Research Article

Subharmonics with Minimal Periods for Convex Discrete Hamiltonian Systems

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We consider the subharmonics with minimal periods for convex discrete Hamiltonian systems. By using variational methods and dual functional, we obtain that the system has a pT -periodic solution for each positive integer p , and solution of system has minimal period pT as H subquadratic growth both at 0 and infinity.

1. Introduction

Consider Hamiltonian systems

$$J\dot{u}(t) + \nabla H(t, u(t)) = 0, \quad u(0) = u(pT), \quad (1)$$

where $u(t) \in \mathbb{R}^{2N}$, $t \in \mathbb{R}$, ∇H stands for the gradient of H with respect to the second variable, and $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$ is the symplectic matrix with I_N the identity in $\mathbb{R}^N$. Moreover, H is T -periodic in the variable t , $p \in \mathbb{N} \setminus \{0\}$.

Classically, solutions for systems (1) are called subharmonics. The first result concerning the subharmonics problem traced back to Birkhoff and Lewis in 1933 (refer to [1]), in which there exists a sequence of subharmonics with arbitrarily large minimal period, using perturbation techniques. More results can be found in [1–5], where H is convex with subquadratic growth both at 0 and infinity. Using Z_p index theory and Clarke duality, Xu and Guo [1, 5] proved that the number of solutions for systems (1) with minimal period pT tends towards infinity as $p \rightarrow \infty$.

For periodic and subharmonic solutions for discrete Hamiltonian systems, Guo and Yu [6, 7] obtained some existence results by means of critical point theory, where they introduced the action functional

$$F(u) = -\frac{1}{2} \sum_{n=1}^{pT} (J\Delta Lu(n-1), u(n)) - \sum_{n=1}^{pT} H(n, Lu(n)). \quad (2)$$

Using Clarke duality, periodic solution of convex discrete Hamiltonian systems with forcing terms has been investigated in [8]. Clarke duality was introduced in 1978 by Clarke [9], and developed by Clarke, Ekeland, and others, see [10–12]. This approach is different from the direct method of variations; some scholars applied it to consider the periodic solutions, subharmonic solutions with prescribed minimal period of Hamiltonian systems; one can refer to [3, 5, 12–14] and references therein. The dynamical behavior of differential and difference equations was studied by using various methods; see [15–19]. We refer the reader to Agarwal [20] for a broad introduction to difference equations.

Motivated by the works of Mawhin and Willem [12] and Xu and Guo [21], we use variational methods and Clarke duality to consider the subharmonics with minimal periods for discrete Hamiltonian systems

$$J\Delta u(n) + \nabla H(n, Lu(n)) = 0, \quad u(n) = u(n + pT), \quad (3)$$

where $u(n) = \begin{pmatrix} u_1(n) \\ u_2(n) \end{pmatrix}$, $Lu(n) = \begin{pmatrix} u_1(n+1) \\ u_2(n) \end{pmatrix}$, $u_i(n) \in \mathbb{R}^N$ ($i = 1, 2$) with N a given positive integer, and $\Delta u(n) = u(n+1) - u(n)$ is the forward difference operator. $p, T \in \mathbb{N} \setminus \{0\}$. Moreover, hamiltonian function H satisfies the following assumption:

(A1) $H : \mathbb{Z} \times \mathbb{R}^{2N} \rightarrow \mathbb{R}$ is continuous differentiable on $\mathbb{R}^{2N}$, $H(n, \cdot)$ convex for each $n \in \mathbb{Z}$ and $H(n+T, u) = H(n, u)$ for each $u \in \mathbb{R}^{2N}$;

(A2) there exist constants $a_1 > 0$, $a_2 > 0$, $1 < \theta < 2$, such that

$$\frac{a_1}{\theta} |u|^\theta \leq H(n, u) \leq \frac{a_2}{\theta} |u|^\theta, \quad u \in \mathbb{R}^{2N}, \quad (4)$$

which implies H subquadratic growth both at 0 and infinity.

Theorem 1. Assume (A1) holds. $H(n, u) \rightarrow +\infty$, $H(n, u)/|u|^2 \rightarrow 0$, as $|u| \rightarrow \infty$ uniformly in $n \in \mathbb{Z}$. Then there exists a pT -periodic solution u_p of (3), such that $\|u_p\|_\infty \triangleq \max_{n \in \mathbb{Z}[1, pT]} \{|u_p(n)|\} \rightarrow \infty$, and the minimal period T_p of u_p tends to $+\infty$ as $p \rightarrow \infty$.

Theorem 2. Under the assumptions (A1) and (A2), if

$$\frac{a_2}{a_1} \leq \begin{cases} \left(\frac{1}{4} \sin \frac{\pi}{pT} \right)^{\theta/2}, & \text{when } pT \text{ is even,} \\ \left(\frac{1}{2} \sin \frac{\pi}{2pT} \right)^{\theta/2}, & \text{when } pT \text{ is odd} \end{cases} \quad (5)$$

for given integer $p > 1$, then the solution of (3) has minimal period pT .

2. Clarke Duality and Eigenvalue Problem

First we introduce a space E_{pT} with dimension $2NpT$ as follows:

$$\begin{aligned} E_{pT} &= \{u = \{u(n)\} \in S \mid u(n + pT) \\ &= u(n), p \in \mathbb{N} \setminus \{0\}, n \in \mathbb{Z}\}, \end{aligned} \quad (6)$$

where

$$\begin{aligned} S &= \left\{ u = \{u(n)\} \mid u(n) = \begin{pmatrix} u_1(n) \\ u_2(n) \end{pmatrix} \in \mathbb{R}^{2N}, \right. \\ &\quad \left. u_j(n) \in \mathbb{R}^N, j = 1, 2, n \in \mathbb{Z} \right\}. \end{aligned} \quad (7)$$

Equipped with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ in E_{pT} as

$$\begin{aligned} \langle u, v \rangle &= \sum_{n=1}^{pT} (u(n), v(n)), \\ \|u\| &= \left(\sum_{n=1}^{pT} |u(n)|^2 \right)^{1/2}, \quad \forall u, v \in E_{pT}, \end{aligned} \quad (8)$$

where $(\cdot, \cdot)$ and $|\cdot|$ denote the usual scalar product and corresponding norm in $\mathbb{R}^{2N}$, respectively. It is easy to see that $(E_{pT}, \langle \cdot, \cdot \rangle)$ is a Hilbert space with $2NpT$ dimension, which can be identified with $\mathbb{R}^{2NpT}$. Then for any $u \in E_{pT}$, it can be written as $u = (u^T(1), u^T(2), \dots, u^T(pT))^T$, where $u(j) = \begin{pmatrix} u_1(j) \\ u_2(j) \end{pmatrix} \in \mathbb{R}^{2N}$, $j \in \mathbb{Z}[1, pT]$, the discrete interval $\{1, 2, \dots, pT\}$ is denoted by $\mathbb{Z}[1, pT]$, and T denotes the transpose of a vector or a matrix.

Denote the subspace $\bar{Y} = \{u \in E_{pT} \mid u(1) = u(2) = \dots = u(pT) \in \mathbb{R}^{2N}\}$. Let Y be the direct orthogonal complement of

E_{pT} to $\bar{Y}$. Thus E_{pT} can be split as $E_{pT} = Y \oplus \bar{Y}$, and for any $u \in E_{pT}$, it can be expressed in the form $u = \bar{u} + \bar{u}$, where $\bar{u} \in Y$, $\bar{u} \in \bar{Y}$.

Next we recall Clarke duality and some lemmas.

The Legendre transform (see [12]) $H^*(t, \cdot)$ of $H(t, \cdot)$ with respect to the second variable is defined by

$$H^*(t, v) = \sup_{u \in \mathbb{R}^{2N}} \{(v, u) - H(t, u)\}, \quad (9)$$

where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^{2N}$. It follows from (A1) and (A2) that

(B1) $H^*(n, \cdot)$ is continuous differentiable on $\mathbb{R}^{2N}$,

(B2) for $\tau = \theta/(\theta - 1)$, $v \in \mathbb{R}^{2N}$, $n \in \mathbb{Z}$, one has

$$\frac{1}{\tau} \left(\frac{1}{a_2} \right)^{\tau-1} |v|^\tau \leq H^*(n, v) \leq \frac{1}{\tau} \left(\frac{1}{a_1} \right)^{\tau-1} |v|^\tau. \quad (10)$$

Associated with variational functional (2), the dual action functional is defined by

$$\begin{aligned} \chi(v) &= \sum_{n=1}^{pT} \frac{1}{2} (L(J\Delta v(n-1)), v(n)) \\ &\quad + \sum_{n=1}^{pT} H^*(n, \Delta v(n)), \quad v \in E_{pT}. \end{aligned} \quad (11)$$

Indeed, by (11), we have $\chi(v + \bar{u}) = \chi(v)$ for any $\bar{u} \in \bar{Y}$ and $v \in Y$. Therefore, χ can be restricted in subspace Y of E_{pT} . Moreover, in terms of Lemma 2.6 in [8] and the following lemma, the critical points of (11) correspond to the subharmonic solutions of (3).

Lemma 3 (see [8, Theorem 1]). Assume that

(H1) $H(n, \cdot) \in C^1(\mathbb{R}^{2N}, \mathbb{R})$, $H(n, \cdot)$ is convex in the second variable for $n \in \mathbb{Z}$,

(H2) there exists $\beta : \mathbb{Z} \rightarrow \mathbb{R}^{2N}$ such that for all $(n, u) \in \mathbb{Z} \times \mathbb{R}^{2N}$, $H(n, u) \geq (\beta(n), u)$, and $\beta(n+T) = \beta(n)$,

(H3) there exist $\alpha \in (0, 2 \sin(\pi/pT))$ and $\gamma : \mathbb{Z} \rightarrow \mathbb{R}^+$, such that for any $(n, u) \in \mathbb{Z} \times \mathbb{R}^{2N}$, $H(n, u) \leq (\alpha/2)|u|^2 + \gamma(n)$, and $\gamma(n+T) = \gamma(n)$,

(H4) for each $u \in \mathbb{R}^{2N}$, $\sum_{n=1}^{pT} H(n, u) \rightarrow +\infty$ as $|u| \rightarrow \infty$.

Then system (3) has at least one periodic solution u , such that $v = -J[u - (1/pT) \sum_{n=1}^{pT} u(n)]$ minimizes the dual action $\chi(v) = \sum_{n=1}^{pT} (1/2)(LJ\Delta v(n-1), v(n)) + \sum_{n=1}^{pT} H^*(n, \Delta v(n))$.

The following lemmas will be useful in our proofs, where Lemma 4 can be proved by means of Euler formula $e^{i\theta} = \cos \theta + i \sin \theta$, and Lemma 5 is a Hölder inequality.

Lemma 4. For any $k \in \mathbb{Z}$, $\sum_{n=1}^{pT} \sin((2k\pi/pT)n) = \sum_{n=1}^{pT} \cos((2k\pi/pT)n) = 0$.

Lemma 5. For any $u_j > 0$, $v_j > 0$, $k \in \mathbb{Z}$, one has $\sum_{j=1}^k u_j v_j \leq (\sum_{j=1}^k u_j^p)^{1/p} (\sum_{j=1}^k v_j^q)^{1/q}$, where $p > 1$, $q > 1$ and $1/p + 1/q = 1$.

Lemma 6 (see [12, proposition 2.2]). Let $H : \mathbb{R}^m \rightarrow \mathbb{R}$ be of C^1 and convex functional, $-\beta \leq H(u) \leq \alpha q^{-1}|u|^q + \gamma$, where $u \in \mathbb{R}^m$, $\alpha > 0$, $q > 1$, $\beta \geq 0$, $\gamma \geq 0$. Then $\alpha^{-p/q} p^{-1} |\nabla H(u)|^p \leq (\nabla H(u), u) + \beta + \gamma$, where $1/p + 1/q = 1$.

In order to know the form of $u \in E_{pT}$, we consider eigenvalue problem

$$LJ\Delta u(n-1) = \lambda u(n), \quad u(n+pT) = u(n), \quad (12)$$

where $u(n) = \begin{pmatrix} u_1(n) \\ u_2(n) \end{pmatrix}$, $Lu(n-1) = \begin{pmatrix} u_1(n) \\ u_2(n-1) \end{pmatrix} \in \mathbb{R}^{2N}$, $n \in \mathbb{Z}$, $\lambda \in \mathbb{R}$. We can rewrite (12) as the following form:

$$\begin{aligned} u_1(n+1) &= (1-\lambda^2)u_1(n) + \lambda u_2(n), \\ u_2(n+1) &= -\lambda u_1(n) + u_2(n), \end{aligned} \quad (13)$$

$$u_1(n+pT) = u_1(n), \quad u_2(n+pT) = u_2(n).$$

Denoting

$$M(\lambda) = \begin{pmatrix} (1-\lambda^2)I_N & \lambda I_N \\ -\lambda I_N & I_N \end{pmatrix}, \quad (14)$$

then problem (12) is equivalent to

$$u(n+1) = M(\lambda)u(n), \quad u(n+pT) = u(n). \quad (15)$$

Letting $u(n) = \mu^n c$ be the solution of (15), for some $c \in \mathbb{C}^{2N}$, we have $\mu c = M(\lambda)c$ and $\mu^{pT} = 1$. Consider the polynomial $|M(\lambda) - \mu I_{2N}| = 0$ and condition $\mu^{pT} = 1$; it follows that

$$\begin{aligned} \mu &= e^{2k\pi i/pT}, \quad \lambda = 2 \sin \frac{k\pi}{pT}, \\ k &\in \mathbb{Z}[-pT+1, pT-1]. \end{aligned} \quad (16)$$

In the following we denote by $\mu_k = e^{2k\pi i/pT}$, $\lambda_k = 2 \sin(k\pi/pT)$, $k \in \mathbb{Z}[-pT+1, pT-1]$, and $\rho \in \mathbb{R}^N$. By $(M(\lambda_k) - \mu_k I_{2N})c = 0$, it follows that

$$c_k = \begin{pmatrix} \rho \\ ie^{(-k\pi i/pT)} \rho \end{pmatrix}. \quad (17)$$

Thus

$$\begin{aligned} u_k(n) &= \mu_k^n c_k = e^{2k\pi ni/pT} \begin{pmatrix} \rho \\ ie^{(-k\pi i/pT)} \rho \end{pmatrix} \\ &= \begin{pmatrix} \cos\left(\frac{2k\pi}{pT}n\right)\rho \\ -\sin\left(\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\right)\rho \end{pmatrix} \\ &\quad + i \begin{pmatrix} \sin\left(\frac{2k\pi}{pT}n\right)\rho \\ \cos\left(\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\right)\rho \end{pmatrix}. \end{aligned} \quad (18)$$

Let

$$\begin{aligned} \xi_k(n) &= \begin{pmatrix} \cos\left(\frac{2k\pi}{pT}n\right)\rho \\ -\sin\left(\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\right)\rho \end{pmatrix}, \\ \eta_k &= \begin{pmatrix} \sin\left(\frac{2k\pi}{pT}n\right)\rho \\ \cos\left(\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\right)\rho \end{pmatrix}. \end{aligned} \quad (19)$$

Obviously, $\xi_k(n)$ and $\eta_k(n)$ satisfy (15). Moreover $LJ\Delta\xi_k(n-1) = \lambda_k\xi_k(n)$, $LJ\Delta\eta_k(n-1) = \lambda_k\eta_k(n)$, $\xi_{2pT+k}(n) = \xi_k(n)$, $\eta_{2pT+k}(n) = \eta_k(n)$, $\xi_{pT-k}(n) = \xi_k(n)$, $\eta_{pT-k}(n) = -\eta_k(n)$.

For $k \neq pT/2$, subspace Y_k is defined by

$$\begin{aligned} Y_k &= \begin{cases} \text{span}\{\xi_k(n), \eta_{k+(pT/2)}(n)\}, & k \in \mathbb{Z}\left[-\frac{pT}{2}+1, \frac{pT}{2}-1\right] \setminus \{0\}, \\ & n \in \mathbb{Z}, \text{ if } pT \text{ is even,} \\ \text{span}\{\xi_k(n), \eta_{k+((pT+1)/2)}(n)\}, & k \in \mathbb{Z}\left[\left[-\frac{pT}{2}\right], \left[\frac{pT}{2}\right]\right] \setminus \{0\}, \\ & n \in \mathbb{Z}, \text{ if } pT \text{ is odd,} \end{cases} \\ &= \begin{cases} \text{span}\{\xi_k(n), \eta_{k+(pT/2)}(n)\}, & k \in \mathbb{Z}\left[-\frac{pT}{2}+1, \frac{pT}{2}-1\right] \setminus \{0\}, \\ & n \in \mathbb{Z}, \text{ if } pT \text{ is even,} \\ \text{span}\{\xi_k(n), \eta_{k+((pT+1)/2)}(n)\}, & k \in \mathbb{Z}\left[\left[-\frac{pT}{2}\right], \left[\frac{pT}{2}\right]\right] \setminus \{0\}, \\ & n \in \mathbb{Z}, \text{ if } pT \text{ is odd,} \end{cases} \end{aligned} \quad (20)$$

where $[\cdot]$ denotes the greatest-integer function and

$$\begin{aligned} Y_{pT/2} &= \text{span}\{\xi_{pT/2}(n), n \in \mathbb{Z}\}, \\ Y_{-pT/2} &= \text{span}\{\xi_{-pT/2}(n), n \in \mathbb{Z}\}. \end{aligned} \quad (21)$$

Therefore,

$$\begin{aligned} Y &= \oplus Y_k, \quad k \in \mathbb{Z}\left[-\frac{pT}{2}, \frac{pT}{2}\right] \setminus \{0\}, \text{ if } pT \text{ is even,} \\ Y &= \oplus Y_k, \quad k \in \mathbb{Z}\left[\left[-\frac{pT}{2}\right], \left[\frac{pT}{2}\right]\right] \setminus \{0\}, \text{ if } pT \text{ is odd.} \end{aligned} \quad (22)$$

Moreover, for any $u = \{u(n)\} \in E_{pT}$, we may express $u(n)$ as

$$\begin{aligned} u(n) &= \sum_{k=-pT+1}^{pT-1} \left[\begin{pmatrix} \cos\left(\frac{2k\pi}{pT}n\right)a_k \\ -\sin\left(\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\right)a_k \end{pmatrix} \right. \\ &\quad \left. + \begin{pmatrix} \sin\left(\frac{2k\pi}{pT}n\right)b_k \\ \cos\left(\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\right)b_k \end{pmatrix} \right], \end{aligned} \quad (23)$$

where $a_k, b_k \in \mathbb{R}^N$.

Since $(\Delta u(n), \Delta u(n)) = -(\Delta^2 u(n-1), u(n))$, we consider eigenvalue problem

$$-\Delta^2 u(n-1) = \lambda u(n), \quad u(n+pT) = u(n), \quad u(n) \in \mathbb{R}^N, \quad (24)$$

where $\Delta^2 u(n-1) = \Delta u(n) - \Delta u(n-1) = u(n+1) - 2u(n) + u(n-1)$. The second order difference equation (24) has complexity solution $u(n) = e^{in\theta} c$ for $c \in \mathbb{C}^N$, where $\theta = 2k\pi/pT$. Moreover, $\lambda = 2 - e^{-i\theta} - e^{i\theta} = 2(1 - \cos \theta) = 4\sin^2(\theta/2)$; that is, $\lambda = 4\sin^2(k\pi/pT)$, $k \in Z[0, pT-1]$.

By the previous, it follows Lemma 7.

Lemma 7. For any $u \in E_{pT}$, one has $-\lambda_{\max} \|u\|^2 \leq \sum_{n=1}^{pT} (LJ\Delta u(n-1), u(n)) \leq \lambda_{\max} \|u\|^2$, and $0 \leq \sum_{n=1}^{pT} |\Delta u(n)|^2 \leq \lambda_{\max}^2 \|u\|^2$, where

$$\begin{aligned} \lambda_{\max} &= \max_{k \in [0, pT-1]} \left\{ 2 \sin \frac{k\pi}{pT} \right\} \\ &= \begin{cases} 2, & \text{if } pT \text{ is even,} \\ 2 \cos \frac{\pi}{2pT}, & \text{if } pT \text{ is odd.} \end{cases} \end{aligned} \quad (25)$$

Moreover, if $u \in Y$, then $4\sin^2(\pi/pT) \|u\|^2 \leq \sum_{n=1}^{pT} |\Delta u(n)|^2 \leq \lambda_{\max}^2 \|u\|^2$.

3. Proofs of Main Results

Lemma 8. Consider

$$\begin{aligned} &\sum_{n=1}^{pT} (LJ\Delta u(n-1), u(n)) \\ &\geq -\left(2 \sin \frac{\pi}{pT}\right)^{-1} \sum_{n=1}^{pT} |\Delta u(n)|^2, \quad \forall u \in E_{pT}. \end{aligned} \quad (26)$$

Proof. Letting $\tilde{u}(n) = u(n) - (1/pT) \sum_{n=1}^{pT} u(n)$, then $\tilde{u} \in Y$. By Lemmas 5 and 7, we have

$$\begin{aligned} &\sum_{n=1}^{pT} (LJ\Delta u(n-1), u(n)) \\ &= \sum_{n=1}^{pT} (LJ\Delta u(n-1), \tilde{u}(n)) \\ &\geq -\left(\sum_{n=1}^{pT} |LJ\Delta u(n-1)|^2\right)^{1/2} \\ &\quad \cdot \left(\sum_{n=1}^{pT} |\tilde{u}(n)|^2\right)^{1/2} \\ &\geq -\left(\sum_{n=1}^{pT} |\Delta u(n)|^2\right)^{1/2} \\ &\quad \cdot \left(2 \sin \frac{\pi}{pT}\right)^{-1} \left(\sum_{n=1}^{pT} |\Delta \tilde{u}(n)|^2\right)^{1/2} \\ &= -\left(2 \sin \frac{\pi}{pT}\right)^{-1} \sum_{n=1}^{pT} |\Delta u(n)|^2. \end{aligned} \quad (27)$$

□

Lemma 9. If there exist $\alpha \in (0, \sin(\pi/pT))$, $\beta \geq 0$ and $\delta > 0$, such that

$$\delta |u| - \beta \leq H(n, u) \leq \frac{\alpha}{2} |u|^2 + \gamma \quad (28)$$

for all $n \in [1, pT]$ and $u \in \mathbb{R}^{2N}$, then each solution of (3) satisfies the inequalities

$$\begin{aligned} \sum_{n=1}^{pT} |\Delta u(n)|^2 &\leq \frac{2\alpha(\beta + \gamma) pT \sin(\pi/pT)}{\sin(\pi/pT) - \alpha}, \\ \sum_{n=1}^{pT} |Lu(n)| &\leq \frac{(\beta + \gamma) pT \sin(\pi/pT)}{\delta(\sin(\pi/pT) - \alpha)}. \end{aligned} \quad (29)$$

Proof. Let u be the solution of (3). By Lemma 6, we have

$$\begin{aligned} \frac{1}{2\alpha} |\nabla H(n, Lu(n))|^2 &\leq (\nabla H(n, Lu(n)), Lu(n)) + \beta + \gamma \\ &= -(J\Delta u(n), Lu(n)) + \beta + \gamma. \end{aligned} \quad (30)$$

Obviously, $|J\Delta u(n)|^2 = (-\nabla H(n, Lu(n)), J\Delta u(n)) = |\nabla H(n, Lu(n))|^2$ by (3), and it follows that $(1/2\alpha) \sum_{n=1}^{pT} |J\Delta u(n)|^2 + \sum_{n=1}^{pT} (J\Delta u(n), Lu(n)) \leq (\beta + \gamma) pT$; that is,

$$\begin{aligned} \frac{1}{2\alpha} \sum_{n=1}^{pT} |\Delta u(n)|^2 + \sum_{n=1}^{pT} (LJ\Delta u(n-1), u(n)) \\ \leq (\beta + \gamma) pT. \end{aligned} \quad (31)$$

By means of Lemma 8, we have

$$\left[\frac{1}{2\alpha} - \left(2 \sin \frac{\pi}{pT}\right)^{-1} \right] \sum_{n=1}^{pT} |\Delta u(n)|^2 \leq (\beta + \gamma) pT, \quad (32)$$

which gives first conclusion.

Now, $H(n, 0) \leq \gamma$ in view of (28); therefore by convex and Lemma 8, we have

$$\begin{aligned} &\delta \sum_{n=1}^{pT} |Lu(n)| - \beta pT \\ &\leq \sum_{n=1}^{pT} H(n, Lu(n)) \\ &\leq \sum_{n=1}^{pT} [H(n, 0) + (\nabla H(n, Lu(n)), Lu(n))] \end{aligned}$$

$$\begin{aligned}
&\leq \gamma pT - \sum_{n=1}^{pT} (J\Delta u(n), Lu(n)) \\
&= \gamma pT - \sum_{n=1}^{pT} (JL\Delta u(n-1), u(n)) \\
&\leq \gamma pT + \left(2 \sin \frac{\pi}{pT}\right)^{-1} \sum_{n=1}^{pT} |\Delta u(n)|^2 \\
&\leq \gamma pT + \frac{\alpha(\beta + \gamma)pT}{\sin(\pi/pT) - \alpha},
\end{aligned} \tag{33}$$

which gives the second conclusion. The proof is completed. $\square$

Proof of Theorem 1. Let $c_1 = \max_{n \in \mathbb{Z}} |H(n, 0)|$. By assumption in Theorem 1, there exists $R > 0$, such that $H(n, u) \geq 1 + c_1$, for $n \in \mathbb{Z}$ and $|u| \geq R$. Moreover, there exist $\alpha \in (0, 2 \sin(\pi/pT))$, $\gamma > 0$ such that

$$H(n, u) \leq \frac{\alpha}{2}|u|^2 + \gamma, \quad \forall (n, u) \in \mathbb{Z} \times \mathbb{R}^{2N}. \tag{34}$$

Thus, by convex of H , for all $(n, u) \in \mathbb{Z} \times \mathbb{R}^{2N}$ with $|u| \geq R$, we have

$$\begin{aligned}
1 + c_1 &\leq H\left(n, \frac{R}{|u|}u\right) \\
&\leq H(n, 0) + \frac{R}{|u|} (H(n, u) - H(n, 0)) \\
&\leq \frac{R}{|u|} H(n, u) + c_1.
\end{aligned} \tag{35}$$

Therefore there exist $\beta > 0$ and $\delta > 0$, such that

$$H(n, u) \geq \delta |u| - \beta, \quad \forall (n, u) \in \mathbb{Z} \times \mathbb{R}^{2N}. \tag{36}$$

Combining the previous argument, by Lemma 3, the system (3) has a pT -periodic solution u_p such that $v_p = -J[u_p - (1/pT) \sum_{n=1}^{pT} u_p(n)] \in Y$ minimizes the dual action

$$\begin{aligned}
\chi_p(v_p) &= \sum_{n=1}^{pT} \frac{1}{2} (LJ\Delta v_p(n-1), v_p(n)) \\
&\quad + \sum_{n=1}^{pT} H^*(n, \Delta v_p(n)) \quad \text{on } E_{pT}.
\end{aligned} \tag{37}$$

It follows that $\Delta u_p(n) = J\Delta v_p(n)$ and $Jv_p(n) = u_p(n) - (1/pT) \sum_{n=1}^{pT} u_p(n)$.

We next prove that $\|u_p\|_\infty \rightarrow \infty$ as $p \rightarrow \infty$.

Suppose not, there exist $c_2 > 0$ and a subsequence $\{p_k\}$ such that

$$p_k \rightarrow \infty, \quad \|u_{p_k}\|_\infty \leq c_2 \quad \text{as } k \rightarrow \infty. \tag{38}$$

In terms of (3), it follows that $\|\Delta u_{p_k}\|_\infty \leq c_3$ for some $c_3 > 0$, and $\|v_{p_k}\|_\infty \leq 2c_2$, $\|\Delta v_{p_k}\|_\infty \leq c_3$. Consequently, by $H^*(n, v) \geq -H(n, 0) \geq -c_1$, we have

$$\begin{aligned}
c_{p_k} &= \chi_{p_k}(v_{p_k}) \\
&= \sum_{n=1}^{p_k T} \frac{1}{2} (LJ\Delta v_{p_k}(n-1), v_{p_k}(n)) \\
&\quad + \sum_{n=1}^{p_k T} H^*(n, \Delta v_{p_k}(n)) \\
&\geq -\frac{1}{2} \sum_{n=1}^{p_k T} |LJ\Delta v_{p_k}(n-1)| |v_{p_k}(n)| - c_1 p_k T \\
&\geq -(\sqrt{2}c_2 c_3 + c_1) p_k T,
\end{aligned} \tag{39}$$

where $n \in Z[1, p_k T]$ and

$$\begin{aligned}
|LJ\Delta v_{p_k}(n-1)| &= \left(|\Delta v_{2,p_k}(n)|^2 + |\Delta v_{1,p_k}(n-1)|^2\right)^{1/2} \\
&\leq \sqrt{2} \|\Delta v_{p_k}\|_\infty \leq \sqrt{2} c_3.
\end{aligned} \tag{40}$$

By (36), if $|v| \leq \delta$, we have $(v, u) - H(n, u) \leq (v, u) - \delta|u| + \beta \leq \beta$, and $H^*(n, v) \leq \beta$. Letting $\rho \in \mathbb{R}^N$ and $|\rho| = 1$, in terms of (12), h_p associated with $\lambda_{-1} = -2 \sin(\pi/pT)$ is given by

$$\begin{aligned}
h_p(n) &= \frac{\delta}{4 \sin(\pi/pT)} \\
&\cdot \begin{pmatrix} \left(\cos \frac{2\pi}{pT} n - \sin \frac{2\pi}{pT} n\right) \rho \\ \left(\sin \frac{2\pi}{pT} \left(n - \frac{1}{2}\right) + \cos \frac{2\pi}{pT} \left(n - \frac{1}{2}\right)\right) \rho \end{pmatrix}
\end{aligned} \tag{41}$$

which belongs to E_{pT} , and

$$\begin{aligned}
&|\Delta h_p(n)|^2 \\
&= \left(\frac{\delta}{4 \sin(\pi/pT)}\right)^2 \\
&\cdot \left| 2 \sin \frac{\pi}{pT} \begin{pmatrix} \left(-\sin \frac{2\pi}{pT} \left(n + \frac{1}{2}\right) - \cos \frac{2\pi}{pT} \left(n + \frac{1}{2}\right)\right) \rho \\ \left(\cos \frac{2\pi}{pT} n - \sin \frac{2\pi}{pT} n\right) \rho \end{pmatrix} \right|^2 \\
&= \frac{1}{4} \left[2 + \sin \frac{2\pi}{pT} (2n+1) - \sin \frac{2\pi}{pT} (2n) \right] \cdot |\rho|^2 \delta^2 \\
&\leq \delta^2.
\end{aligned} \tag{42}$$

Moreover, by Lemma 4 we have

$$\begin{aligned}
& \sum_{n=1}^{pT} |h_p(n)|^2 \\
&= \sum_{n=1}^{pT} \left(\frac{\delta}{4 \sin(\pi/pT)} \right)^2 \\
&\quad \cdot \left(2 + \sin \frac{2\pi}{pT} (2n-1) - \sin \frac{2\pi}{pT} (2n) \right) |\rho|^2 \\
&= \left(\frac{\delta}{4 \sin(\pi/pT)} \right)^2 2|\rho|^2 pT = \frac{\delta^2 pT}{8 \sin^2(\pi/pT)}.
\end{aligned} \tag{43}$$

Thus $c_p = \chi_p(h_p) \leq \sum_{n=1}^{pT} (1/2)(LJ\Delta h_p(n-1), h_p(n)) + \beta pT = \sum_{n=1}^{pT} (1/2)(-2 \sin(\pi/pT)) |h_p(n)|^2 + \beta pT = -\delta^2 pT/8 \sin(\pi/pT) + \beta pT$. Combining (39), we have $8(\sqrt{2}c_3 + c_1 + \beta_1) \sin(\pi/p_k T) \geq \delta^2$, which is impossible as k large. So the claim $\lim_{p \rightarrow \infty} \|u_p\|_\infty = \infty$ is valid.

It remains only to prove that the minimal period T_p of u_p tends to $+\infty$ as $p \rightarrow \infty$.

If not, there exists $T > 0$ and a sequence $\{p_k\}$ such that the minimal period T_{p_k} of u_{p_k} satisfies $1 \leq T_{p_k} \leq T$. By assumption in Theorem 1, there exists $\alpha \in (0, \sin(\pi/T))$ and $\gamma > 0$ such that

$$H(n, u) \leq \frac{\alpha}{2} |u|^2 + \gamma, \quad \forall (n, u) \in \mathbb{Z} \times \mathbb{R}^{2N}. \tag{44}$$

By (36) and Lemma 9 with pT replaced by T_{p_k} , we get

$$\sum_{n=1}^{T_{p_k}} |\Delta u_{p_k}(n)|^2 \leq \frac{2\alpha(\beta + \gamma) T_{p_k} \sin(\pi/T_{p_k})}{\sin(\pi/T_{p_k}) - \alpha} \tag{45}$$

$$\leq \frac{2\alpha(\beta + \gamma) T \sin(\pi/T)}{\sin(\pi/T) - \alpha},$$

$$\sum_{n=1}^{T_{p_k}} |Lu_{p_k}(n)| \leq \frac{(\beta + \gamma) T_{p_k} \sin(\pi/T_{p_k})}{\delta(\sin(\pi/T_{p_k}) - \alpha)} \tag{46}$$

$$\leq \frac{(\beta + \gamma) T_{p_k} \sin(\pi/T)}{\delta(\sin(\pi/T) - \alpha)}.$$

Write $u_{p_k} = \tilde{u}_{p_k} + \bar{u}_{p_k}$, where $\bar{u}_{p_k} = (1/T_{p_k}) \sum_{n=1}^{T_{p_k}} u_{p_k}(n) = (1/T_{p_k}) \sum_{n=1}^{T_{p_k}} Lu_{p_k}(n) \in \bar{Y}$. Inequality (46) implies that

$$\begin{aligned}
\|\bar{u}_{p_k}\|_\infty &\triangleq \max_{n \in \mathbb{Z}[1, T_{p_k}]} \{|\bar{u}_{p_k}|\} \\
&\leq \frac{1}{T_{p_k}} \sum_{n=1}^{T_{p_k}} |Lu_{p_k}(n)| \leq \frac{(\beta + \gamma) \sin(\pi/T)}{\delta(\sin(\pi/T) - \alpha)}.
\end{aligned} \tag{47}$$

By Lemma 7 and (45), it follows that

$$\begin{aligned}
\|\tilde{u}_{p_k}\|^2 &= \sum_{n=1}^{T_{p_k}} |\tilde{u}_{p_k}(n)|^2 \\
&\leq \left(2 \sin \frac{\pi}{T_{p_k}} \right)^{-1} \sum_{n=1}^{T_{p_k}} |\Delta u_{p_k}(n)|^2 \\
&\leq (2 \sin(\pi/T))^{-1} \frac{2\alpha(\beta + \gamma) T \sin(\pi/T)}{\sin(\pi/T) - \alpha} \\
&\leq \frac{\alpha(\beta + \gamma) T}{\sin(\pi/T) - \alpha},
\end{aligned} \tag{48}$$

which implies that $\{\|\tilde{u}_{p_k}\|_\infty\}$ is bounded, therefore $\{\|u_{p_k}\|_\infty\}$ is bounded; a contradiction with the second claim $\lim_{p \rightarrow \infty} \|u_p\|_\infty = \infty$. This completes the proof. $\square$

Proof of Theorem 2. Under the assumptions (A1) and (A2), all conditions in Theorem 1 are satisfied. Then, for each integer $p > 1$, there exists a pT -periodic solution u of (3) such that $v = -J[u - (1/pT) \sum_{n=1}^{pT} u(n)] \in Y$ minimizes the dual action

$$\begin{aligned}
\chi(v) &= \sum_{n=1}^{pT} \frac{1}{2} (LJ\Delta v(n-1), v(n)) \\
&\quad + \sum_{n=1}^{pT} H^*(n, \Delta v(n)) \quad \text{on } E_{pT}.
\end{aligned} \tag{49}$$

If the critical point v of dual action functional χ has minimal period $pT/l \in \mathbb{N} \setminus \{0\}$, where $l \in \mathbb{N} \setminus \{0\}$, then by Lemma 7 with pT replaced by pT/l , we have the following estimate:

$$4\sin^2 \frac{l\pi}{pT} \sum_{n=1}^{pT} |v(n)|^2 \leq \sum_{n=1}^{pT} |\Delta v(n)|^2. \tag{50}$$

By Lemma 5 and the previous inequality, we have

$$\begin{aligned}
& \sum_{n=1}^{pT} (LJ\Delta v(n-1), v(n)) \\
&\geq - \left(\sum_{n=1}^{pT} |LJ\Delta v(n-1)|^2 \right)^{1/2} \\
&\quad \cdot \left(\sum_{n=1}^{pT} |v(n)|^2 \right)^{1/2} \\
&\geq - \left(\sum_{n=1}^{pT} |\Delta v(n)|^2 \right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
& \cdot \left(2 \sin \frac{l\pi}{pT} \right)^{-1} \left(\sum_{n=1}^{pT} |\Delta v(n)|^2 \right)^{1/2} \\
& = - \left(2 \sin \frac{l\pi}{pT} \right)^{-1} \sum_{n=1}^{pT} |\Delta v(n)|^2 \\
& \geq - \left(2 \sin \frac{l\pi}{pT} \right)^{-1} (pT)^{(1-2/\tau)} \left(\sum_{n=1}^{pT} |\Delta v(n)|^\tau \right)^{2/\tau}, \quad (51)
\end{aligned}$$

where $\tau = \theta/(\theta - 1) > 2$ for $1 < \theta < 2$. It follows from assumption (B2) that

$$H^*(n, \Delta v(n)) \geq \frac{1}{\tau} \left(\frac{1}{a_2} \right)^{\tau-1} |\Delta v(n)|^\tau, \quad (52)$$

thus

$$\chi(v) \geq - \left(2 \sin \frac{l\pi}{pT} \right)^{-1} (pT)^{(\tau-2)/\tau} \left(\sum_{n=1}^{pT} |\Delta v(n)|^\tau \right)^{2/\tau} \quad (53)$$

$$\begin{aligned}
& + \frac{1}{\tau} \left(\frac{1}{a_2} \right)^{\tau-1} \sum_{n=1}^{pT} |\Delta v(n)|^\tau \\
& \geq \frac{(1/\tau - 1/2) pT (a_2^2)^{(\tau-1)/(\tau-2)}}{(\sin(l\pi/pT))^{\tau/(\tau-2)}}. \quad (54)
\end{aligned}$$

One can obtain the previous inequality by minimizing in (53) with respect to $(\sum_{n=1}^{pT} |\Delta v(n)|^\tau)^{1/\tau}$, and the minimum is attained at $(pT)^{1/\tau} (a_2)^{(\tau-1)/(\tau-2)} / (\sin(l\pi/pT))^{1/(\tau-2)}$.

On the other hand, let

$$v(n) = \frac{1}{\sqrt{pT}} \begin{pmatrix} \cos \frac{2k\pi}{pT} n \cdot a_k \\ -\sin \frac{2k\pi}{pT} \left(n - \frac{1}{2} \right) \cdot a_k \end{pmatrix}, \quad (55)$$

where $a_k \in \mathbb{R}^N$, $k \in Z[[-pT/2], [pT/2]] \setminus \{0\}$. Then $v \in Y_k$, and

$$\Delta v(n) = -2 \sin \frac{k\pi}{pT} \frac{1}{\sqrt{pT}} \begin{pmatrix} \sin \frac{2k\pi}{pT} \left(n + \frac{1}{2} \right) \cdot a_k \\ \cos \frac{2k\pi}{pT} n \cdot a_k \end{pmatrix}. \quad (56)$$

Taking $a_k = (d, 0, \dots, 0)^T \in \mathbb{R}^N$, where $d \in \mathbb{R}$, by Lemma 4, it follows that

$$\begin{aligned}
& \sum_{n=1}^{pT} (LJ \Delta v(n-1), v(n)) \\
& = \sum_{n=1}^{pT} [-\Delta v_2(n) v_1(n) + \Delta v_1(n-1) v_2(n)] \\
& = \sum_{n=1}^{pT} \frac{1}{pT} \cdot 2 \sin \frac{k\pi}{pT} \\
& \quad \cdot \left(\cos^2 \frac{2k\pi}{pT} n \cdot |d|^2 + \sin^2 \frac{2k\pi}{pT} \left(n - \frac{1}{2} \right) \cdot |d|^2 \right) \\
& = \lambda_k \cdot |d|^2, \quad (57)
\end{aligned}$$

where $\lambda_k = 2 \sin(k\pi/pT)$ and

$$\begin{aligned}
& \sum_{n=1}^{pT} |\Delta v(n)|^\tau \\
& = \sum_{n=1}^{pT} |\lambda_k|^\tau (pT)^{-\tau/2} \\
& \quad \cdot \left(\sin^2 \frac{2k\pi}{pT} \left(n + \frac{1}{2} \right) + \cos^2 \frac{2k\pi}{pT} n \right)^{\tau/2} |d|^\tau \\
& \leq \lambda_{\max}^\tau \cdot (pT)^{1-(\tau/2)} \cdot 2^{\tau/2} |d|^\tau. \quad (58)
\end{aligned}$$

Therefore, taking $k = -[pT/2]$, by eigenvalue problem (24) and (B2), it follows that

$$\begin{aligned}
\chi(v) & = \frac{1}{2} \sum_{n=1}^{pT} (LJ \Delta v(n-1), v(n)) \\
& \quad + \sum_{n=1}^{pT} H^*(n, \Delta v(n)) \\
& \leq -\frac{1}{2} \lambda_{\max} \cdot |d|^2 \\
& \quad + \frac{1}{\tau} \left(\frac{1}{a_1} \right)^{\tau-1} \sum_{n=1}^{pT} |\Delta v(n)|^\tau \\
& \leq -\frac{1}{2} \lambda_{\max} \cdot |d|^2 + \frac{1}{\tau} \left(\frac{1}{a_1} \right)^{\tau-1} \lambda_{\max}^\tau \\
& \quad \cdot (pT)^{1-(\tau/2)} \cdot 2^{\tau/2} |d|^\tau. \quad (59)
\end{aligned}$$

Let $f(\rho)$ equal the right-hand side of (59) where $\rho = |d|$. It is easy to see that the absolute minimum of f is attained at $\rho_{\min} = (a_1)^{(\tau-1)/(\tau-2)} (pT)^{1/2} / [\lambda_{\max}^{(\tau-1)/(\tau-2)} \cdot 2^{\tau/2(\tau-2)}]$ and given

by $f_{\min} = (1/\tau - 1/2)pT(a_1^2)^{(\tau-1)/(\tau-2)}/(2\lambda_{\max})^{\tau/(\tau-2)}$. Hence, by (19), let

$$\begin{aligned}\xi(n) &= \xi_{-[pT/2]}(n) \\ &= \begin{pmatrix} \cos \frac{2k\pi}{pT} n \cdot \rho \\ -\sin \frac{2k\pi}{pT} \left(n - \frac{1}{2}\right) \cdot \rho \end{pmatrix},\end{aligned}\quad (60)$$

where $\rho \in \mathbb{R}^N$, $k = -[pT/2]$.

If pT is even, then $\xi(n) = (1, 1)^T \cdot (-1)^n \rho$. Set

$$\begin{aligned}Y_{\rho_{\min}} &= \{v \in Y_{-[pT/2]} : v(n) = \xi(n), \\ \rho &= (d, 0, \dots, 0)^T \in \mathbb{R}^N, d \in \mathbb{R}\}.\end{aligned}\quad (61)$$

For $v \in Y_{\rho_{\min}}$, we have

$$\chi(v) \leq f_{\min}.\quad (62)$$

Combining (54), (59), and (62), we have

$$\begin{aligned}&\frac{(1/\tau - 1/2)pT(a_2^2)^{(\tau-1)/(\tau-2)}}{(\sin(l\pi/pT))^{\tau/(\tau-2)}} \\ &\leq \frac{(1/\tau - 1/2)pT(a_1^2)^{(\tau-1)/(\tau-2)}}{(2\lambda_{\max})^{\tau/(\tau-2)}}.\end{aligned}\quad (63)$$

By $\tau > 2$, and $\theta = \tau/(\tau - 1)$, it follows that

$$\frac{\sin(l\pi/pT)}{(2\lambda_{\max})} \leq (a_2/a_1)^{2/\theta}.\quad (64)$$

For integer $p > 1$, $T \geq 1$, $l \in \mathbb{N} \setminus \{0\}$, $pT/l \in \mathbb{N} \setminus \{0\}$, we have $0 < l\pi/pT \leq \pi$, $0 < \pi/pT \leq \pi/2$.

If pT is even, then $\lambda_{\max} = 2$. By assumption $a_2/a_1 \leq ((1/4)\sin(\pi/pT))^{\theta/2}$ we have $\sin(l\pi/pT) \leq \sin(\pi/pT)$, which implies that $l = 1$ or $l = pT - 1$. If $pT > 2$, then $pT/l = pT/(pT - 1) \notin \mathbb{N}$. So we have $l = 1$.

If pT is odd, then $\lambda_{\max} = 2\cos(\pi/2pT)$. By assumption $a_2/a_1 \leq ((1/2)\sin(\pi/2pT))^{\theta/2}$, we have $\sin(l\pi/pT) \leq \sin(\pi/pT)$, so $l = 1$. This completes the proof. $\square$

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