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Research Article

Some Spaces of Double Sequences Obtained through Invariant Mean and Related Concepts

S. A. Mohiuddine and Abdullah Alotaibi

Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Correspondence should be addressed to S. A. Mohiuddine; mohiuddine@gmail.com

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We introduce some double sequences spaces involving the notions of invariant mean (or σ-mean) and σ-convergence for double sequences while the idea of σ-convergence for double sequences was introduced by Çakan et al. 2006, by using the notion of invariant mean. We determine here some inclusion relations and topological results for these new double sequence spaces.

1. Preliminaries, Background, and Notation

In 1900, Pringsheim [1] presented the following notion of convergence for double sequences. A double sequence $x = (x_{jk})$ is said to *converge to the limit L in Pringsheim's sense* (shortly, *p-convergent to L*) if for every $\varepsilon > 0$ there exists an integer N such that $|x_{jk} - L| < \varepsilon$ whenever $j, k > N$. In this case, L is called the *p-limit* of x.

A double sequence $x = (x_{jk})$ of real or complex numbers is said to be *bounded* if $\|x\|_\infty = \sup_{j,k}|x_{jk}| < \infty$. The space of all bounded double sequences is denoted by $\mathscr{M}_u$.

If $x \in \mathscr{M}_u$ and is p-convergent to L, then x is said to be *boundedly p-convergent to L* (shortly, *bp-convergent to L*). In this case, L is called the *bp-limit* of x. The assumption of p-convergent was made because a double sequence on which p-convergent is not necessarily bounded.

In general, for any notion of convergence ν, the space of all ν-convergent double sequences will be denoted by $\mathscr{C}_\nu$, the space of all ν-convergent to 0 double sequences by $\mathscr{C}_{\nu 0}$ and the limit of a ν-convergent double sequence x by $\nu\text{-}\lim_{j,k} x_{jk}$, where $\nu \in \{p, bp\}$.

Let Ω denote the vector space of all double sequences with the vector space operations defined coordinatewise. Vector subspaces of Ω are called *double sequence spaces*. In addition to the above-mentioned double sequence spaces, we consider the double sequence space as

$$\mathscr{L}_u := \left\{ x \in \Omega \mid \|x\|_1 := \sum_{j,k} \left| x_{jk} \right| < \infty \right\} \tag{1}$$

of all absolutely summable double sequences.

All considered double sequence spaces are supposed to contain

$$\Phi := \operatorname{span}\left\{ \mathbf{e}^{\mathbf{jk}} \mid j, k \in \mathbb{N} \right\}, \tag{2}$$

where

$$\mathbf{e}_{\mathbf{il}}^{\mathbf{jk}} = \begin{cases} 1; & \text{if } (j,k) = (i,\ell), \\ 0; & \text{otherwise.} \end{cases} \tag{3}$$

We denote the pointwise sums $\sum_{j,k} \mathbf{e}^{\mathbf{jk}}$, $\sum_j \mathbf{e}^{\mathbf{jk}}$ $(k \in \mathbb{N})$, and $\sum_k \mathbf{e}^{\mathbf{jk}}$ $(j \in \mathbb{N})$ by $\mathbf{e}$, $\mathbf{e}^{\mathbf{k}}$, and $\mathbf{e}_{\mathbf{j}}$, respectively.

Let σ be a one-to-one mapping from the set $\mathbb{N}$ of natural numbers into itself. A continuous linear functional φ on the space ℓ_∞ of bounded single sequences is said to be an *invariant mean* or a *σ-mean* if and only if (i) $\varphi(x) \geq 0$ when

the sequence $x = (x_k)$ has $x_k \geq 0$ for all k, (ii) $\varphi(e) = 1$, where $e = (1, 1, 1, \ldots)$, and (iii) $\varphi(x) = \varphi((x_{\sigma(k)}))$ for all $x \in \ell_\infty$.

Throughout this paper we consider the mapping σ which has no finite orbits, that is, $\sigma^p(k) \neq k$ for all integer $k \geq 0$ and $p \geq 1$, where $\sigma^p(k)$ denotes the pth iterate of σ at k. Note that a σ-mean extends the limit functional on the space c of convergent single sequences in the sense that $\varphi(x) = \lim x$ for all $x \in c$, (see [2]). Consequently, $c \subset V_\sigma$ the set of bounded sequences all of whose σ-means are equal. We say that a sequence $x = (x_k)$ is *σ-convergent* if and only if $x \in V_\sigma$. Using this concept, Schaefer [3] defined and characterized σ-conservative, σ-regular, and σ-coercive matrices for single sequences. If σ is translation then V_σ is reduced to the set f of almost convergent sequences [4]. Recently, Mohiuddine [5] has obtained an application of almost convergence for single sequences in approximation theorems and proved some related results.

In 2006, Çakan et al. [6] presented the following definition of σ-convergence for double sequences and further studied by Mursaleen and Mohiuddine [7–9]. A double sequence $x = (x_{jk})$ of real numbers is said to be *σ-convergent* to a number L if and only if $x \in \mathscr{V}_\sigma$, where

$$\mathscr{V}_\sigma = \left\{x \in \mathscr{M}_u : \lim_{p,q\to\infty} \zeta_{pqst}(x) = L \text{ uniformly in } s,t; \; L = \sigma\text{-}\lim x\right\},$$

$$\zeta_{pqst}(x) = \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{\sigma^j(s),\sigma^k(t)}, \tag{4}$$

while here the limit means bp-limit. Let us denote by $\mathscr{V}_\sigma$ the space of σ-convergent double sequences $x = (x_{jk})$. For $\sigma(n) = n+1$, the set $\mathscr{V}_\sigma$ is reduced to the set $\mathscr{F}$ of almost convergent double sequences [10]. Note that $\mathscr{C}_{bp} \subset \mathscr{V}_\sigma \subset \mathscr{M}_u$.

Maddox [11] has defined the concepts of strong almost convergence and M-convergence for single sequences and established inclusion relation between strong almost convergence, M-convergence, and almost convergence for single sequence. Başarir [12] extended the notion of strong almost convergence from single sequences to double sequences and proved some interesting results involving this idea and the notion of almost convergence for double sequences. In the recent past, Mursaleen and Mohiuddine [13] presented the notions of absolute and strong σ-convergence for double sequences. A bounded double sequence $x = (x_{jk})$ is said to be *strongly σ-convergent* if there exists a number ℓ such that

$$\frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} \left|x_{\sigma^j(s),\sigma^k(t)} - \ell\right| \longrightarrow 0 \quad \text{as } p,q \longrightarrow \infty \text{ uniformly in } s,t, \tag{5}$$

while here the limit means bp-limit. In this case, we write $[\mathscr{V}_\sigma]$-$\lim x = \ell$. Let us denote by $[\mathscr{V}_\sigma]$ the set of all strongly σ-convergent sequences $x = (x_{jk})$. If σ is translation then $[\mathscr{V}_\sigma]$ is reduced to the set $[\mathscr{F}]$ of strong almost convergence double sequences due to Başarir [12].

For more details of spaces for single and double sequences and related concepts, we refer to [14–31] and references therein.

In this paper, we define and study some new spaces involving the idea of invariant mean and σ-convergence for double sequences and establish a relation between these spaces. Further, we extend above spaces to more general spaces by considering the double sequences $\alpha = (\alpha_{jk})$ such that $\alpha_{jk} > 0$ for all j, k and $\sup_{j,k}\alpha_{jk} = H < \infty$ and prove some topological results.

2. The Double Sequence Spaces

We construct the following spaces involving the idea of invariant mean and σ-convergence for double sequences:

$$\begin{aligned}
\mathscr{W}_\sigma &= \left\{x = (x_{jk}) : \frac{1}{(m+1)(n+1)} \times \sum_{p=0}^{m}\sum_{q=0}^{n} \zeta_{pqst}(x - \ell E) \longrightarrow 0 \text{ as } m,n \longrightarrow \infty, \right. \\
&\qquad \left. \text{uniformly in } s,t, \text{ for some } \ell\right\}, \\
[\mathscr{W}_\sigma] &= \left\{x = (x_{jk}) : \frac{1}{(m+1)(n+1)} \times \sum_{p=0}^{m}\sum_{q=0}^{n} \left|\zeta_{pqst}(x - \ell E)\right| \longrightarrow 0 \text{ as } m,n \longrightarrow \infty, \right. \\
&\qquad \left. \text{uniformly in } s,t, \text{ for some } \ell\right\}, \\
\mathscr{W}'_\sigma &= \left\{x = (x_{jk}) : \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \left|\xi_{pqst} - \xi_{p-1,q,s,t} - \xi_{p,q-1,s,t} + \xi_{p-1,q-1,s,t}\right| \right. \\
&\qquad \left. \text{converges uniformly in } s,t\right\}, \\
\mathscr{W}''_\sigma &= \left\{x = (x_{jk}) : \sup_{s,t} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \left|\xi_{pqst} - \xi_{p-1,q,s,t} - \xi_{p,q-1,s,t} \right.\right. \\
&\qquad \left.\left. + \xi_{p-1,q-1,s,t}\right| < \infty\right\},
\end{aligned} \tag{6}$$

where $E = (e_{jk})$ with $e_{jk} = 1$ for all j,k;

$$\xi_{mnst} = \xi_{mnst}(x) = \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m}\sum_{q=0}^{n} \zeta_{pqst}(x),$$

$$\begin{aligned}\xi_{0,0,s,t}(x) &= \zeta_{0,0,s,t} = x_{s,t},\\ \xi_{-1,0,s,t}(x) &= \zeta_{-1,0,s,t}(x) = x_{s-1,t},\\ \xi_{0,-1,s,t}(x) &= \zeta_{0,-1,s,t}(x) = x_{s,t-1},\\ \xi_{-1,-1,s,t}(x) &= \zeta_{-1,-1,s,t}(x) = x_{s-1,t-1}.\end{aligned} \tag{7}$$

Remark 1. If $[\mathscr{W}_\sigma]$-$\lim x = \ell$, that is,

$$\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\left|\zeta_{pqst}-\ell\right| \longrightarrow 0 \tag{8}$$

as $m, n \to \infty$, uniformly in s, t; then

$$\begin{aligned}&\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\left|\frac{1}{p+1}\sum_{j=0}^{p}\zeta_{jqst}-\ell\right| \longrightarrow 0,\\ &\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\left|\frac{1}{q+1}\sum_{k=0}^{q}\zeta_{pkst}-\ell\right| \longrightarrow 0.\end{aligned} \tag{9}$$

We remark that by using Abel's transformation for single series

$$\sum_{i=1}^{m} v_i\left(u_i \mp u_{i+1}\right) = \sum_{i=1}^{m} u_i\left(v_i \mp v_{i-1}\right) \mp u_{m+1}v_m. \tag{10}$$

We get Abel's transformation for double series

$$\begin{aligned}&\sum_{j=1}^{p}\sum_{k=1}^{q} v_{jk}\left(u_{jk}-u_{j+1,k}-u_{j,k+1}+u_{j+1,k+1}\right)\\ &\quad= \sum_{j=1}^{p}\sum_{k=1}^{q} u_{jk}\left(\Delta_{11}v_{jk}\right) - \sum_{j=1}^{p} u_{j,q+1}\left(\Delta_{10}v_{jq}\right)\\ &\quad- \sum_{k=1}^{q} u_{p+1,k}\left(\Delta_{01}v_{pk}\right) + u_{p+1,q+1}v_{pq},\end{aligned} \tag{11}$$

where

$$\begin{aligned}\Delta_{10}v_{jq} &= v_{jq}-v_{j-1,q}, \qquad \Delta_{01}v_{pk} = v_{pk}-v_{p,k-1}\\ \Delta_{11}v_{jk} &= v_{jk}-v_{j-1,k}-v_{j,k-1}+v_{j-1,k-1}.\end{aligned} \tag{12}$$

In the recent past, Altay and Başar [32] also presented another form of Abel's transformation for double series.

3. Inclusion Relations

In the following theorem, we establish a relationship between spaces defined in Section 2. Before proceeding further, first we prove the following lemmas which will be used to prove our inclusion relations.

Lemma 2. *Consider that* $[\mathscr{W}_\sigma]$-$\lim x = \ell$ *if and only if*

(L1) $\mathscr{W}_\sigma$-$\lim x = \ell$;

(L2) $(1/uv)\sum_{m=1}^{u}\sum_{n=1}^{v}|\Omega^{\dagger}_{mnst} - \ell| \to 0$ *as* $u, v \to \infty$ *(uniformly in s, t);*

(L3) $(1/uv)\sum_{m=1}^{u}\sum_{n=1}^{v}|\Omega^{\ddagger}_{mnst} - \ell| \to 0$ *as* $u, v \to \infty$ *(uniformly in s, t);*

(L4) $(1/uv)\sum_{m=1}^{u}\sum_{n=1}^{v}|\zeta_{mnst} + \xi_{mnst} - \Omega^{\dagger}_{mnst} - \Omega^{\ddagger}_{mnst}| \to 0$ *as* $u, v \to \infty$ *(uniformly in s, t),*

where

$$\begin{aligned}\Omega^{\dagger}_{mnst} &= \frac{1}{(m+1)}\sum_{p=0}^{m}\zeta_{pnst},\\ \Omega^{\ddagger}_{mnst} &= \frac{1}{(n+1)}\sum_{q=0}^{n}\zeta_{mqst}.\end{aligned} \tag{13}$$

Proof. Suppose that $[\mathscr{W}_\sigma]$-$\lim x = \ell$. Thus, we have $\mathscr{W}_\sigma$-$\lim x = \ell$, that is, (L1) holds. We see that conditions (L2) and (L3) follows from the Remark 1. Write

$$\begin{aligned}&\frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v}\left|\zeta_{mnst}+\xi_{mnst}-\Omega^{\dagger}_{mnst}-\Omega^{\ddagger}_{mnst}\right|\\ &\quad= \frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v}\left|\zeta_{mnst}-\ell+\xi_{mnst}-\ell-\Omega^{\dagger}_{mnst}+\ell-\Omega^{\ddagger}_{mnst}+\ell\right|\\ &\quad\le \frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v}\left|\zeta_{mnst}-\ell\right| + \frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v}\left|\xi_{mnst}-\ell\right|\\ &\qquad+ \frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v}\left|\Omega^{\dagger}_{mnst}-\ell\right| + \frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v}\left|\Omega^{\ddagger}_{mnst}-\ell\right|\\ &\quad= \Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4 \quad (\text{say}).\end{aligned} \tag{14}$$

By our assumption, that is, $[\mathscr{W}_\sigma]$-$\lim x = \ell$, $\Sigma_1 \to 0$ as $u, v \to \infty$ uniformly in s, t. The condition (L1) implies that ξ_{mnst} tends to zero as m, n tending to ∞ uniformly in s, t; therefore $\Sigma_2 \to 0$ as $u, v \to \infty$ uniformly in s, t and $\Sigma_3, \Sigma_4 \to 0$ as $u, v \to \infty$ uniformly in s, t by the conclusion (L2) and (L3), respectively. Thus, (14) tends to zero as $u, v \to \infty$ uniformly in s, t, that is, (L4) holds.

Conversely, let (L1)–(L4) hold. Then,

$$\begin{aligned}&\frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v}\left|\zeta_{mnst}-\ell\right|\\ &\quad\le \frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v}\left|\zeta_{mnst}+\xi_{mnst}-\Omega^{\dagger}_{mnst}-\Omega^{\ddagger}_{mnst}\right|\\ &\qquad+ \frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v}\left|\xi_{mnst}-\ell\right|\\ &\qquad+ \frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v}\left|\Omega^{\dagger}_{mnst}-\ell\right|\\ &\qquad+ \frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v}\left|\Omega^{\ddagger}_{mnst}-\ell\right|\end{aligned} \tag{15}$$

$\to 0$ $(u, v \to \infty)$ uniformly in s, t. □

Lemma 3. *One has*

$$\zeta_{mnst}+\xi_{mnst}-\Omega^{\dagger}_{mnst}-\Omega^{\ddagger}_{mnst} = mn\left[\xi_{mnst}-\xi_{m-1,n,s,t}-\xi_{m,n-1,s,t}+\xi_{m-1,n-1,s,t}\right]. \tag{16}$$

Proof. Since

$$\begin{aligned}\xi_{mnst}&-\xi_{m-1,n,s,t}-\xi_{m,n-1,s,t}+\xi_{m-1,n-1,s,t}\\ &=\left[\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\zeta_{pqst}-\frac{1}{m(n+1)}\sum_{p=0}^{m-1}\sum_{q=0}^{n}\zeta_{pqst}\right]\\ &\quad-\left[\frac{1}{(m+1)n}\sum_{p=0}^{m}\sum_{q=0}^{n-1}\zeta_{pqst}-\frac{1}{mn}\sum_{p=0}^{m-1}\sum_{q=0}^{n-1}\zeta_{pqst}\right].\end{aligned} \tag{17}$$

First, we solve the expression in the first bracket

$$\begin{aligned}&\left[\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\zeta_{pqst}-\frac{1}{m(n+1)}\sum_{p=0}^{m-1}\sum_{q=0}^{n}\zeta_{pqst}\right]\\ &=\frac{1}{m(m+1)(n+1)}\\ &\quad\times\left[\sum_{q=0}^{n}\left(m\sum_{p=0}^{m}\zeta_{pqst}-(m+1)\sum_{p=0}^{m-1}\zeta_{pqst}\right)\right]\\ &=\frac{1}{m(m+1)(n+1)}\sum_{q=0}^{n}\left[m\zeta_{mqst}-\sum_{p=0}^{m-1}\zeta_{pqst}\right]\\ &=\frac{1}{m(m+1)(n+1)}\sum_{q=0}^{n}\left[(m+1)\zeta_{mqst}-\sum_{p=0}^{m}\zeta_{pqst}\right]\\ &=\frac{1}{m(n+1)}\sum_{q=0}^{n}\left[\zeta_{mqst}-\frac{1}{(m+1)}\sum_{p=0}^{m}\zeta_{pqst}\right]\\ &=\frac{1}{m(n+1)}\sum_{q=0}^{n}\zeta_{mqst}-\frac{1}{m(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\zeta_{pqst}\\ &=\frac{1}{m(n+1)}\sum_{q=0}^{n}\zeta_{mqst}-\frac{1}{m}\xi_{mnst}.\end{aligned} \tag{18}$$

Now, the expression in the second bracket

$$\begin{aligned}&\frac{1}{(m+1)n}\sum_{p=0}^{m}\sum_{q=0}^{n-1}\zeta_{pqst}-\frac{1}{mn}\sum_{p=0}^{m-1}\sum_{q=0}^{n-1}\zeta_{pqst}\\ &=\frac{1}{n}\sum_{q=0}^{n-1}\left[\frac{1}{m(m+1)}\left\{m\sum_{p=0}^{m}\zeta_{pqst}-(m+1)\sum_{p=0}^{m-1}\zeta_{pqst}\right\}\right]\\ &=\frac{1}{n}\sum_{q=0}^{n-1}\left[\frac{1}{m(m+1)}\left\{m\zeta_{mqst}-\sum_{p=0}^{m-1}\zeta_{pqst}\right\}\right]\\ &=\frac{1}{n}\sum_{q=0}^{n-1}\left[\frac{1}{m(m+1)}\left\{(m+1)\zeta_{mqst}-\sum_{p=0}^{m}\zeta_{pqst}\right\}\right]\\ &=\frac{1}{mn}\sum_{q=0}^{n-1}\zeta_{mqst}-\frac{1}{m(m+1)n}\sum_{p=0}^{m}\sum_{q=0}^{n-1}\zeta_{pqst}\\ &=\frac{1}{mn}\sum_{q=0}^{n-1}\zeta_{mqst}-\frac{1}{m}\xi_{m,n-1,s,t}.\end{aligned} \tag{19}$$

Substituting (18) and (19) in (17), we get

$$\begin{aligned}\xi_{mnst}&-\xi_{m-1,n,s,t}-\xi_{m,n-1,s,t}+\xi_{m-1,n-1,s,t}\\ &=\frac{1}{m(n+1)}\sum_{q=0}^{n}\zeta_{mqst}-\frac{1}{mn}\sum_{q=0}^{n-1}\zeta_{mqst}-\frac{1}{m}\xi_{mnst}\\ &\quad+\frac{1}{m}\xi_{m,n-1,s,t}\\ &=\frac{1}{mn(n+1)}\left[n\sum_{q=0}^{n}\zeta_{mqst}-(n+1)\sum_{q=0}^{n-1}\zeta_{mqst}\right]\\ &\quad-\frac{1}{m}\left(\xi_{mnst}-\xi_{m,n-1,s,t}\right)\\ &=\frac{1}{mn(n+1)}\left[n\zeta_{mnst}-\sum_{q=0}^{n-1}\zeta_{mqst}\right]\\ &\quad-\frac{1}{m}\left(\xi_{mnst}-\xi_{m,n-1,s,t}\right)\\ &=\frac{1}{mn(n+1)}\left[(n+1)\zeta_{mnst}-\sum_{q=0}^{n}\zeta_{mqst}\right]\\ &\quad-\frac{1}{m}\left(\xi_{mnst}-\xi_{m,n-1,s,t}\right)\\ &=\frac{1}{mn}\left[\zeta_{mnst}-\frac{1}{(n+1)}\sum_{q=0}^{n}\zeta_{mqst}\right]\\ &\quad-\frac{1}{m}\left(\xi_{mnst}-\xi_{m,n-1,s,t}\right).\end{aligned} \tag{20}$$

We know that

$$\begin{aligned}\xi_{mnst}&=\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\zeta_{pqst}\\ &=\frac{1}{(m+1)(n+1)}\left[\sum_{p=0}^{m-1}\sum_{q=0}^{n}\zeta_{pqst}+\sum_{q=0}^{n}\zeta_{mqst}\right],\\ \xi_{m-1,n,s,t}&=\frac{1}{m(n+1)}\sum_{p=0}^{m-1}\sum_{q=0}^{n}\zeta_{pqst}.\end{aligned} \tag{21}$$

From (21), we have

$$(m+1)\,\xi_{mnst} - m\xi_{m-1,n,s,t} = \frac{1}{(n+1)}\sum_{q=0}^{n}\zeta_{mqst}. \tag{22}$$

Thus, (20) becomes

$$\begin{aligned}
&\xi_{mnst} - \xi_{m-1,n,s,t} - \xi_{m,n-1,s,t} + \xi_{m-1,n-1,s,t} \\
&\quad = \frac{1}{mn}\left[\zeta_{mnst} - (m+1)\,\xi_{mnst} + m\xi_{m-1,n,s,t}\right] \\
&\qquad - \frac{1}{m}\left(\xi_{mnst} - \xi_{m,n-1,s,t}\right) \\
&\quad = \frac{1}{mn}\left[\zeta_{mnst} - \xi_{mnst} - m\left(\xi_{mnst} - \xi_{m-1,n,s,t}\right)\right. \\
&\qquad \left. -n\left(\xi_{mnst} - \xi_{m,n-1,s,t}\right)\right].
\end{aligned} \tag{23}$$

Also (22) can be written as

$$m\left(\xi_{mnst} - \xi_{m-1,n,s,t}\right) = \frac{1}{(n+1)}\sum_{q=0}^{n}\zeta_{mqst} - \xi_{mnst}. \tag{24}$$

Similarly, we can write

$$n\left(\xi_{mnst} - \xi_{m,n-1,s,t}\right) = \frac{1}{(m+1)}\sum_{q=0}^{m}\zeta_{pnst} - \xi_{mnst}. \tag{25}$$

Using (24) and (25) in (23), we get

$$\begin{aligned}
&\xi_{mnst} - \xi_{m-1,n,s,t} - \xi_{m,n-1,s,t} + \xi_{m-1,n-1,s,t} \\
&\quad = \frac{1}{mn}\left[\zeta_{mnst} + \xi_{mnst} - \frac{1}{(m+1)}\sum_{p=0}^{m}\zeta_{pnst} - \frac{1}{(n+1)}\sum_{q=0}^{n}\zeta_{mqst}\right].
\end{aligned} \tag{26}$$

This implies that

$$\begin{aligned}
&\zeta_{mnst} + \xi_{mnst} - \Omega^{\dagger}_{mnst} - \Omega^{\ddagger}_{mnst} \\
&\quad = mn\left[\xi_{mnst} - \xi_{m-1,n,s,t} - \xi_{m,n-1,s,t} + \xi_{m-1,n-1,s,t}\right].
\end{aligned} \tag{27}$$

□

Theorem 4. *One has the following inclusions and the limit is preserved in each case:*

(i) $[\mathscr{V}_\sigma] \subset [\mathscr{W}_\sigma] \subset \mathscr{W}_\sigma$,

(ii) $\mathscr{W}'_\sigma \subset [\mathscr{W}_\sigma]$ *if the conditions (L2) and (L3) of Lemma 2 hold,*

(iii) $\mathscr{W}'_\sigma \subset \mathscr{W}''_\sigma$.

Proof. (i) Let $x \in [\mathscr{V}_\sigma]$ with $[\mathscr{V}_\sigma]$-$\lim x = \ell$, say. Then,

$$\zeta_{pqst}\left(|x-\ell|\right) \longrightarrow 0 \quad \text{as } p,q \longrightarrow \infty, \text{ uniformly in } s,t. \tag{28}$$

This implies that

$$\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\zeta_{pqst}\left(|x-\ell|\right) \longrightarrow 0 \quad \text{as } p,q \longrightarrow \infty, \text{ uniformly in } s,t. \tag{29}$$

Also, we have

$$\begin{aligned}
&\frac{1}{(m+1)(n+1)}\left|\sum_{p=0}^{m}\sum_{q=0}^{n}\zeta_{pqst}\left(x-\ell\right)\right| \\
&\quad \le \frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\left|\zeta_{pqst}\left(x-\ell\right)\right| \\
&\quad \le \frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\zeta_{pqst}\left(|x-\ell|\right).
\end{aligned} \tag{30}$$

Hence, $[\mathscr{V}_\sigma] \subset [\mathscr{W}_\sigma] \subset \mathscr{W}_\sigma$ and

$$[\mathscr{W}_\sigma]\text{-}\lim x = \mathscr{W}_\sigma\text{-}\lim x = \ell. \tag{31}$$

(ii) We have to show that $\mathscr{W}'_\sigma \subset [\mathscr{W}_\sigma]$. If $x \in \mathscr{W}'_\sigma$, then we have

$$\begin{aligned}
\mathfrak{T}_{mnst} &= \sum_{p=m}^{\infty}\sum_{q=n}^{\infty}\left|\xi_{pqst} - \xi_{p-1,q,s,t} - \xi_{p,q-1,s,t} + \xi_{p-1,q-1,s,t}\right| \\
&\longrightarrow 0
\end{aligned} \tag{32}$$

as $m,n \to \infty$, uniformly in s, t; and

$$\xi_{pqst} \longrightarrow \ell \quad (\text{say}) \text{ as } p,q \longrightarrow \infty, \text{uniformly in } s,t, \tag{33}$$

that is, $\mathscr{W}_\sigma$-$\lim x = \ell$.

In order to prove that $x \in [\mathscr{W}_\sigma]$, it is enough to show that condition (L4) of Lemma 2 holds. Now,

$$\begin{aligned}
&\mathfrak{T}_{mnst} - \mathfrak{T}_{m,n+1,s,t} - \mathfrak{T}_{m+1,n,s,t} + \mathfrak{T}_{m+1,n+1,s,t} \\
&\quad = \sum_{p=m}^{\infty}\sum_{q=n}^{\infty}\left|\xi_{pqst} - \xi_{p-1,q,s,t} - \xi_{p,q-1,s,t} + \xi_{p-1,q-1,s,t}\right| \\
&\qquad - \sum_{p=m}^{\infty}\sum_{q=n+1}^{\infty}\left|\xi_{pqst} - \xi_{p-1,q,s,t} - \xi_{p,q-1,s,t} + \xi_{p-1,q-1,s,t}\right| \\
&\qquad - \sum_{p=m+1}^{\infty}\sum_{q=n}^{\infty}\left|\xi_{pqst} - \xi_{p-1,q,s,t} - \xi_{p,q-1,s,t} + \xi_{p-1,q-1,s,t}\right| \\
&\qquad + \sum_{p=m+1}^{\infty}\sum_{q=n+1}^{\infty}\left|\xi_{pqst} - \xi_{p-1,q,s,t} - \xi_{p,q-1,s,t} + \xi_{p-1,q-1,s,t}\right| \\
&\quad = \sum_{p=m}^{\infty}\left[\sum_{q=n}^{\infty} - \sum_{q=n+1}^{\infty}\right] \\
&\qquad \times \left|\xi_{pqst} - \xi_{p-1,q,s,t} - \xi_{p,q-1,s,t} + \xi_{p-1,q-1,s,t}\right| \\
&\qquad - \sum_{p=m+1}^{\infty}\left[\sum_{q=n}^{\infty} - \sum_{q=n+1}^{\infty}\right] \\
&\qquad \times \left|\xi_{pqst} - \xi_{p-1,q,s,t} - \xi_{p,q-1,s,t} + \xi_{p-1,q-1,s,t}\right|
\end{aligned}$$

$$= \sum_{p=m}^{\infty} \left|\xi_{pnst} - \xi_{p-1,n,s,t} - \xi_{p,n-1,s,t} + \xi_{p-1,n-1,s,t}\right| - \sum_{p=m+1}^{\infty} \left|\xi_{pnst} - \xi_{p-1,n,s,t} - \xi_{p,n-1,s,t} + \xi_{p-1,n-1,s,t}\right|$$
$$= \left[\sum_{p=m}^{\infty} - \sum_{p=m+1}^{\infty}\right] \left|\xi_{pnst} - \xi_{p-1,n,s,t} - \xi_{p,n-1,s,t} + \xi_{p-1,n-1,s,t}\right|$$
$$= \left|\xi_{mnst} - \xi_{m-1,n,s,t} - \xi_{m,n-1,s,t} + \xi_{m-1,n-1,s,t}\right|. \tag{34}$$

Replacing m and n by p and q, respectively, we have

$$\mathfrak{I}_{pqst} - \mathfrak{I}_{p,q+1,s,t} - \mathfrak{I}_{p+1,q,s,t} + \mathfrak{I}_{p+1,q+1,s,t} = \left|\xi_{pqst} - \xi_{p-1,q,s,t} - \xi_{p,q-1,s,t} + \xi_{p-1,q-1,s,t}\right|. \tag{35}$$

By Lemma 3, we have

$$\zeta_{pqst} + \xi_{pqst} - \Omega^{\dagger}_{pqst} - \Omega^{\ddagger}_{pqst} = pq\left[\xi_{pqst} - \xi_{p-1,q,s,t} - \xi_{p,q-1,s,t} + \xi_{p-1,q-1,s,t}\right]. \tag{36}$$

So that we have

$$\frac{1}{mn}\sum_{p=1}^{m}\sum_{q=1}^{n}\left|\zeta_{pqst} + \xi_{pqst} - \Omega^{\dagger}_{pqst} - \Omega^{\ddagger}_{pqst}\right| = \frac{1}{mn}\sum_{p=1}^{m}\sum_{q=1}^{n} pq\left[\mathfrak{I}_{pqst} - \mathfrak{I}_{p,q+1,s,t} - \mathfrak{I}_{p+1,q,s,t} + \mathfrak{I}_{p+1,q+1,s,t}\right]. \tag{37}$$

By using Abel's transformation for double series in the right hand side of above equation, we have

$$\frac{1}{mn}\sum_{p=1}^{m}\sum_{q=1}^{n}\left|\zeta_{pqst} + \xi_{pqst} - \Omega^{\dagger}_{pqst} - \Omega^{\ddagger}_{pqst}\right| = \frac{1}{mn}\left[\sum_{p=1}^{m}\sum_{q=1}^{n}\mathfrak{I}_{pqst} - m\sum_{q=1}^{n}\mathfrak{I}_{m+1,q,s,t} - n\sum_{p=1}^{m}\mathfrak{I}_{p,n+1,s,t} + mn\mathfrak{I}_{m+1,n+1,s,t}\right] \tag{38}$$

$\to 0$ as $m, n \to \infty$, uniformly in s, t (by (32)). Hence, by Lemma 2, $x \in [\mathscr{W}_{\sigma}]$.

(iii) Let $x \in \mathscr{W}'_{\sigma}$, and we have to show that

$$\sup_{s,t}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\left|\xi_{pqst} - \xi_{p-1,q,s,t} - \xi_{p,q-1,s,t} + \xi_{p-1,q-1,s,t}\right| \le K, \tag{39}$$

where K is an absolute constant. Since $x \in \mathscr{W}'_{\sigma}$, there exist integers p_0, q_0 such that

$$\sum_{p>p_0}\sum_{q>q_0}\left|\xi_{pqst} - \xi_{p-1,q,s,t} - \xi_{p,q-1,s,t} + \xi_{p-1,q-1,s,t}\right| < 1, \quad \forall s, t. \tag{40}$$

Hence, it is left to show that for fixed p, q

$$\left|\xi_{pqst} - \xi_{p-1,q,s,t} - \xi_{p,q-1,s,t} + \xi_{p-1,q-1,s,t}\right| \le K, \quad \forall s, t. \tag{41}$$

From (40), we have

$$\left|\xi_{pqst} - \xi_{p-1,q,s,t} - \xi_{p,q-1,s,t} + \xi_{p-1,q-1,s,t}\right| < 1, \tag{42}$$

for every fixed $p > p_0$, $q > q_0$ and for all s, t. Since

$$m(m+1)n(n+1)\left(\xi_{mnst} - \xi_{m-1,n,s,t} - \xi_{m,n-1,s,t} + \xi_{m-1,n-1,s,t}\right) = \sum_{p=1}^{m}\sum_{q=1}^{n} pq\left(\zeta_{pqst} - \zeta_{p-1,q,s,t} - \zeta_{p,q-1,s,t} + \zeta_{p-1,q-1,s,t}\right). \tag{43}$$

Accordingly,

$$\begin{aligned}
mn\Big[&(m+1)(n+1) \times \left(\xi_{mnst} - \xi_{m-1,n,s,t} - \xi_{m,n-1,s,t} + \xi_{m-1,n-1,s,t}\right) \\
&- (m-1)(n+1) \times \left(\xi_{m-1,n,s,t} - \xi_{m-2,n,s,t} - \xi_{m-1,n-1,s,t} + \xi_{m-2,n-1,s,t}\right) \\
&- (m+1)(n-1) \times \left(\xi_{m,n-1,s,t} - \xi_{m-1,n-1,s,t} - \xi_{m,n-2,s,t} + \xi_{m-1,n-2,s,t}\right) \\
&+ (m-1)(n-1) \times \left(\xi_{m-1,n-1,s,t} - \xi_{m-2,n-1,s,t} - \xi_{m-1,n-2,s,t} + \xi_{m-2,n-2,s,t}\right)\Big] \\
&= \sum_{q=1}^{n} q\left[\sum_{p=1}^{m} p\left(\zeta_{pqst} - \zeta_{p-1,q,s,t} - \zeta_{p,q-1,s,t} + \zeta_{p-1,q-1,s,t}\right) - \sum_{p=1}^{m-1} p\left(\zeta_{pqst} - \zeta_{p-1,q,s,t} - \zeta_{p,q-1,s,t} + \zeta_{p-1,q-1,s,t}\right)\right] \\
&- \sum_{q=1}^{n-1} q\left[\sum_{p=1}^{m} p\left(\zeta_{pqst} - \zeta_{p-1,q,s,t} - \zeta_{p,q-1,s,t} + \zeta_{p-1,q-1,s,t}\right) - \sum_{p=1}^{m-1} p\left(\zeta_{pqst} - \zeta_{p-1,q,s,t} - \zeta_{p,q-1,s,t} + \zeta_{p-1,q-1,s,t}\right)\right] \\
&= \sum_{q=1}^{n} q\left[m\left(\zeta_{mqst} - \zeta_{m-1,q,s,t} - \zeta_{m,q-1,s,t} + \zeta_{m-1,q-1,s,t}\right)\right] - \sum_{q=1}^{n-1} q\left[m\left(\zeta_{mqst} - \zeta_{m-1,q,s,t} - \zeta_{m,q-1,s,t} + \zeta_{m-1,q-1,s,t}\right)\right].
\end{aligned} \tag{44}$$

This implies that

$$\begin{aligned}
&(m+1)(n+1)(\xi_{mnst}-\xi_{m-1,n,s,t}-\xi_{m,n-1,s,t}+\xi_{m-1,n-1,s,t})\\
&\quad-(m-1)(n+1)\\
&\quad\times(\xi_{m-1,n,s,t}-\xi_{m-2,n,s,t}-\xi_{m-1,n-1,s,t}+\xi_{m-2,n-1,s,t})\\
&\quad-(m+1)(n-1)\\
&\quad\times(\xi_{m,n-1,s,t}-\xi_{m-1,n-1,s,t}-\xi_{m,n-2,s,t}+\xi_{m-1,n-2,s,t})\\
&\quad+(m-1)(n-1)\\
&\quad\times(\xi_{m-1,n-1,s,t}-\xi_{m-2,n-1,s,t}-\xi_{m-1,n-2,s,t}+\xi_{m-2,n-2,s,t})\\
&=[\zeta_{mnst}-\zeta_{m-1,n,s,t}-\zeta_{m,n-1,s,t}+\zeta_{m-1,n-1,s,t}].
\end{aligned}\tag{45}$$

Using (42) and (45), we have

$$|\zeta_{mnst}-\zeta_{m-1,n,s,t}-\zeta_{m,n-1,s,t}+\zeta_{m-1,n-1,s,t}|\le K(m,n)\tag{46}$$

for every fixed $m>p_0$, $n>q_0$ and for all s, t, where $K(m,n)$ is a constant depending upon m,n.

Now, for any given infinite double series $\sum_s\sum_t a_{st}$ denoted as "a", let us write

$$x_{jk}=\sum_{s=0}^{j}\sum_{t=0}^{k}a_{st},\quad j,k=1,2,\ldots\tag{47}$$

and σ be monotonically increasing. For simplicity in notation, we denote

$$\begin{aligned}
\phi_{mnst}&=\zeta_{mnst}(a)-\zeta_{m-1,n,s,t}(a)-\zeta_{m,n-1,s,t}(a)\\
&\quad+\zeta_{m-1,n-1,s,t}(a).
\end{aligned}\tag{48}$$

Again from the definition of ζ_{mnst}, it is easy to obtain

$$\phi_{mnst}=\frac{1}{m(m+1)n(n+1)}\sum_{j=1}^{m}\sum_{k=1}^{n}jk\left(\sum_{u=d_j}^{d'_j}\sum_{v=h_k}^{h'_k}a_{uv}\right)\tag{49}$$

for all $m,n\ge1$ and $\phi_{0,0,s,t}(a)=a_{st}$ with $d_j=\sigma^{j-1}(s)+1$, $d'_j=\sigma^j(s)$, $h_k=\sigma^{k-1}(t)+1$, $h'_k=\sigma^k(t)$. Further, we calculate

$$\begin{aligned}
&m(m+1)n(n+1)\phi_{mnst}-m(m-1)n(n+1)\phi_{m-1,n,s,t}\\
&\quad-m(m+1)n(n-1)\phi_{m,n-1,s,t}\\
&\quad+m(m-1)n(n-1)\phi_{m-1,n-1,s,t}\\
&=\left\{\sum_{j=1}^{m}\sum_{k=1}^{n}jk-\sum_{j=1}^{m-1}\sum_{k=1}^{n}jk-\sum_{j=1}^{m}\sum_{k=1}^{n-1}jk+\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}jk\right\}\\
&\quad\times\left(\sum_{u=d_j}^{d'_j}\sum_{v=h_k}^{h'_k}a_{uv}\right)\\
&=\left\{\sum_{k=1}^{n}k\left[\sum_{j=1}^{m}j-\sum_{j=1}^{m-1}j\right]-\sum_{k=1}^{n-1}k\left[\sum_{j=1}^{m}j+\sum_{j=1}^{m-1}j\right]\right\}\\
&\quad\times\left(\sum_{u=d_j}^{d'_j}\sum_{v=h_k}^{h'_k}a_{uv}\right)\\
&=\left\{m\sum_{k=1}^{n}k-m\sum_{k=1}^{n-1}k\right\}\left(\sum_{u=d_m}^{d'_m}\sum_{v=h_k}^{h'_k}a_{uv}\right)\\
&=mn\left(\sum_{u=d_m}^{d'_m}\sum_{v=h_n}^{h'_n}a_{uv}\right).
\end{aligned}\tag{50}$$

Thus, we have

$$\begin{aligned}
\sum_{u=d_m}^{d'_m}\sum_{v=h_n}^{h'_n}a_{uv}&=(m+1)(n+1)\phi_{m,n,s,t}\\
&\quad-(m-1)(n+1)\phi_{m-1,n,s,t}\\
&\quad-(m+1)(n-1)\phi_{m,n-1,s,t}\\
&\quad+(m-1)(n-1)\phi_{m-1,n-1,s,t}.
\end{aligned}\tag{51}$$

Hence, it follows from (46) that for each fixed $m>p_0$, $n>q_0$,

$$\left|\sum_{u=d_m}^{d'_m}\sum_{v=h_n}^{h'_n}a_{uv}\right|\le K(m,n),\quad\forall s,t.\tag{52}$$

Hence, it follows from (52) that

$$|a_{uv}|\le K\quad\forall u,v,\tag{53}$$

where K is independent of u,v. By (49), we have

$$\begin{aligned}
&|\zeta_{mnst}-\zeta_{m-1,n,s,t}-\zeta_{m,n-1,s,t}+\zeta_{m-1,n-1,s,t}|\\
&\quad\le K,\quad\forall m,n,s,t.
\end{aligned}\tag{54}$$

Also from (43) and (54), we have

$$|\xi_{pqst}-\xi_{p-1,q,s,t}-\xi_{p,q-1,s,t}+\xi_{p-1,q-1,s,t}|\le K,\quad\forall p,q,s,t.\tag{55}$$

□

4. Topological Results

Here, we extend the spaces $[\mathscr{W}_\sigma]$, $\mathscr{W}'_\sigma$, $\mathscr{W}''_\sigma$ to more general spaces, respectively, denoted by $[\mathscr{W}_\sigma(\alpha)]$, $\mathscr{W}'_\sigma(\alpha)$, $\mathscr{W}''_\sigma(\alpha)$,

where $\alpha = (\alpha_{jk})$ is a double sequence of positive real numbers for all j, k and $\sup_{j,k}\alpha_{jk} = H < \infty$. First, we recall the notion of paranorm as follows.

A *paranorm* is a function $g : X \to \mathbb{R}$ defined on a linear space X such that for all $x, y, z \in X$

(P1) $g(x) = 0$ if $x = \theta$

(P2) $g(-x) = g(x)$

(P3) $g(x + y) \leq g(x) + g(y)$

(P4) If (α_n) is a sequence of scalars with $\alpha_n \to \alpha_0$ $(n \to \infty)$ and $x_n, a \in X$ with $x_n \to a$ $(n \to \infty)$ in the sense that $g(x_n - a) \to 0$ $(n \to \infty)$, then $\alpha_n x_n \to \alpha_0 a$ $(n \to \infty)$, in the sense that $g(\alpha_n x_n - \alpha_0 a) \to 0$ $(n \to \infty)$.

A paranorm g for which $g(x) = 0$ implies $x = \theta$ is called a *total paranorm* on X, and the pair (X, g) is called a *total paranormed space*. Note that each seminorm (norm) p on X is a paranorm (total) but converse needs not be true.

A paranormed space (X, g) is a topological linear space with the topology given by the paranorm g.

Now, we define the following spaces:

$$\begin{aligned}
[\mathscr{W}_\sigma(\alpha)] = \Bigg\{ & x = (x_{jk}) : \frac{1}{(m+1)(n+1)} \\
& \times \sum_{p=0}^{m}\sum_{q=0}^{n} \left|\zeta_{pqst}(x - \ell E)\right|^{\alpha_{pq}} \longrightarrow 0, \\
& \text{as } m, n \longrightarrow \infty, \text{ uniformly in } s, t \Bigg\}, \\
\mathscr{W}'_\sigma(\alpha) = \Bigg\{ & x = (x_{jk}) : \\
& \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \left|\xi_{pqst} - \xi_{p-1,q,s,t} - \xi_{p,q-1,s,t} + \xi_{p-1,q-1,s,t}\right|^{\alpha_{pq}} \\
& \text{converges uniformly in } s, t \Bigg\}, \\
\mathscr{W}''_\sigma(\alpha) = \Bigg\{ & x = (x_{jk}) : \\
& \sup_{s,t} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \left|\xi_{pqst} - \xi_{p-1,q,s,t} - \xi_{p,q-1,s,t} + \xi_{p-1,q-1,s,t}\right|^{\alpha_{pq}} \\
& < \infty \Bigg\}.
\end{aligned} \tag{56}$$

We remark that if $\alpha = (\alpha_{jk})$ is a constant sequence, then we write $[\mathscr{W}_{\sigma,\alpha}]$ in place of $[\mathscr{W}_\sigma(\alpha)]$.

Theorem 5. *Let $\alpha = (\alpha_{pq})$ be a bounded double sequence of strictly positive real numbers. Then, $[\mathscr{W}_\sigma(\alpha)]$ is a complete linear topological space paranormed by*

$$g(x) = \sup_{m,n,s,t}\left(\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\left|\zeta_{pqst}(x)\right|^{\alpha_{pq}}\right)^{1/M}, \tag{57}$$

where $M = \max(1, \sup_{p,q}\alpha_{pq})$. If $\alpha \geq 1$ then $[\mathscr{W}_{\sigma,\alpha}]$ is a Banach space and $[\mathscr{W}_{\sigma,\alpha}]$ is a α-normed space if $0 < \alpha < 1$.

Proof. Let (x_{pq}) and (y_{pq}) be two double sequences. Then,

$$\left|x_{pq} + y_{pq}\right|^{\alpha_{pq}} \leq K\left(\left|x_{pq}\right|^{\alpha_{pq}} + \left|y_{pq}\right|^{\alpha_{pq}}\right), \tag{58}$$

where $K = \max(1, 2^{H-1})$ and $H = \sup_{p,q}\alpha_{pq}$. Since

$$|\lambda|^{\alpha_{pq}} \leq \max\left(1, |\lambda|^H\right), \tag{59}$$

therefore

$$\begin{aligned}
& \frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\left|\zeta_{pqst}(\lambda x + \mu y)\right|^{\alpha_{pq}} \\
& \leq KK_1\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\left|\zeta_{pqst}(x)\right|^{\alpha_{pq}} \\
& + KK_2\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\left|\zeta_{pqst}(y)\right|^{\alpha_{pq}},
\end{aligned} \tag{60}$$

where $K_1 = \max(1, |\lambda|^H)$ and $K_2 = \max(1, |\mu|^H)$. From (60), we have that if $x, y \in [\mathscr{W}_\sigma(\alpha)]$, then $\lambda x + \mu y \in [\mathscr{W}_\sigma(\alpha)]$. Thus, $[\mathscr{W}_\sigma(\alpha)]$ is a linear space. Without loss of generality, we can take

$$[\mathscr{W}_\sigma(\alpha)]\text{-}\lim x = 0. \tag{61}$$

Clearly, $g(0) = 0$, $g(-x) = g(x)$. From (58) and Minkowski's inequality, we have

$$\begin{aligned}
& \left(\sum_{p=0}^{m}\sum_{q=0}^{n}\left|\zeta_{pqst}(x+y)\right|^{\alpha_{pq}}\right)^{1/M} \\
& \leq \left[\sum_{p=0}^{m}\sum_{q=0}^{n}\left(\left|\zeta_{pqst}(x)\right|^{\alpha_{pq}/M} + \left|\zeta_{pqst}(y)\right|^{\alpha_{pq}/M}\right)\right]^{1/M} \\
& \leq \left(\sum_{p=0}^{m}\sum_{q=0}^{n}\left|\zeta_{pqst}(x)\right|^{\alpha_{pq}}\right)^{1/M} \\
& + \left(\sum_{p=0}^{m}\sum_{q=0}^{n}\left|\zeta_{pqst}(y)\right|^{\alpha_{pq}}\right)^{1/M}.
\end{aligned} \tag{62}$$

Hence,

$$g(x+y) \leq g(x) + g(y). \tag{63}$$

Since $\alpha = (\alpha_{mn})$ is bounded away from zero, there exists a constant $\delta > 0$ such that $\alpha_{mn} \geq \delta$ for all m, n. Now for $|\lambda| \leq 1$, $|\lambda|^{\alpha_{st}} \leq |\lambda|^{\delta}$ and so

$$g(\lambda x) \leq |\lambda|^{\delta/M} g(x), \tag{64}$$

that is, the scaler multiplication is continuous. Hence, g is a paranorm on $[\mathscr{W}_{\sigma}(\alpha)]$.

Let (x^b) be a Cauchy sequence in $[\mathscr{W}_{\sigma}(\alpha)]$, that is,

$$g\left(x^d - x^b\right) \longrightarrow 0 \quad \text{as } b, d \longrightarrow \infty. \tag{65}$$

Since

$$\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \left|\zeta_{pqst}\left(x^d - x^b\right)\right|^{\alpha_{pq}} \leq \left[g\left(x^d - x^b\right)\right]^M, \tag{66}$$

it follows that

$$\left|\zeta_{pqst}\left(x^d - x^b\right)\right|^{\alpha_{pq}} \longrightarrow 0 \quad \text{as } b, d \longrightarrow \infty \ \forall p, q, s, t. \tag{67}$$

In particular,

$$\left|\zeta_{0,0,s,t}\left(x^d - x^b\right)\right| = \left|x_{st}^d - x_{st}^b\right| \longrightarrow 0 \quad \text{as } b, d \longrightarrow \infty \text{ for fixed } s, t. \tag{68}$$

Hence, (x^d) is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, there exists $x = (x_{jk}) \in \mathbb{C}$ such that $x^d \to x$ coordinatewise as $d \to \infty$. It follows from (66) that given $\epsilon > 0$, there exists d_0 such that

$$\left(\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \left|\zeta_{pqst}\left(x^d - x^b\right)\right|^{\alpha_{pq}}\right)^{1/M} < \epsilon, \tag{69}$$

for $b, d > d_0$. Now taking $b \to \infty$ and $\sup_{s,t,m,n}$ in (69), we have $g(x^d - x) \leq \epsilon$ for $d > d_0$. This proves that $x^d \to x$ and $x = (x_{jk}) \in [\mathscr{W}_{\sigma}(\alpha)]$. Hence $[\mathscr{W}_{\sigma}(\alpha)]$ is complete. If α is a constant then it is easy to prove the rest of the theorem. □

Theorem 6. *One has the following:*

$\mathscr{W}'_{\sigma}(\alpha)$ is a complete paranormed space, paranormed by

$$h(x) = \sup_{s,t}\left(\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \left|\xi_{pqst} - \xi_{p-1,q,s,t} - \xi_{p,q-1,s,t} + \xi_{p-1,q-1,s,t}\right|^{\alpha_{pq}}\right)^{1/M} \tag{70}$$

which is defined on $\mathscr{W}'_{\sigma}(\alpha)$. If $\alpha \geq 1$ then $\mathscr{W}'_{\sigma,\alpha}$ is a Banach space and if $0 < \alpha < 1$, $\mathscr{W}_{\sigma,\alpha}$ is α-normed space.

Proof. In order for the paranorm in (70) be defined, we require that

$$\mathscr{W}'_{\sigma}(\alpha) \subset \mathscr{W}''_{\sigma}(\alpha) \tag{71}$$

which is proved in the next theorem (i.e., Theorem 7). Using the standard technique as in the previous theorem, we can prove that h is subadditive.

Now, we have to prove the continuity of scalar multiplication. Suppose that $x = (x_{jk}) \in \mathscr{W}'_{\sigma}(\alpha)$. Then, for $\epsilon > 0$ there exist integers $M, N > 0$ such that

$$\sum_{p=M}^{\infty} \sum_{q=N}^{\infty} \left|\xi_{pqst}(x) - \xi_{p-1,q,s,t}(x) - \xi_{p,q-1,s,t}(x) + \xi_{p-1,q-1,s,t}(x)\right|^{\alpha_{pq}} < \epsilon. \tag{72}$$

If $|\lambda| \leq 1$, then by (72) we have

$$\begin{aligned}\sum_{p=M}^{\infty} \sum_{q=N}^{\infty} &\left|\xi_{pqst}(\lambda x) - \xi_{p-1,q,s,t}(\lambda x) - \xi_{p,q-1,s,t}(\lambda x) + \xi_{p-1,q-1,s,t}(\lambda x)\right|^{\alpha_{pq}} \\ &\leq \sum_{p=M}^{\infty} \sum_{q=N}^{\infty} \left|\xi_{pqst}(x) - \xi_{p-1,q,s,t}(x) - \xi_{p,q-1,s,t}(x) + \xi_{p-1,q-1,s,t}(x)\right|^{\alpha_{pq}} < \epsilon.\end{aligned} \tag{73}$$

Since for fixed M, N

$$\sum_{p=0}^{M-1} \sum_{q=0}^{N-1} \left|\xi_{pqst}(\lambda x) - \xi_{p-1,q,s,t}(\lambda x) - \xi_{p,q-1,s,t}(\lambda x) + \xi_{p-1,q-1,s,t}(\lambda x)\right|^{\alpha_{pq}} \longrightarrow 0 \tag{74}$$

as $\lambda \to 0$, it follows from (73) and (74) that for fixed $x = (x_{jk})$, $h(\lambda x) \to 0$ as $\lambda \to 0$. Also, since $|\lambda|^{\alpha_{pq}} \leq \max(1, |\lambda|^H)$ implies that

$$h(\lambda x) \leq \left(\sup |\lambda|^{\alpha_{pq}}\right)^{1/M} h(x). \tag{75}$$

It follows that for fixed λ, $h(\lambda x) \to 0$ as $x \to 0$. This proves the continuity of scalar multiplication. Hence, h is a paranorm. The proof of the completeness of $\mathscr{W}'_{\sigma}(\alpha)$ can be achieved by using the same technique as in Theorem 5. □

Theorem 7. *Suppose that $\alpha = (\alpha_{pq})$ is bounded double sequence of strictly positive real numbers. Then,*

(i) *$\mathscr{W}''_{\sigma}(\alpha)$ is a complete linear space paranormed by the function h defined in (70). In particular, if α is a constant, $\mathscr{W}''_{\sigma,\alpha}$ is a Banach space for $\alpha \geq 1$ and α-normed space for $0 < \alpha < 1$,*

(ii) *$\mathscr{W}'_{\sigma}(\alpha)$ is a closed subspace of $\mathscr{W}''_{\sigma}(\alpha)$.*

Proof. (i) Proceeding along the same lines as in Theorem 6, except for the proof of continuity of scalar multiplication. If $x = (x_{jk}) \in \mathscr{W}''_{\sigma}(\alpha)$, we cannot assert (72) as in the case when $x = (x_{jk}) \in \mathscr{W}'_{\sigma}(\alpha)$. Since α is bounded away from zero, there

exists $\delta > 0$ such that $\alpha_{mn} \geq \delta$ for all m, n. Hence, $|\lambda| \leq 1$ implies $|\lambda|^{\alpha_{pq}} \leq |\lambda|^{\delta}$. Since

$$h(\lambda x) \leq |\lambda|^{\alpha_{pq}} h(x), \tag{76}$$

continuity of scalar multiplication follows.

(ii) For this, first we have to show that

$$\mathscr{W}'_{\sigma}(\alpha) \subset \mathscr{W}''_{\sigma}(\alpha). \tag{77}$$

Let $x = (x_{jk}) \in \mathscr{W}'_{\sigma}(\alpha)$. Then, there exist integers M, N such that

$$\sum_{p=M}^{\infty} \sum_{q=N}^{\infty} \left| \xi_{pqst}(x) - \xi_{p-1,q,s,t}(x) - \xi_{p,q-1,s,t}(x) + \xi_{p-1,q-1,s,t}(x) \right|^{\alpha_{pq}} < 1. \tag{78}$$

By an argument similar to Theorem 4(iii), we obtain (77). Since $\mathscr{W}'_{\sigma}(\alpha)$ and $\mathscr{W}''_{\sigma}(\alpha)$ are complete with the same metric, we have (ii). □

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