \( j, k > N \). In this case, \( L \) is called the \( p \)-limit of \( x \).

A double sequence \( x = (x_{jk}) \) of real or complex numbers is said to be bounded if \( \|x\|_1 = \sup_{j,k} |x_{jk}| < \infty \). The space of all bounded double sequences is denoted by \( \mathcal{M}_u \).

If \( x \in \mathcal{M}_u \) and is \( p \)-convergent to \( L \), then \( x \) is said to be boundedly \( p \)-convergent to \( L \) (shortly, \( bp \)-convergent to \( L \)). In this case, \( L \) is called the \( bp \)-limit of \( x \). The assumption of \( p \)-convergent was made because a double sequence on which \( p \)-convergent is not necessarily bounded.

In general, for any notion of convergence \( v \), the space of all \( v \)-convergent double sequences will be denoted by \( \mathcal{E}_v \), the space of all \( v \)-convergent to \( 0 \) double sequences by \( \mathcal{E}_{v0} \) and the limit of a \( v \)-convergent double sequence \( x \) by \( \nu \)-lim\( _{j,k} x_{jk} \), where \( \nu \in \{p, bp\} \).

Let \( \Omega \) denote the vector space of all double sequences with the vector space operations defined coordinatewise. Vector subspaces of \( \Omega \) are called double sequence spaces.

Let \( \sigma \) be a one-to-one mapping from the set \( \mathbb{N} \) of natural numbers into itself. A continuous linear functional \( \varphi \) on the space \( \ell_{\infty} \) of bounded single sequences is said to be an invariant mean or a \( \sigma \)-mean if and only if (i) \( \varphi(x) \geq 0 \) when...
the sequence $x = (x_k)$ has $x_k \geq 0$ for all $k$, (ii) $\varphi(e) = 1$, where $e = (1, 1, 1, \ldots)$, and (iii) $\varphi(x) = \varphi((x_{\sigma(k)}))$ for all $x \in E_{\sigma}$.

Throughout this paper we consider the mapping $\sigma$ which has no finite orbits, that is, $\sigma^p(k) \neq k$ for all integer $k \geq 0$ and $p \geq 1$, where $\sigma^p(k)$ denotes the $p$th iterate of $\sigma$ at $k$. Note that a $\sigma$-mean extends the limit functional on the space $c$ of convergent single sequences in the sense that $\varphi(x) = \lim x$ for all $x \in c$, (see [2]). Consequently, $c \subset V_{\sigma}$. Using this concept, Schaefer [3] defined and characterized $\sigma$-conservative, $\sigma$-regular, and $\sigma$-coercive matrices for single sequences. If $\sigma$ is translation then $V_{\sigma}$ is reduced to the set of almost convergent sequences [4]. Recently, Mohiuddine [5] has obtained an application of almost convergence for single sequences in approximation theorems and proved some related results.

In 2006, Çakan et al. [6] presented the following definition of $\sigma$-convergence for double sequences and further studied by Mursaleen and Mohiuddine [7–9]. A double sequence $x = (x_{j,k})$ of real numbers is said to be $\sigma$-convergent to a number $L$ if and only if $x \in \mathcal{Y}_{\sigma}$, where

$$
\mathcal{Y}_{\sigma} = \left\{ x \in \mathcal{M}_u : \lim_{p,q \rightarrow \infty} \xi_{p,q}(x) = L \text{ uniformly in } s,t; \right\}
$$

$$
L = \sigma-\lim_{p,q \rightarrow \infty} x_{j,k} = \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{\sigma(j),\sigma(k)}
$$

while here the limit means $bp$-limit. Let us denote by $\mathcal{Y}_{\sigma}$ the space of $\sigma$-convergent double sequences $x = (x_{j,k})$. For $\sigma(n) = n+1$, the set $\mathcal{Y}_{\sigma}$ is reduced to the set $\mathcal{F}$ of almost convergent double sequences [10]. Note that $\mathcal{B}_{bp} \subset \mathcal{Y}_{\sigma} \subset \mathcal{M}_u$.

Maddox [11] has defined the concepts of strong almost convergence and $M$-convergence for single sequences and established inclusion relation between strong almost convergence, $M$-convergence, and almost convergence for single sequence. Başarır [12] extended the notion of strong almost convergence from single sequences to double sequences and proved some interesting results involving this idea and the notion of almost convergence for double sequences. In the recent past, Mursaleen and Mohiuddine [13] presented the notions of absolute and strong $\sigma$-convergence for double sequences. A bounded double sequence $x = (x_{j,k})$ is said to be strongly $\sigma$-convergent if there exists a number $\ell$ such that

$$
\frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} |x_{\sigma(j),\sigma(k)} - \ell| \rightarrow 0
$$

as $p, q \rightarrow \infty$ uniformly in $s,t$.

For more details of spaces for single and double sequences and related concepts, we refer to [14–31] and references therein.

In this paper, we define and study some new spaces involving the idea of invariant mean and $\sigma$-convergence for double sequences and establish a relation between these spaces. Further, we extend above spaces to more general spaces by considering the double sequences $\alpha = (\alpha_{j,k})$ such that $\alpha_{j,k} > 0$ for all $j, k$ and $\sup_{j,k} |\alpha_{j,k}| = H < \infty$ and prove some topological results.

### 2. The Double Sequence Spaces

We construct the following spaces involving the idea of invariant mean and $\sigma$-convergence for double sequences:

$$
\mathcal{W}_{\sigma} = \left\{ x = (x_{j,k}) : \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \xi_{p,q}(x - \ell E) \rightarrow 0 \text{ as } m, n \rightarrow \infty, \right\}
$$

uniformly in $s,t$, for some $\ell$,

$$
\mathcal{W}_{\sigma}^{+} = \left\{ x = (x_{j,k}) : \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} |\xi_{p,q}(x - \ell E)| \rightarrow 0 \text{ as } m, n \rightarrow \infty, \right\}
$$

uniformly in $s,t$, for some $\ell$,

$$
\mathcal{W}_{\sigma}^{\alpha} = \left\{ x = (x_{j,k}) : \sum_{p=0}^{\infty} \left( \sum_{q=0}^{\infty} |\xi_{p,q}(x - \ell E)| - \xi_{p,1-q,1-t} - \xi_{p,q-1,1-t} + \xi_{p-1,q,1-t} \right) \rightarrow 0 \right\},
$$

$$
\mathcal{W}_{\sigma}^{\alpha^*} = \left\{ x = (x_{j,k}) : \sup_{j,k} \left( \sum_{p=0}^{\infty} \left( \sum_{q=0}^{\infty} |\xi_{p,q}(x - \ell E)| - \xi_{p,1-q,1-t} - \xi_{p,q-1,1-t} \right) \right) < \infty \right\}
$$

where $E = (e_{j,k})$ with $e_{j,k} = 1$ for all $j, k$;

$$
\xi_{\alpha_{j,k}} = \xi_{\alpha_{j,k}}(x) = \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \xi_{p,q}(x),
$$
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\[
\xi_{0,0,\ell}(x) = \xi_{0,0,\ell} = x_{\ell}, \\
\xi_{-1,0,\ell}(x) = \xi_{-1,0,\ell}(x) = x_{-1,\ell}, \\
\xi_{0,-1,\ell}(x) = \xi_{0,-1,\ell}(x) = x_{\ell,-1}, \\
\xi_{-1,-1,\ell}(x) = \xi_{-1,-1,\ell}(x) = x_{-1,-1,\ell}.
\] (7)

**Remark 1.** If \([\mathcal{W}_\sigma]-\lim x = \ell\), that is,

\[
\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \left| \xi_{p,q}\right| \rightarrow 0 \quad (m,n \rightarrow \infty) \quad \text{uniformly in } s, t.
\] (8)

as \(m,n \rightarrow \infty\), uniformly in \(s, t\), then

\[
\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \left| \frac{1}{p+1} \sum_{j=0}^{p} \xi_{jg-2} \right| \rightarrow 0,
\] (9)

We remark that by using Abel’s transformation for single series

\[
\sum_{i=1}^{m} v_i (u_i + u_{i+1}) = \sum_{i=1}^{m} u_i (v_{i-1} + v_i) + u_{m+1} v_m.
\] (10)

We get Abel’s transformation for double series

\[
p \sum_{j=1}^{q} v_{jk} \left( u_{jk} - u_{j+1,k} - u_{j+1,k+1} + u_{j+1,k-1} \right)
\]

\[
= p \sum_{j=1}^{q} u_{j+1,k} \left( \Delta_{11} v_{jk} + \Delta_{10} v_{j+1,k} \right) - p \sum_{j=1}^{q} u_{j+1,k+1} \left( \Delta_{10} v_{j+1,k} \right)
\]

\[
= q \sum_{k=1}^{1} u_{j+1,k} \left( \Delta_{01} v_{jk} \right) + v_{p+1,q} - v_{p,q},
\] (11)

where

\[
\Delta_{10} v_{j} j = v_{j} - v_{j-1,j}, \quad \Delta_{01} v_{j} = v_{j} - v_{j-1,j-1}
\]

\[
\Delta_{11} v_{j} j = v_{j} - v_{j-1,j}, \quad v_{j} - v_{j-1,j-1}.
\] (12)

In the recent past, Altay and Başar [32] also presented another form of Abel’s transformation for double series.

3. Inclusion Relations

In the following theorem, we establish a relationship between spaces defined in Section 2. Before proceeding further, first we prove the following lemmas which will be used to prove our inclusion relations.

**Lemma 2.** Consider that \([\mathcal{W}_\sigma]-\lim x = \ell\) if and only if

\[ (L1) \quad \mathcal{W}_\sigma-\lim x = \ell; \]

\[ (L2) \quad (1/uv) \sum_{m=1}^{v} \sum_{n=1}^{u} \left| \Omega_{mnt - \ell} \right| \rightarrow 0 \quad \text{as } u, v \rightarrow \infty \quad \text{(uniformly in } s, t); \]

\[ (L3) \quad (1/uv) \sum_{m=1}^{v} \sum_{n=1}^{u} |\Omega_{mnt - \ell}| \rightarrow 0 \quad \text{as } u, v \rightarrow \infty \quad \text{(uniformly in } s, t); \]

\[ (L4) \quad (1/uv) \sum_{m=1}^{v} \sum_{n=1}^{u} |\xi_{mnt} + \Omega_{mnt} + \Omega_{mnt} \Omega_{mnt} | \rightarrow 0 \quad \text{as } u, v \rightarrow \infty \quad \text{(uniformly in } s, t), \]

where

\[
\Omega_{mnt} = \frac{1}{(m+1)} \sum_{p=0}^{m} \xi_{p,n}.
\] (13)

\[
\Omega_{mnt} = \frac{1}{(n+1)} \sum_{q=0}^{n} \xi_{m,q}.
\]

**Proof.** Suppose that \([\mathcal{W}_\sigma]-\lim x = \ell\). Thus, we have \(\mathcal{W}_\sigma-\lim x = \ell\), that is, \((L1)\) holds. We see that conditions \((L2)\) and \((L3)\) follows from the Remark 1. Write

\[
\frac{1}{uv} \sum_{m=1}^{u} \sum_{n=1}^{v} \left| \xi_{mnt} + \Omega_{mnt} - \Omega_{mnt} \Omega_{mnt} \right|
\]

\[
= \frac{1}{uv} \sum_{m=1}^{u} \sum_{n=1}^{v} \left| \xi_{mnt} - \ell \right| + \frac{1}{uv} \sum_{m=1}^{u} \sum_{n=1}^{v} \left| \Omega_{mnt} - \ell \right|
\]

\[
\leq \frac{1}{uv} \sum_{m=1}^{u} \sum_{n=1}^{v} \left| \xi_{mnt} - \ell \right| + \frac{1}{uv} \sum_{m=1}^{u} \sum_{n=1}^{v} \left| \Omega_{mnt} - \ell \right|
\]

\[
= \Sigma_{1} + \Sigma_{2} + \Sigma_{3} + \Sigma_{4} \quad \text{(say)}.
\] (14)

By our assumption, that is, \([\mathcal{W}_\sigma]-\lim x = \ell\), \(\Sigma_{1} \rightarrow 0\) as \(u, v \rightarrow \infty\) uniformly in \(s, t\). The condition \((L1)\) implies that \(\xi_{mnt}\) tends to zero as \(m, n\) tending to \(\infty\) uniformly in \(s, t\); therefore \(\Sigma_{2} \rightarrow 0\) as \(u, v \rightarrow \infty\) uniformly in \(s, t\) and \(\Sigma_{3}, \Sigma_{4} \rightarrow 0\) as \(u, v \rightarrow \infty\) uniformly in \(s, t\) by the conclusion \((L2)\) and \((L3)\), respectively. Thus, \((14)\) tends to zero as \(u, v \rightarrow \infty\) uniformly in \(s, t\), that is, \((L4)\) holds.

Conversely, let \((L1)-(L4)\) hold. Then,

\[
\frac{1}{uv} \sum_{m=1}^{u} \sum_{n=1}^{v} \left| \xi_{mnt} - \ell \right|
\]

\[
\leq \frac{1}{uv} \sum_{m=1}^{u} \sum_{n=1}^{v} \left| \xi_{mnt} + \Omega_{mnt} - \Omega_{mnt} \Omega_{mnt} \right|
\]

\[
= \frac{1}{uv} \sum_{m=1}^{u} \sum_{n=1}^{v} \left| \xi_{mnt} - \ell \right| + \frac{1}{uv} \sum_{m=1}^{u} \sum_{n=1}^{v} \left| \Omega_{mnt} - \ell \right|
\]

\[
= \Sigma_{1} + \Sigma_{2} + \Sigma_{3} + \Sigma_{4} \quad \text{(say)}.
\] (15)
Lemma 3. One has

\[ \lambda_{m,n} + \lambda_{m,n} - \lambda_{m,n} - \lambda_{m,n} = m \left[ \lambda_{m,n} - \lambda_{m-1,n} - \lambda_{m,n-1} + \lambda_{m-1,n-1} \right]. \]  

(16)

Proof. Since

\[ \lambda_{m,n} = \lambda_{m-1,n} + \lambda_{m,n-1} + \lambda_{m-1,n-1} \]

(17)

First, we solve the expression in the first bracket

\[
\left[ \frac{1}{m+1} \sum_{p=0}^{m} \sum_{q=0}^{n} \lambda_{p,q} - \frac{1}{m+1} \sum_{p=0}^{m-1} \sum_{q=0}^{n} \lambda_{p,q} \right]
\]

\[
= \frac{1}{(m+1)(n+1)} \sum_{p=0}^{n} \left( \sum_{q=0}^{m} \lambda_{p,q} - (m+1) \sum_{p=0}^{m-1} \lambda_{p,q} \right)
\]

\[= \frac{1}{m(n+1)} \sum_{q=0}^{n} \left( \lambda_{m, q} - \lambda_{m-1, q} \right) \]

\[= \frac{1}{m} \lambda_{m,n} - \lambda_{m,n-1} \]

(18)

Now, the expression in the second bracket

\[
\frac{1}{m+1} \sum_{p=0}^{m} \sum_{q=0}^{n} \lambda_{p,q} - \frac{1}{m} \sum_{p=0}^{m-1} \sum_{q=0}^{n} \lambda_{p,q}
\]

\[= \frac{1}{m+1} \sum_{p=0}^{m} \left( \sum_{q=0}^{n} \lambda_{p,q} - (m+1) \sum_{p=0}^{m-1} \lambda_{p,q} \right)
\]

\[= \frac{1}{m} \lambda_{m,n} - \lambda_{m,n-1} \]

(19)

Substituting (18) and (19) in (17), we get

\[ \lambda_{m,n} = \lambda_{m-1,n} + \lambda_{m,n-1} + \lambda_{m-1,n-1} \]

\[= \frac{1}{m(n+1)} \sum_{q=0}^{n} \lambda_{m,n} - \frac{1}{m} \sum_{p=0}^{m-1} \sum_{q=0}^{n} \lambda_{p,q}
\]

\[+ \frac{1}{m} \lambda_{m,n} - \frac{1}{m} \lambda_{m,n-1} \]

\[= \frac{1}{m} \lambda_{m,n} - \lambda_{m,n-1} \]

(20)

We know that

\[ \lambda_{m,n} = \frac{1}{m+1} \sum_{p=0}^{m} \sum_{q=0}^{n} \lambda_{p,q} \]

\[= \frac{1}{m+1} \sum_{p=0}^{m-1} \sum_{q=0}^{n} \lambda_{p,q} + \sum_{q=0}^{n} \lambda_{m, q} \]

(21)
From (21), we have
\[(m + 1)\xi_{mnst} - m\xi_{m-1,n,st} = \frac{1}{(n + 1)} \sum_{q=0}^{n} \xi_{pqst}.\]  
(22)
Thus, (20) becomes
\[
\xi_{mnst} - \xi_{m-1,n,st} = \xi_{m,n-1,st} + \xi_{m,n-1,1,ts} \nonumber
\]
\[- \frac{1}{m} \left[ \xi_{mnst} - (m + 1)\xi_{mnst} + m\xi_{m-1,n,st} \right] \nonumber
\]
\[= \frac{1}{m} \left[ \xi_{mnst} - (m + 1)\xi_{mnst} + \xi_{m-1,n,st} \right] \nonumber
\]
(23)
\[= \frac{1}{m} \left[ \xi_{mnst} - \xi_{mnst} - m(\xi_{mnst} - \xi_{m-1,n,st}) \right] \nonumber
\]
\[= \frac{1}{m} \left[ \xi_{mnst} - \xi_{mnst} - n(\xi_{mnst} - \xi_{m-1,n,st}) \right] \nonumber.
\]
Also (22) can be written as
\[m(\xi_{mnst} - \xi_{m-1,n,st}) = \frac{1}{(n + 1)} \sum_{q=0}^{n} \xi_{pqst} - \xi_{mnst}.\]  
(24)
Similarly, we can write
\[n(\xi_{mnst} - \xi_{m,n-1,ts}) = \frac{1}{(m + 1)} \sum_{q=0}^{m} \xi_{pqst} - \xi_{mnst}.\]  
(25)
Using (24) and (25) in (23), we get
\[
\xi_{mnst} - \xi_{m-1,n,st} = \xi_{m,n-1,st} + \xi_{m,n-1,1,ts} \nonumber
\]
\[= \frac{1}{m} \left[ \xi_{mnst} + \xi_{mnst} - \frac{1}{m} \sum_{p=0}^{m} \xi_{pqst} - \frac{1}{n} \sum_{q=0}^{n} \xi_{pqst} \right] \nonumber.
\]
This implies that
\[\xi_{mnst} + \xi_{mnst} - \Omega_{mnst} = \frac{1}{m} \sum_{p=0}^{m} \xi_{pqst} - \frac{1}{n} \sum_{q=0}^{n} \xi_{pqst} \nonumber.
\]
(26)
\[\xi_{mnst} + \xi_{mnst} - \Omega_{mnst} = \frac{1}{m} \sum_{p=0}^{m} \xi_{pqst} - \frac{1}{n} \sum_{q=0}^{n} \xi_{pqst}.\]
(27)

**Theorem 4.** One has the following inclusions and the limit is preserved in each case:

(i) \([V_\sigma] \subset [W_\sigma] \subset W_\sigma',\)

(ii) \(W_\sigma' \subset [W_\sigma] \subset [W_\sigma],\)

(iii) \(W_\sigma' \subset W_\sigma'\).

**Proof.** (i) Let \(x \in [V_\sigma]\) with \([V_\sigma]\)-lim \(x = \ell\), say. Then,
\[\xi_{pqst} (|x - \ell|) \rightarrow 0 \text{ as } p, q \rightarrow \infty, \text{ uniformly in } s, t.\]  
(28)
This implies that
\[
\frac{1}{(m + 1)(n + 1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \xi_{pqst} (|x - \ell|) \rightarrow 0 
\]  
as \(p, q \rightarrow \infty, \text{ uniformly in } s, t.\)  
(29)
Also, we have
\[
\frac{1}{(m + 1)(n + 1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \xi_{pqst} (|x - \ell|) \nonumber
\]
\[\leq \frac{1}{(m + 1)(n + 1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \xi_{pqst} (|x - \ell|) \nonumber
\]
\[\leq \frac{1}{(m + 1)(n + 1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \xi_{pqst} (|x - \ell|).\]  
(30)
Hence, \([V_\sigma] \subset [W_\sigma] \subset W_\sigma'\) and
\[\lim_{x \rightarrow \ell} \xi_{pqst} = \xi_{pqst} \quad \text{as } p, q \rightarrow \infty, \text{ uniformly in } s, t,\]  
(33)
that is, \(W_\sigma'\)-lim \(x = \ell\).

In order to prove that \(x \in [W_\sigma]\), it is enough to show that condition (L4) of Lemma 2 holds.
\[\frac{1}{m} \sum_{p=0}^{m} \sum_{q=0}^{n} \xi_{pqst} - \xi_{p-1,q,s,t} - \xi_{p,q-1,s,t} + \xi_{p-1,q-1,s,t}\]
\[\rightarrow 0\]
as \(m, n \rightarrow \infty, \text{ uniformly in } s, t;\) and
\[\xi_{pqst} \rightarrow \ell \quad \text{as } p, q \rightarrow \infty, \text{ uniformly in } s, t,\]  
(33)
that is, \(W_\sigma'\)-lim \(x = \ell\).

Now,
\[\frac{1}{m} \sum_{p=0}^{m} \sum_{q=0}^{n} \xi_{pqst} - \xi_{p-1,q,s,t} - \xi_{p,q-1,s,t} + \xi_{p-1,q-1,s,t}\]
\[
\leq \frac{1}{m} \sum_{p=0}^{m} \sum_{q=0}^{n} \xi_{pqst} (|x - \ell|) \nonumber
\]
\[\leq \frac{1}{m} \sum_{p=0}^{m} \sum_{q=0}^{n} \xi_{pqst} (|x - \ell|).\]  
(30)
Hence, \([V_\sigma] \subset [W_\sigma] \subset W_\sigma'\) and
\[\lim_{x \rightarrow \ell} \xi_{pqst} = \xi_{pqst} \quad \text{as } p, q \rightarrow \infty, \text{ uniformly in } s, t,\]  
(33)
that is, \(W_\sigma'\)-lim \(x = \ell\).

In order to prove that \(x \in [W_\sigma]\), it is enough to show that condition (L4) of Lemma 2 holds.
\[\sum_{p=1}^{m} \left[ \xi_{p,1,q,1} - \xi_{p,1,n,1} + \xi_{p,1,n,1} \right]
- \sum_{p=m+1}^{\infty} \left[ \xi_{p,1,q,1} - \xi_{p,1,n,1} + \xi_{p,1,n,1} \right]
= \left[ \sum_{p=m}^{\infty} - \sum_{p=m+1}^{\infty} \right] \left[ \xi_{p,1,q,1} - \xi_{p,1,n,1} + \xi_{p,1,n,1} \right]
= [\xi_{m,1,q,1} - \xi_{m,1,n,1} + \xi_{m,1,n,1}]. \quad (34)\]

Replacing \(m\) and \(n\) by \(p\) and \(q\), respectively, we have
\[\mathcal{F}_{p,q} - \mathcal{F}_{p,q+1} + \mathcal{F}_{p+1,q} = \mathcal{F}_{p,q} - \mathcal{F}_{p,q+1} + \mathcal{F}_{p+1,q+1}, \quad (35)\]

By Lemma 3, we have
\[\dot{\xi}_{p,q} + \dot{\xi}_{p+1,q} - \Omega_{p,q} - \Omega_{p+1,q}
= pq \left[ \xi_{p,1,q,1} - \xi_{p,1,n,1} + \xi_{p,1,n,1} \right]. \quad (36)\]

So that we have
\[\frac{1}{mn} \sum_{p=1}^{m} \sum_{q=1}^{n} \left[ \xi_{p,1,q,1} - \xi_{p,1,n,1} + \xi_{p,1,n,1} \right]
= \frac{1}{mn} \sum_{p=1}^{m} \sum_{q=1}^{n} pq \left[ \mathcal{F}_{p,q} - \mathcal{F}_{p,q+1} + \mathcal{F}_{p+1,q+1} \right]. \quad (37)\]

By using Abel’s transformation for double series in the right hand side of above equation, we have
\[\frac{1}{mn} \sum_{p=1}^{m} \sum_{q=1}^{n} \left[ \xi_{p,1,q,1} - \xi_{p,1,n,1} + \xi_{p,1,n,1} \right]
= \left[ \mathcal{F}_{p,1,q,1} + \mathcal{F}_{p,1,n,1} \right] - n \mathcal{F}_{m+1,q,1} + mn \mathcal{F}_{m+1,n,1}. \quad (38)\]

Hence, it is left to show that for fixed \(p, q\)
\[\left| \xi_{p,q} - \xi_{p,q-1,1} + \xi_{p,q-1,1} \right| \leq K, \quad \forall s, t. \quad (41)\]

From (40), we have
\[\left| \xi_{p,q} - \xi_{p,q-1,1} + \xi_{p,q-1,1} \right| < 1, \quad (42)\]

for every fixed \(p > p_0, q > q_0\) and for all \(s, t\). Since
\[m(m+1)n(n+1)(\xi_{m,n} - \xi_{m-1,n,1} + \xi_{m-1,n,1} + \xi_{m-1,n,1})
= \sum_{p=1}^{m} \sum_{q=1}^{n} pq \left( \xi_{p,q} - \xi_{p,q-1,1} + \xi_{p,q-1,1} \right). \quad (43)\]

Accordingly,
\[mn \left[ (m+1)(n+1) \right] \times \left( \xi_{m,n} - \xi_{m-1,n,1} + \xi_{m-1,n,1} \right)
- (m-1)(n+1) \times \left( \xi_{m-1,n,1} - \xi_{m-2,n,1} + \xi_{m-2,n,1} \right)
- (m+1)(n-1) \times \left( \xi_{m,n-1,1} - \xi_{m-1,n,1} + \xi_{m-1,n,1} \right)
+ (m-1)(n-1) \times \left( \xi_{m-1,n,1} - \xi_{m-2,n,1} + \xi_{m-2,n,1} \right)
= \sum_{q=1}^{n} q \left[ \sum_{p=1}^{m} p \left( \xi_{p,q} - \xi_{p,q-1,1} + \xi_{p,q-1,1} \right)
- \sum_{p=1}^{m-1} p \left( \xi_{p,q} - \xi_{p,q-1,1} + \xi_{p,q-1,1} \right) \right]
- \sum_{q=1}^{n-1} q \left[ \sum_{p=1}^{m} p \left( \xi_{p,q} - \xi_{p,q-1,1} + \xi_{p,q-1,1} \right)
- \sum_{p=1}^{m-1} p \left( \xi_{p,q} - \xi_{p,q-1,1} + \xi_{p,q-1,1} \right) \right]
= \sum_{q=1}^{n} q \left[ m \left( \xi_{m,q} - \xi_{m,q-1,1} + \xi_{m,q-1,1} \right) \right]. \quad (44)\]
This implies that

\[(m + 1)(n + 1) (\xi_{mnst} - \xi_{m-1,n,s,t} - \xi_{m,n-1,s,t} + \xi_{m-1,n-1,s,t})
- (m - 1)(n + 1)\] 
\[\times (\xi_{m-1,n,s,t} - \xi_{m-2,n,s,t} - \xi_{m-1,n-1,s,t} + \xi_{m-2,n-1,s,t})\] 
\[- (m + 1)(n - 1)\] 
\[\times (\xi_{m,n-1,s,t} - \xi_{m-1,n-1,s,t} - \xi_{m,n-2,s,t} + \xi_{m-1,n-2,s,t})\] 
\[- (m - 1)(n - 1)\] 
\[\times (\xi_{m-1,n-1,s,t} - \xi_{m-2,n-1,s,t} - \xi_{m-1,n-2,s,t} + \xi_{m-2,n-2,s,t})\] 
\[= [\xi_{mnst} - \xi_{m-1,n,s,t} - \xi_{m,n-1,s,t} + \xi_{m-1,n-1,s,t}].\] (45)

Using (42) and (45), we have

\[|\xi_{mnst} - \xi_{m-1,n,s,t} - \xi_{m,n-1,s,t} + \xi_{m-1,n-1,s,t}| \leq K(m, n)\] (46)

for every fixed \(m > p_0, n > q_0\) and for all \(s, t\), where \(K(m, n)\) is a constant depending upon \(m, n\).

Now, for any given infinite double series \(\sum \sum a_{st}\) denoted as "\(d\)", let us write

\[x_{jk} = \sum_{s=0}^{j} \sum_{t=0}^{k} a_{st}, \quad j, k = 1, 2, \ldots\] (47)

and \(\sigma\) be monotonically increasing. For simplicity in notation, we denote

\[\phi_{mnst} = \xi_{mnst} (a) - \xi_{m-1,n,s,t} (a) - \xi_{m,n-1,s,t} (a) + \xi_{m-1,n-1,s,t} (a).\] (48)

Again from the definition of \(\xi_{mnst}\), it is easy to obtain

\[\phi_{mnst} = \frac{1}{m(m + 1)n(n + 1)} \sum_{j=1}^{m} \sum_{k=1}^{n} \left( \sum_{u=d_j}^{d_j'} \sum_{v=d_k}^{d_k'} a_{uv} \right)\] (49)

for all \(m, n \geq 1\) and \(\phi_{0,0,0,0}(a) = a_{00}\) with \(d_j = \sigma^{-1}(s) + 1, d_j' = \sigma^j(s), h_k = \sigma^{-k}(t) + 1, h_k' = \sigma^k(t)\). Further, we calculate

\[m(m + 1)n(n + 1)\phi_{mnst} - m(m - 1)n(n + 1)\phi_{m-1,n,s,t}\] 
\[- m(m + 1)n(n - 1)\phi_{m,n-1,s,t}\] 
\[+ m(m - 1)n(n - 1)\phi_{m-1,n-1,s,t}\] 
\[= \left\{ \sum_{j=1}^{m} \sum_{k=1}^{n} \sum_{u=d_j}^{d_j'} \sum_{v=d_k}^{d_k'} a_{uv} \right\}\] 
\times \left\{ \sum_{d_j = h_k}^{d_j'} \sum_{d_k = h_k'} a_{uv} \right\}\] (50)

Thus, we have

\[\sum_{d_j = h_k}^{d_j'} \sum_{d_k = h_k'} a_{uv} = (m + 1)(n + 1)\phi_{mnst}\]
\[- (m - 1)(n + 1)\phi_{m-1,n,s,t}\] 
\[- (m + 1)(n - 1)\phi_{m,n-1,s,t}\] 
\[+ (m - 1)(n - 1)\phi_{m-1,n-1,s,t}\]. (51)

Hence, it follows from (46) that for each fixed \(m > p_0, n > q_0\),

\[\left| \sum_{u=d_j}^{d_j'} \sum_{v=d_k}^{d_k'} a_{uv} \right| \leq K(m, n), \forall s, t.\] (52)

Hence, it follows from (52) that

\[|a_{uv}| \leq K, \forall u, v,\] (53)

where \(K\) is independent of \(u, v\). By (49), we have

\[|\xi_{mnst} - \xi_{m-1,n,s,t} - \xi_{m,n-1,s,t} + \xi_{m-1,n-1,s,t}| \leq K, \forall m, n, s, t.\] (54)

Also from (43) and (54), we have

\[|\xi_{mnst} - \xi_{p,q,s,t} - \xi_{p,q-1,s,t} + \xi_{p-1,q,s,t}| \leq K, \forall p, q, s, t.\] (55)

4. Topological Results

Here, we extend the spaces \([W_\sigma], W_\sigma', W_\sigma''\) to more general spaces, respectively, denoted by \([W_\sigma(\alpha)], W_\sigma'(\alpha), W_\sigma''(\alpha),\)
Let \( \alpha = (\alpha_{jk}) \) be a bounded double sequence of strictly positive real numbers. Then, \([\mathcal{W}_\alpha]\) is a complete linear topological space paranormed by

\[
g(x) = \sup_{m,n} \left( \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} |k_{pq}\alpha|^{1/M} \right)^{1/M},
\]

where \( M = \max(1, \sup_{p,q} \alpha_{pq}) \). If \( \alpha \geq 1 \) then \([\mathcal{W}_\alpha]\) is a Banach space and \([\mathcal{W}_\alpha]\) is a \( \alpha \)-normed space if \( 0 < \alpha < 1 \).

**Proof.** Let \((x_{pq})\) and \((y_{pq})\) be two double sequences. Then,

\[
|x_{pq} + y_{pq}|^{\alpha_{pq}} \leq K \left( |x_{pq}|^{\alpha_{pq}} + |y_{pq}|^{\alpha_{pq}} \right),
\]

where \( K = \max(1,2^{H-1}) \) and \( H = \sup_{p,q} \alpha_{pq} \). Since

\[
|\lambda|^{\alpha_{pq}} \leq \max \left( 1, |\lambda|^{H} \right),
\]

therefore

\[
\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} |k_{pq}\alpha|^{1/M} \leq KK_1 \left( \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} |k_{pq}\alpha|^{1/M} \right)^{1/M} + KK_2 \left( \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} |k_{pq}\alpha|^{1/M} \right)^{1/M},
\]

where \( K_1 = \max(1, |\lambda|^{H}) \) and \( K_2 = \max(1, |\mu|^{H}) \). From (60), we have that if \( x, y \in [\mathcal{W}_\alpha] \), then \( \lambda x + \mu y \in [\mathcal{W}_\alpha] \). Thus, \([\mathcal{W}_\alpha]\) is a linear space. Without loss of generality, we can take

\[
[\mathcal{W}_\alpha] \cdot \lim x = 0.
\]

Clearly, \( g(0) = 0, g(-x) = g(x) \). From (58) and Minkowski’s inequality, we have

\[
\left( \sum_{p=0}^{m} \sum_{q=0}^{n} |k_{pq}\alpha|^{1/M} \right)^{1/M} \leq \left( \sum_{p=0}^{m} \sum_{q=0}^{n} |k_{pq}\alpha|^{1/M} \right)^{1/M} + \left( \sum_{p=0}^{m} \sum_{q=0}^{n} |k_{pq}\alpha|^{1/M} \right)^{1/M},
\]

Hence,

\[
g(x+y) \leq g(x) + g(y).
\]
Since $\alpha = (\alpha_{mn})$ is bounded away from zero, there exists a constant $\delta > 0$ such that $\alpha_{mn} \geq \delta$ for all $m, n$. Now for $|\lambda| \leq 1$, $|\lambda|^m \leq |\lambda|^\delta$ and so
\[
g(\lambda x) \leq |\lambda|^{\delta/M} g(x),
\]
that is, the scalar multiplication is continuous. Hence, $g$ is a paranorm on $[W_\sigma(\alpha)]$.

Let $(x^b)$ be a Cauchy sequence in $[W_\sigma(\alpha)]$, that is,
\[
g(x^d - x^b) \to 0 \quad \text{as } b, d \to \infty.
\]
Since
\[
\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \left| \xi_{p,q}(x^d - x^b) \right|^{|\sigma|} \leq \left[ g(x^d - x^b) \right]^M,
\]
it follows that
\[
\left| \xi_{p,q}(x^d - x^b) \right|^{|\sigma|} \to 0 \quad \text{as } b, d \to \infty \forall p, q, s, t.
\]
In particular,
\[
\left| \xi_{0,0,s,t}(x^d - x^b) \right| = \left| x_{s,t}^d - x_{s,t}^b \right| \to 0
\]
as $b, d \to \infty$ for fixed $s, t$.

Hence, $(x^d)$ is a Cauchy sequence in $C$. Since $C$ is complete, there exists $x = (x_{jk}) \in C$ such that $x^d \to x$ coordinatewise as $d \to \infty$. It follows from (66) that given $\epsilon > 0$, there exists $d_0$ such that
\[
\left( \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \left| \xi_{p,q}(x^d - x^b) \right|^{|\sigma|} \right)^{1/M} < \epsilon,
\]
for $b, d > d_0$. Now taking $b \to \infty$ and sup $s,t,m,n$ in (69), we have $g(x^d - x) \leq \epsilon$ for $d > d_0$. This proves that $x^d \to x$ and $x = (x_{jk}) \in [W_\sigma(\alpha)]$. Hence $[W_\sigma(\alpha)]$ is complete. If $\alpha$ is a constant then it is easy to prove the rest of the theorem. \(\square\)

**Theorem 6.** One has the following:

$W'_\sigma(\alpha)$ is a complete paranormed space, paranormed by

\[
h(x) = \sup_{s,t} \left( \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \left| \xi_{p,q} - \xi_{p-1,q,s,t} \right|^{|\sigma|} + \left| \xi_{p,q,s,t} - \xi_{p-1,q-1,s,t} \right|^{|\sigma|} \right)^{1/M}
\]
which is defined on $W'_\sigma(\alpha)$. If $\alpha \geq 1$ then $W'_\sigma(\alpha)$ is a Banach space and if $0 < \alpha < 1$, $W'_\sigma(\alpha)$ is $\alpha$-normed space.

**Proof.** In order for the paranorm in (70) to be defined, we require that
\[
W'_\sigma(\alpha) \subset W''_\sigma(\alpha)
\]
which is proved in the next theorem (i.e., Theorem 7). Using the standard technique as in the previous theorem, we can prove that $h$ is subadditive.

Now, we have to prove the continuity of scalar multiplication. Suppose that $x = (x_{jk}) \in W''_\sigma(\alpha)$. Then, for $\epsilon > 0$ there exist integers $M, N > 0$ such that
\[
\sum_{p=M}^{\infty} \sum_{q=N}^{\infty} \left| \xi_{p,q} - \xi_{p-1,q,s,t} \right|^{|\sigma|} + \left| \xi_{p,q,s,t} - \xi_{p-1,q-1,s,t} \right|^{|\sigma|} < \epsilon.
\]
If $|\lambda| \leq 1$, then by (72) we have
\[
\sum_{p=M}^{\infty} \sum_{q=N}^{\infty} \left| \xi_{p,q} - \xi_{p-1,q,s,t} \right|^{|\sigma|} + \left| \xi_{p,q,s,t} - \xi_{p-1,q-1,s,t} \right|^{|\sigma|} < \epsilon.
\]
Since for fixed $M, N$
\[
\sum_{p=0}^{M-1} \sum_{q=0}^{N-1} \left| \xi_{p,q} - \xi_{p-1,q,s,t} \right|^{|\sigma|} + \left| \xi_{p,q,s,t} - \xi_{p-1,q-1,s,t} \right|^{|\sigma|} \to 0
\]
as $\lambda \to 0$, it follows from (73) and (74) that for fixed $x = (x_{jk})$, $h(\lambda x) \to 0$ as $\lambda \to 0$. Also, since $|\lambda|\sigma_{\alpha} \leq \max(1, |\lambda|^\delta)$ implies that
\[
h(\lambda x) \leq (\sup |\lambda|\sigma_{\alpha})^{1/M} h(x).
\]
It follows that for fixed $\lambda$, $h(\lambda x) \to 0$ as $x \to 0$. This proves the continuity of scalar multiplication. Hence, $h$ is a paranorm. The proof of the completeness of $W'_\sigma(\alpha)$ can be achieved by using the same technique as in Theorem 5. \(\square\)

**Theorem 7.** Suppose that $\alpha = (\alpha_{pq})$ is bounded double sequence of strictly positive real numbers. Then,

(i) $W''_\sigma(\alpha)$ is a complete linear space paranormed by the function $h$ defined in (70). In particular, if $\alpha$ is a constant, $W''_{\sigma,\alpha}$ is a Banach space for $\alpha \geq 1$ and $\alpha$-normed space for $0 < \alpha < 1$,

(ii) $W'_\sigma(\alpha)$ is a closed subspace of $W''_\sigma(\alpha)$.

**Proof.** (i) Proceeding along the same lines as in Theorem 6, except for the proof of continuity of scalar multiplication. If $x = (x_{jk}) \in W''_\sigma(\alpha)$, we cannot assert (72) as in the case when $x = (x_{jk}) \in W'_\sigma(\alpha)$. Since $\alpha$ is bounded away from zero, there
exists $\delta > 0$ such that $\alpha_{mn} \geq \delta$ for all $m, n$. Hence, $|\lambda| \leq 1$ implies $|\lambda|^{\alpha_{mn}} \leq |\lambda|^\delta$. Since

$$h(\lambda x) \leq |\lambda|^{\alpha_{mn}} h(x),$$

(76)

continuity of scalar multiplication follows.

(ii) For this, first we have to show that

$$\mathcal{W}^d_\alpha(\alpha) \subset \mathcal{W}^d_\alpha(\alpha).$$

(77)

Let $x = (x_{jk}) \in \mathcal{W}^d_\alpha(\alpha)$. Then, there exist integers $M$, $N$ such that

$$\sum_{p=M}^{\infty} \sum_{q=N}^{\infty} |\xi_{p,q}(x) - \xi_{p-1,q-1}(x) - \xi_{p-1,q}(x)|^{\alpha_{pq}} < 1.$$ 

(78)

By an argument similar to Theorem 4(iii), we obtain (77). Since $\mathcal{W}^d_\alpha(\alpha)$ and $\mathcal{W}^d_\alpha(\alpha)$ are complete with the same metric, we have (ii). $\square$

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