A Generalization of Lacunary Equistatistical Convergence of Positive Linear Operators

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1. Introduction

Approximation theory has important applications in the theory of polynomial approximation, in various areas of functional analysis, numerical solutions of integral and differential equations [1–6]. In recent years, with the help of the concept of statistical convergence, various statistical approximation results have been proved [7]. In the usual sense, every convergent sequence is statistically convergent, but its converse is not always true. And, statistical convergent sequences do not need to be bounded.

Recently, Aktuğlu and Gezer [8] generalized the idea of statistical convergence to lacunary equi-statistical convergence. In this paper, we first study some Korovkin type approximation theorems via lacunary equi-statistical convergence in $H_{w}^{2}$ spaces. Then using the modulus of continuity, we study rates of lacunary equi-statistically convergence in $H_{w}^{2}$. A Korovkin type approximation theorem by means of lacunary equi-statistical convergence was given in [8]. We can state this theorem now. An operator $L$ defined on a linear space of functions $Y$ is called linear if $L(\alpha f + \beta g, x) = \alpha L(f, x) + \beta L(g, x)$, for all $f, g \in Y, \alpha, \beta \in \mathbb{R}$ and is called positive, if $L(f, x) \geq 0$, for all $f \in Y, f \geq 0$. Let $X$ be a compact subset of $\mathbb{R}$, and let $C(X)$ be the space of all continuous real valued functions on $X$.

A lacunary sequence $\theta = \{k_r\}$ is an integer sequence such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \rightarrow \infty \quad \text{as} \quad r \rightarrow \infty. \quad (3)$$

In this paper the intervals determined by $\theta$ will be denoted by $I_r = (k_{r-1}, k_r]$, and the ratio $k_r/k_{r-1}$ will be abbreviated by $q_r$. Let $\theta$ be a lacunary sequence then $\{f_r\}_{r \in \mathbb{N}}$ is said to be lacunary equi-statistically convergent to $f$ on $X$ and denoted by $f_r \rightarrow f$ (equistat) if for every $\epsilon > 0$, the sequence of real valued functions $\{s_r, \epsilon\}_{n \in \mathbb{N}}$ defined by

$$s_r(\epsilon) := \frac{1}{h_r} \left| \left\{ m \in I_r : f_m(x) - f(x) \geq \epsilon \right\} \right| \quad (4)$$

uniformly converges to zero function on $X$, which means that

$$\lim_{n \rightarrow \infty} \left\| s_r(\epsilon) \right\|_{C(X)} = 0. \quad (5)$$

A Korovkin type approximation theorem by means of lacunary equi-statistical convergence was given in [8]. We can state this theorem now. An operator $L$ defined on a linear space of functions $Y$ is called linear if $L(\alpha f + \beta g, x) = \alpha L(f, x) + \beta L(g, x)$, for all $f, g \in Y, \alpha, \beta \in \mathbb{R}$ and is called positive, if $L(f, x) \geq 0$, for all $f \in Y, f \geq 0$. Let $X$ be a compact subset of $\mathbb{R}$, and let $C(X)$ be the space of all continuous real valued functions on $X$. 

$$p_r, \epsilon := \frac{1}{r} \left| \left\{ m \leq r : f_m(x) - f(x) \geq \epsilon \right\} \right| \quad (1)$$

converges uniformly to the zero function on $X$, which means that

$$\lim_{r \rightarrow \infty} \left\| p_r, \epsilon \right\|_{C(X)} = 0. \quad (2)$$

In this paper we consider some analogs of the Korovkin approximation theorem via lacunary equistatistical convergence. In particular we study lacunary equi-statistical convergence of approximating operators on $H_{w}^{2}$ spaces, the spaces of all real valued continuous functions $f$ defined on $K = [0, \infty)^{m}$ and satisfying some special conditions.

$$\lim_{r \rightarrow \infty} \left\| s_r, \epsilon \right\|_{C(X)} = 0. \quad (5)$$
Lemma 1 (see [8]). Let \( \theta \) be a lacunary sequence, and let \( L_r : C(X) \to C(X) \) be a sequence of positive linear operators satisfying
\[
L_r(t^r, x) \to x^r, \quad (\theta\text{-equistat}), \quad \nu = 0, 1, 2, \quad (6)
\]
then for all \( f \in C(X) \),
\[
L_r(f, x) \to f, \quad (\theta\text{-equistat}). \quad (7)
\]

We turn to introducing some notation and the basic definitions used in this paper. Throughout this paper \( I = [0, \infty) \). Let
\[
C(I) := \{ f : f \text{ is a real-valued continuous function on } I \},
\]
and define the lacunary sequence \( \theta \) to satisfy
\[
\delta_2 > 0, \quad \delta_1 = 1 + \frac{\varepsilon}{\delta_2}, \quad (9)
\]
Refer to [9]. Let \( K := I^2 = [0, \infty) \times [0, \infty) \), then the norm on \( C_B(K) \) is given by
\[
\|f\| := \sup_{(x,y) \in K} |f(x,y)|, \quad f \in C_B(K),
\]
and also denote the valued of \( Lf \) at a point \( (x, y) \in K \) is denoted by \( L(f;x, y) \) [10, 11].

\( \underline{w}_2(f; \delta_1, \delta_2) \) and \( \overline{w}_2(f; \delta_1, \delta_2) \) are the type of modulus of continuity for the functions of two variables satisfying the following properties: for any real numbers \( \delta_1, \delta_2, \delta_1', \delta_2', \) and \( \delta_2'' > 0 \),

(i) \( \underline{w}_2(f; \delta_1, \delta_2) \) and \( \overline{w}_2(f; \delta_1', \delta_2') \) are nonnegative increasing functions on \([0, \infty)\),
(ii) \( \underline{w}_2(f; \delta_1' + \delta_2', \delta_2) \leq \underline{w}_2(f; \delta_1', \delta_2') + \underline{w}_2(f; \delta_1'', \delta_2'), \)
(iii) \( \overline{w}_2(f; \delta_1, \delta_1' + \delta_2') \leq \overline{w}_2(f; \delta_1, \delta_1') + \overline{w}_2(f; \delta_1, \delta_2'), \)
(iv) \( \lim_{\delta_1 \to 0} \underline{w}_2(f; \delta_1, \delta_2) = 0. \)

The space \( H_{w_2} \) is of all real-valued functions \( f \) defined on \( K \) and satisfying
\[
|f(u, v) - f(x, y)| \leq w_2\left(f; \left|\frac{u}{1+u} - \frac{x}{1+x}\right|, \left|\frac{v}{1+v} - \frac{y}{1+y}\right|\right), \quad (13)
\]
It is clear that any function in \( H_{w_2} \) is continuous and bounded on \( K \).

2. Lacunary Equistatistical Approximation

In this section, using the concept of Lacunary equistatistical convergence, we give a Korovkin type result for a sequence of positive linear operators defined on \( C(I^m) \), the space of all continuous real valued functions on the subset \( I^m \) of \( R^m \), and the real \( m \)-dimensional space. We first consider the case of \( m = 2 \). Following [7] we can state the following theorem.

Theorem 2. Let \( \theta = \{k_n\} \) be a lacunary sequence, and let \( \{L_n\} \) be a sequence of positive linear operators from \( H_{w_2} \) into \( C_B(K) \). \( L_n \) is satisfying \( L_n(f ; x , y) \rightarrow f(x, y) \) (\( \theta\)-equistat), \( \nu = 0, 1, 2 \), where \( f_m(u, v) \in H_{w_2}, k = 0, 1, 2, 3 \),
\[
f_0(u, v) = 1,
\]
\[
f_1(u, v) = \frac{u}{u + 1},
\]
\[
f_2(u, v) = \frac{v}{v + 1}, \quad (14)
\]
\[
f_3(u, v) = \left(\frac{u}{u + 1}\right)^2 + \left(\frac{v}{v + 1}\right)^2,
\]
then for all \( f \in H_{w_2} \),
\[
L_r(f ; x , y) \rightarrow f(x, y) \) (\( \theta\)-equistat). \quad (15)
\]
Proof. Let \((x, y) \in K \) be a fixed point, \( f \in H_{w_2} \), and assume that \( (14) \) holds. For every \( \varepsilon > 0 \), there exist \( \delta_1, \delta_2 > 0 \) such that \[ |f(u, v) - f(x, y)| < \varepsilon \] holds for all \((u, v) \in K \) satisfying
\[
\left|\frac{u}{u + 1} - \frac{x}{1 + x}\right| < \delta_1, \quad \left|\frac{v}{v + 1} - \frac{y}{1 + y}\right| < \delta_2. \quad (16)
\]
Let
\[
K_{\delta_1, \delta_2} := \{(u, v) \in K : \left|\frac{u}{u + 1} - \frac{x}{1 + x}\right| < \delta_1, \left|\frac{v}{v + 1} - \frac{y}{1 + y}\right| < \delta_2 \}.
\]
Hence,
\[
|f(u, v) - f(x, y)| = \left|f(u, v) - f(x, y)\right|_{K_{\delta_1, \delta_2}}(u, v) \quad (17)
\]
\[
< \varepsilon + 2M_{K_{\delta_1, \delta_2}}(u, v),
\]
where \( \chi_{K} \) denotes the characteristic function of the set \( P \). Observe that
\[
\chi_{K_{\delta_1, \delta_2}}(u, v) \leq \frac{1}{\delta_1}\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2 + \frac{1}{\delta_2}\left(\frac{v}{v+1} - \frac{y}{y+1}\right)^2. \quad (19)
\]
Using (18), (19), and $M := \|f\|$ we have
\[
\left| f(u, v) - f(x, y) \right|
\leq \varepsilon + \frac{2M}{\delta^2} \left\{ \left( \frac{u}{1+u} - \frac{x}{1+x} \right)^2 + \left( \frac{v}{v+1} - \frac{y}{y+1} \right)^2 \right\},
\]
where $\delta := \min\{\delta_1, \delta_2\}$.

By the linearity and positivity of the operators $\{L_r\}$ and by (18), we have
\[
L_r \left( \left( f_1 - \frac{u}{1+u} f_0 \right)^2 + \left( f_2 - \frac{v}{1+v} f_0 \right)^2; x, y \right)
\leq L_r \left( f_3; x, y \right)
\]
\[
- 2 \left[ \frac{x}{1+x} L_r \left( f_1; x, y \right) + \frac{y}{1+y} L_r \left( f_2; x, y \right) \right] L_r \left( f_3; x, y \right)
\]
\[
- 2 \left[ \frac{x}{1+x} L_r \left( f_1; x, y \right) + \frac{y}{1+y} L_r \left( f_2; x, y \right) \right]
\]
\[
+ \left( \frac{x}{1+x} \right)^2 + \left( \frac{y}{1+y} \right)^2 \right\} L_r \left( f_0; x, y \right).
\]
Hence, we get
\[
\left| L_r \left( f_1; x, y \right) - f(x, y) \right|
\leq L_r \left( f(u, v) - f(x, y); x, y \right)
\]
\[
+ \left| f(x, y) \right| \left| L_r \left( f_0; x, y \right) - f_0(x, y) \right|
\]
\[
\leq L_r \left( \varepsilon + \frac{2M}{\delta^2} \left\{ \left( \frac{u}{1+u} - \frac{x}{1+x} \right)^2 + \left( \frac{v}{v+1} - \frac{y}{y+1} \right)^2 \right\}; x, y \right)
\]
\[
+ M \left| L_r \left( f_0; x, y \right) - f_0(x, y) \right|
\]
\[
\leq L_r \left( \varepsilon + \frac{2M}{\delta^2} \left\{ \left( f_1 - \frac{x}{1+x} \cdot f_0 \right)^2 + \left( f_2 - \frac{y}{1+y} \cdot f_0 \right)^2 \right\}; x, y \right)
\]
\[
+ M \left| L_r \left( f_0; x, y \right) - f_0(x, y) \right|
\]
\[
= L_r \left( \varepsilon; x, y \right) + L_r \left( \frac{2M}{\delta^2} \left( f_1 - \frac{x}{1+x} \cdot f_0 \right)^2 \right.
\]
\[
+ \left( f_2 - \frac{y}{1+y} \cdot f_0 \right)^2; x, y \right)
\]
\[
+ M \left| L_r \left( f_0; x, y \right) - f_0(x, y) \right|
\]
\[
\leq \varepsilon + \frac{2M}{\delta^2} \left| L_r \left( f_3; x, y \right) - f_3(x, y) \right|
\]
\[
+ \frac{4M}{\delta^2} \left| L_r \left( f_1; x, y \right) - f_1(x, y) \right|
\]
\[
+ \frac{4M}{\delta^2} \left| L_r \left( f_2; x, y \right) - f_2(x, y) \right|
\]
\[
+ \left( \varepsilon + M + \frac{4M}{\delta^2} \right) \left| L_r \left( f_0; x, y \right) - f_0(x, y) \right|
\]
\[
= \frac{2M}{\delta^2} \left| L_r \left( f_3; x, y \right) - f_3(x, y) \right|
\]
\[
+ \frac{4M}{\delta^2} \left| L_r \left( f_1; x, y \right) - f_1(x, y) \right|
\]
\[
+ \frac{4M}{\delta^2} \left| L_r \left( f_2; x, y \right) - f_2(x, y) \right|
\]
\[
+ \varepsilon + N \left| L_r \left( f_0; x, y \right) - f_0(x, y) \right|,
\]
where $N := \varepsilon + M + 4M/\delta^2$. For a given $\mu > 0$, choose $\varepsilon > 0$ such that $\varepsilon < \mu$. Define the following sets:
\[
D_{\mu} (x, y) := \left\{ m \in \mathbb{N} : |L_m (f; x, y) - f (x, y)| \geq \mu \right\},
\]
\[
D_{\mu}^\nu (x, y) := \left\{ m \in \mathbb{N} : |L_m (f; x, y) - f (x, y)| \geq \frac{\mu - \varepsilon}{4N} \right\},
\]
where $\nu = 0, 1, 2, 3$. Then from (22) we clearly have
\[
D_{\mu} (x, y) \subseteq \bigcup_{\nu=0}^3 D_{\mu}^\nu (x, y).
\]
Therefore define the following real valued functions:
\[
s_{r,\mu} (x, y) := \frac{1}{h_r} \left\{ \left\{ m \in I_r : |L_m (f; x, y) - f (x, y)| \geq \mu \right\} \right\},
\]
\[
s_{r,\mu}^\nu (x, y) := \frac{1}{h_r} \left\{ \left\{ m \in I_r : |L_m (f; x, y) - f (x, y)| \geq \frac{\mu - \varepsilon}{4N} \right\} \right\},
\]
where $\nu = 0, 1, 2, 3$. Then by the monotonicity and (24) we get
\[
s_{r,\mu} (x, y) \leq \frac{1}{h_r} \sum_{\nu=0}^3 s_{r,\mu}^\nu (x, y)
\]
for all $x \in X$, and this implies the inequality
\[
\| s_{r,\mu}^\nu \|_K \leq \frac{1}{h_r} \sum_{\nu=0}^3 \| s_{r,\mu}^\nu \|_K.
\]
Taking limit in (27) as $r \to \infty$ and using (14) we have
\[
\lim_{r \to \infty} \| s_{r,\mu}^\nu \|_K = 0.
\]
Then for all \( f \in H_{w_2} \), we conclude that
\[
L_r (f; x, y) \rightarrow f(x, y), \ (\theta\text{-equistat}). \quad (29)
\]

Now replace \( I^2 \) by \( I^m = [0, \infty) \times \cdots \times [0, \infty) \) and by an induction, we consider the modulus of continuity type function \( w_2 \), then let \( H_{w_2} \) be the space of all real-valued functions \( f \in H \)
\[
\| f (u_1, u_2, \ldots, u_m) - f (x_1, x_2, \ldots, x_m) \| \\
\leq w_2 \left( f \left( \frac{u_1}{u_1 + 1} - \frac{x_1}{x_1 + 1}, \ldots, \frac{u_m}{u_m + 1} - \frac{x_m}{x_m + 1} \right) \right). \quad (30)
\]

Therefore, using a similar technique in the proof of Lemma 1 one can obtain the following result immediately.

**Theorem 3.** Let \( \theta = \{k_r\} \) be a lacunary sequence, and let \( \{L_r\} \) be a sequence of positive linear operators from \( H_{w_2} \) into \( C_\theta(I^m) \). \( L_r \) is satisfying
\[
L_r (f; x, y) \rightarrow f(x, y), \ (\theta\text{-equistat}), \quad \nu = 0, 1, 2, \ldots, m + 1,
\]
where \( f_k (u_1, u_2, \ldots, u_m) \in H_{w_2}, \ k = 0, 1, 2, \ldots, m + 1, \)
\[
f_0 (u_1, u_2, \ldots, u_m) = 1,
\]
\[
f_1 (u_1, u_2, \ldots, u_m) = \frac{u_1}{u_1 + 1},
\]
\[
\vdots
\]
\[
f_m (u_1, u_2, \ldots, u_m) = \frac{u_m}{u_m + 1},
\]
\[
f_{m+1} (u_1, u_2, \ldots, u_m) = \left( \frac{u_1}{u_1 + 1} \right)^2 + \left( \frac{u_2}{u_2 + 1} \right)^2 + \ldots + \left( \frac{u_m}{u_m + 1} \right)^2.
\]
\[
(32)
\]
Then for all \( f \in H_{w_2} \),
\[
L_r (f; u_1, u_2, \ldots, u_m) \rightarrow f(u_1, u_2, \ldots, u_m), \ (\theta\text{-equistat}). \quad (33)
\]

Assume that \( I = [0, \infty), K := I \times I \). One considers the following positive linear operators defined on \( H_{w_2} (K) \):
\[
B_n (f; x, y) = \frac{1}{(1 + x)^n (1 + y)^n} \times \sum_{k=0}^{n} \sum_{l=0}^{n} f \left( \frac{k}{n - k + 1}, \frac{l}{n - l + 1} \right) \binom{n}{k} \binom{n}{l} x^k y^l,
\]
\[
(34)
\]
where \( f \in H_{w_2}, (x, y) \in K \) and \( n \in \mathbb{N} \).

**Lemma 4.** Let \( \theta = \{k_r\} \) be a lacunary sequence, and let
\[
B_n (f; x, y) = \frac{1}{(1 + x)^n (1 + y)^n} \times \sum_{k=0}^{n} \sum_{l=0}^{n} f \left( \frac{k}{n - k + 1}, \frac{l}{n - l + 1} \right) \binom{n}{k} \binom{n}{l} x^k y^l.
\]
\[
(35)
\]
be a sequence of positive linear operators from \( H_{w_2} \) into \( C_\theta(K) \). If \( B_n \) is satisfying
\[
B_n (f; u; x, y) \rightarrow f(x, y), \ (\theta\text{-equistat}), \quad \nu = 0, 1, 2, 3,
\]
\[
f_0 (u, v) = 1,
\]
\[
f_1 (u, v) = \frac{u}{u + 1},
\]
\[
f_2 (u, v) = \frac{v}{v + 1},
\]
\[
f_3 (u, v) = \left( \frac{u}{u + 1} \right)^2 + \left( \frac{v}{v + 1} \right)^2,
\]
\[
(36)
\]
then for all \( f \in H_{w_2} \),
\[
B_n (f; x, y) \rightarrow f(x, y), \ (\theta\text{-equistat}). \quad (37)
\]

**Proof.** Assume that (36) holds, and let \( f \in H_{w_2} \). Since
\[
(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \quad (1 + y)^n = \sum_{l=0}^{n} \binom{n}{l} y^l,
\]
\[
(38)
\]
it is clear that, for all \( n \in \mathbb{N} \),
\[
B_n (f_0; x, y) = \frac{1}{(1 + x)^n (1 + y)^n} \sum_{k=0}^{n} \sum_{l=0}^{n} \binom{n}{k} x^k \binom{n}{l} y^l
\]
\[
(39)
\]
Now, by assumption we have
\[
B_n (f_0; x, y) \rightarrow f_0 (x, y), \ (\theta\text{-equistat}). \quad (40)
\]
Using the definition of $B_n$, we get

\begin{align*}
B_n(f_1;x,y) &= \frac{1}{(1+x)^n(1+y)^n} \\
& \quad \times \sum_{k=0}^{n} \frac{k}{(n-k+1)} \left(\binom{n}{k} x^k \binom{n}{l} y^l \sum_{i=0}^{l} \binom{n}{i} x^i y^i \right) \\
&= \frac{1}{(1+x)^n(1+y)^n} \sum_{k=0}^{n} k \left(\binom{n}{k} x^k \sum_{l=0}^{\infty} \binom{n}{l} y^l \right)
\end{align*}

(41)

To see this, by the definition of $B_n$, we first write

\begin{align*}
B_n(f_3;x,y) &= \frac{1}{(1+x)^n(1+y)^n} \\
& \quad \times \left[ k^2 \left(\frac{1}{(n+1)^2} + \frac{l^2}{(n+1)^2} \right) \right] \\
&= \frac{1}{(1+x)^n(1+y)^n} \sum_{k=0}^{n} k^2 \left(\binom{n}{k} x^k \sum_{l=0}^{\infty} \binom{n}{l} y^l \right)
\end{align*}

(42)

Then,

\begin{align*}
B_n(f_3;x,y) &= \frac{n(n-1)}{(n+1)^2} \frac{x^2}{(x+1)^2} + \frac{n}{(n+1)^2} \frac{x}{(x+1)^2} + \frac{y}{(y+1)^2}
\end{align*}

(43)

\begin{align*}
\left| B_n(f_3;x,y) - f_3(x,y) \right| &= \left| \frac{x^2}{(x+1)^2} + \frac{y^2}{(y+1)^2} \right| \frac{n(n-1)}{(n+1)^2} - 1
\end{align*}

(44)

The fact that $\lim_{n \to \infty} \frac{n(n-1)}{(n+1)^2} = 1$ and using a similar technique as in the proof of Lemma 1, we get

\begin{align*}
\lim_{n \to \infty} \frac{1}{h_n} \hat{\int}_{\mathcal{I}} \left| \left| \left| m \in \mathcal{I} : \left| B_n(f_3;x,y) - f_3(x,y) \right| \geq \varepsilon \right| \right| \right| = 0.
\end{align*}

(45)

Hence we have

\begin{align*}
B_n(f_3;x,y) \to f_3(x,y), \quad (\theta\text{-equistat}).
\end{align*}

(46)

Also we have

\begin{align*}
B_n(f_3;x,y) \to f_3(x,y), \quad (\theta\text{-equistat}).
\end{align*}

(47)
3. Rates of Lacunary Equistatistical Convergence

In this section we study the order of lacunary equi-statistical convergence of a sequence of positive linear operators acting on $H_w(K)$, where $K = I^m$. To achieve this we first consider the case of $m = 2$.

**Definition 5.** A sequence $\{f_r\}$ is called lacunary equi-statistically convergent to a function $f$ with rate $0 < \beta < 1$ if for every $\epsilon > 0$,

$$
\lim_{r \to \infty} \frac{s_{r,\epsilon}(x, y)}{r^{-\beta}} = 0,
$$

(53)

where $s_{r,\epsilon}(x, y)$ is given in Lemma 1. In this case it is denoted by

$$
f_r - f = o(r^{-\beta}), \quad (\text{equi-stat}) \quad \text{on} \quad K = I \times I.
$$

(54)

**Lemma 6.** Let $\{f_r\}$ and $\{g_r\}$ be two sequences of functions in $H_{w_2}(K)$, with

$$
f_r - f = o(r^{-\beta_1}), \quad (\text{equi-stat}),
$$

$$
g_r - g = o(r^{-\beta_2}), \quad (\text{equi-stat}).
$$

(55)

Then one has

$$
(f_r + g_r) - (f + g) = o(r^{-\beta}), \quad (\text{equi-stat}),
$$

(56)

where $\beta = \min\{\beta_1, \beta_2\}$.

**Proof.** Assume that $f_r - f = o(r^{-\beta_1})$, (equi-stat) and $g_r - g = o(r^{-\beta_2})$, (equi-stat) on $K$. For all $\epsilon > 0$, consider the following functions:

$$
s_{r,\epsilon}(x, y) := \frac{1}{h_r} \left| \left\{ n \in I_r : \left| (f_n + g_n)(x, y) - (f + g)(x, y) \right| \geq \epsilon \right\} \right|,
$$

$$
s_{r,\epsilon}^1(x, y) := \frac{1}{h_r} \left| \left\{ n \in I_r : \left| f_n(x) - f(x) \right| \geq \frac{\epsilon}{2} \right\} \right|,
$$

$$
s_{r,\epsilon}^2(x, y) := \frac{1}{h_r} \left| \left\{ n \in I_r : \left| g_n(x) - g(x) \right| \geq \frac{\epsilon}{2} \right\} \right|.
$$

(57)

Then we have

$$
\lim_{r \to \infty} \frac{\|s_{r,\epsilon}(x, y)\|_{H_{w_2}(K)}}{r^{-\beta}} = \lim_{r \to \infty} \frac{\|s_{r,\epsilon}^1(x, y)\|_{r^{-\beta}}}{r^{-\beta_1}}
$$

$$
+ \lim_{r \to \infty} \frac{\|s_{r,\epsilon}^2(x, y)\|_{r^{-\beta}}}{r^{-\beta_1}},
$$

(58)

and hence

$$
\frac{\|s_{r,\epsilon}(x, y)\|_{H_{w_2}(K)}}{r^{-\beta}} \leq \frac{\|s_{r,\epsilon}^1(x, y)\|_{H_{w_2}(K)}}{r^{-\beta_1}} + \frac{\|s_{r,\epsilon}^2(x, y)\|_{H_{w_2}(K)}}{r^{-\beta_1}}.
$$

(59)

Taking limit as $r \to \infty$ and using the assumption complete the proof.

Now we give the rate of lacunary equi-statistical convergence of a positive linear operators $L_r(f; x, y)$ to $f(x, y)$ with the help of modulus of continuity.

**Theorem 7.** Let $K = I \times I$, and let $L_r : H_{w_2}(K) \to H_{w_2}(K)$ be a sequence of positive linear operators. Assume that

(i) $L_r(f_0; x, y) - f_0 = o(r^{-\beta_1})$, (equi-stat) on $K$,

(ii) $\omega(f; \delta_{r,x}, \delta_{r,y}) = o(r^{-\beta_2})$, (equi-stat) on $K$ with

$$
\delta_{r,x} = \left| L_r \left( \left( \frac{u}{1+u} - \frac{x}{1+x} \right)^2, x \right) \right|,
$$

$$
\delta_{r,y} = \left| L_r \left( \left( \frac{v}{1+v} - \frac{y}{1+y} \right)^2, y \right) \right|.
$$

(60)

Then

$$
L_r(f; x, y) - f(x, y) = o(r^{-\beta}), \quad (\text{equi-stat}) \quad \text{on} \quad K,
$$

(61)

where $\beta = \min\{\beta_1, \beta_2\}$.

**Proof.** Let $f \in H_{w_2}(K)$ and $x \in K$. Use

$$
\|L_r(f; x, y) - f(x, y)\|
$$

$$
\leq L_r \left( \|f(u, v) - f(x, y)\| ; x, y \right) + \|f(x, y)\| L_r(f_0; x, y) - f(x, y) \right|
$$

$$
\leq L_r \left( w_2 \left( f_r \left( \frac{u}{1+u} - \frac{x}{1+x} \right) ; x, y \right) \right)
$$

$$
\leq (1 + L_r(f_0; x, y)) w_2 \left( f; \delta_{r,x}, \delta_{r,y} \right)
$$

$$
+ M \left| L_r(f_0; x, y) - f_0(x, y) \right|
$$

$$
= 2w_2 \left( f; \delta_{r,x}, \delta_{r,y} \right) + M \left| L_r(f_0; x, y) - f_0(x, y) \right|
$$

$$
+ w_2 \left( f; \delta_{r,x}, \delta_{r,y} \right) \left| L_r(f_0; x, y) - f_0(x, y) \right|,
$$

(62)

where $M = \|f\|_{H_{w_2}(K)}$. Using inequality (62), conditions (i) and (ii) we get

$$
\lim_{r \to \infty} \frac{1}{h_r} \left| \left\{ r \in I_r : |L_r(f; x, y) - f(x, y)| \geq \epsilon \right\} \right| = 0,
$$

(63)
so we have

\[ L_r(f; x, y) - f(x, y) = o(r^\beta), \quad (\theta\text{-equistat}) \text{ on } K. \]  

(64)

Finally we give the rate of lacunary equi-statistical convergence for the operators \( L_r(f; x, y) \) by using the Peetre's \( K \)-functional.

Theorem 8. Let \( f \in H_w(K) \) and \( \{K(f; \delta_{r,x}, \delta_{r,y})\} \) be the sequence of Peetre's \( K \)-functional. If

\[ \delta_{r,x} = \left\| L_r \left( \left( f_1 - \frac{x}{1 + x} \right); x, y \right) \right\|_{C_{\Theta}(K)} \]
\[ + \left\| L_r \left( \left( f_1 - \frac{x}{1 + x} \right)^2; x, y \right) \right\|_{C_{\Theta}(K)}, \]

\[ \delta_{r,y} = \left\| L_r \left( \left( f_2 - \frac{y}{1 + y} \right); x, y \right) \right\|_{C_{\Theta}(K)} \]
\[ + \left\| L_r \left( \left( f_2 - \frac{y}{1 + y} \right)^2; x, y \right) \right\|_{C_{\Theta}(K)}, \]

\[ \lim_{r \to \infty} \delta_{r,x} = 0, \quad (\theta\text{-equistat}) \]
\[ \lim_{r \to \infty} \delta_{r,y} = 0, \quad (\theta\text{-equistat}) \]

on \( x, y \in K \), then

\[ \left\| L_r(f; x, y) - f(x, y) \right\|_{C_{\Theta}(K)} \leq K \left( f; \delta_{r,x}, \delta_{r,y} \right). \]  

(68)

Proof. For each \( g \in H_w(K) \), we get

\[ \left\| L_r(g; x, y) - g(x, y) \right\|_{C_{\Theta}(K)} \]
\[ \leq \left\| g \right\|_{C_{\Theta}(K)} \left\| L_r \left( \left( f_1 - \frac{x}{1 + x} \right); x, y \right) \right\|_{C_{\Theta}(K)} \]
\[ + \left\| L_r \left( \left( f_1 - \frac{x}{1 + x} \right)^2; x, y \right) \right\|_{C_{\Theta}(K)} \]
\[ + \left\| g \right\|_{C_{\Theta}(K)} \left\| L_r \left( \left( f_2 - \frac{y}{1 + y} \right); x, y \right) \right\|_{C_{\Theta}(K)} \]
\[ + \left\| L_r \left( \left( f_2 - \frac{y}{1 + y} \right)^2; x, y \right) \right\|_{C_{\Theta}(K)} \]
\[ = \left( \delta_{r,x}, \delta_{r,y} \right) \right\| g \right\|_{C_{\Theta}(K)} \]  

(69)

\[ \square \]
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