Research Article

Determination of a Control Parameter for the Difference Schrödinger Equation

Allaberen Ashyralyev¹,² and Mesut Urun¹,³

¹ Department of Mathematics, Fatih University, Buyukcekmece, 34500 Istanbul, Turkey
² ITTU, 32 Gerogly Street, 74400 Ashgabat, Turkmenistan
³ Department of Mathematics, Murat Education Institution, 34353 Istanbul, Turkey

Correspondence should be addressed to Mesut Urun; mesuturun@gmail.com

Received 28 July 2013; Accepted 18 September 2013

The first order of accuracy difference scheme for the numerical solution of the boundary value problem for the differential equation with parameter \( p \),

\[
i(\frac{du(t)}{dt}) + Au(t) + iu(t) = f(t) + p, \quad 0 < t < T,
\]

\[
u(0) = \varphi, \quad u(T) = \psi
\]

in a Hilbert space \( H \) with self-adjoint positive definite operator \( A \) is constructed. The well-posedness of this difference scheme is established. The stability inequalities for the solution of difference schemes for three different types of control parameter problems for the Schrödinger equation are obtained.

1. Introduction: Difference Scheme

The theory and applications of well-posedness of inverse problems for partial differential equations have been studied extensively in a large cycle of papers (see, e.g., [1–24] and the references therein).

Our goal in this paper is to investigate Schrödinger equations with parameter. In the paper [25], the boundary value problem for the differential equation with parameter \( p \)

\[
i\frac{du(t)}{dt} + Au(t) + iu(t) = f(t) + p, \quad 0 < t < T,
\]

\[
u(0) = \varphi, \quad u(T) = \psi
\]

in a Hilbert space \( H \) with self-adjoint positive definite operator \( A \) was studied. The well-posedness of this problem was established. The stability inequalities for the solution of three determinations of control parameter problems for the Schrödinger equation were obtained. In the present paper, the first order of accuracy Rothe difference scheme

\[
i\tau^{-1}(u_k - u_{k-1}) + A u_k + i u_k = \varphi_k + p, \quad \varphi_k = f(t_k),
\]

\[
t_k = k\tau, \quad 1 \leq k \leq N, \quad N\tau = T
\]

for the approximate solution of the boundary value problem (1) for the differential equation with parameter \( p \) is presented. It is easy to see that

\[
u_k = \varphi_k + (A + il)^{-1}p,
\]

\[
p = (A + il)(\psi - \nu_N),
\]

where \( \{\nu_k\}_{k=0}^{N} \) is the solution of the following single-step difference scheme:

\[
i\tau^{-1}(\nu_k - \nu_{k-1}) + A\nu_k + i\nu_k = \varphi_k, \quad \varphi_k = f(t_k),
\]

\[
t_k = k\tau, \quad 1 \leq k \leq N, \quad N\tau = T
\]

\[
\nu_0 = \varphi, \quad \nu_N = \psi
\]

The theorem on well-posedness of difference problem (2) is proved. In practice, the stability inequalities for the solution of difference schemes for the approximate solution of three different types of control parameter problems are obtained.

The paper is organized as follows. Section 1 is the introduction. In Section 2, the main theorem on stability of difference problem (2) is established. In Section 3, theorems on the stability inequalities for the solution of difference schemes for the approximate solution of three different types of control
2. The Main Theorem on Stability

In this section, we will study the stability of difference scheme (2).

Let \([0, T] = [t_k = kr, k = 1, \ldots, N, N\tau = T]\) be the uniform grid space with step size \(\tau > 0\), where \(N\) is a fixed positive integer. Throughout the present paper, \(\mathcal{G}([0, T], H)\) denotes the linear space of grid functions \( \varphi^\tau = \{ \varphi_k \}_{k=1}^N \) with values in the Hilbert space \(H\). Let \( \mathcal{C}_\tau(H) = \mathcal{C}([0, T], H) \) be the Banach space of bounded grid functions with the norm

\[
\| \varphi^\tau \|_{\mathcal{C}_\tau(H)} = \max_{1 \leq k \leq N} \| \varphi_k \|_H. \tag{5}
\]

Let us start with a lemma we need below. We denote that \(R = ((1+\tau)I - i\tau A)^{-1}\) is the step operator of problem (2).

**Lemma 1.** Assume that \(A\) is a positive definite self-adjoint operator. The operator \(I - R^N\) has an inverse \(T_\tau = (I - R^N)^{-1}\) and the following estimate is satisfied:

\[
\| T_\tau \|_{H \rightarrow H} \leq M (\delta) . \tag{6}
\]

**Proof.** The proof of estimate (6) is based on the triangle inequality and the estimate

\[
\left\| (1 - R^N)^{-1} \right\|_{H \rightarrow H} \leq \sup_{\delta \in \mathbb{R}} \frac{1}{1 - \left| (1 + \tau (1 - i\mu))^{-N} \right|} \leq \frac{1}{1 - \left( (1 + \tau)^2 + (\tau\delta)^2 \right)^{N/2}} \leq \mu (\delta) . \tag{7}
\]

Now, let us obtain the formula for the solution of problem (2). It is clear that the first order of accuracy difference scheme

\[
i \tau^{-1} (u_k - u_{k-1}) + Au_k + iu_k = p + \varphi_k, \quad \varphi_k = f (t_k) , \tag{8}
\]

\[
t_k = kr, \quad 1 \leq k \leq N, \quad N\tau = T, \quad u_0 = \varphi
\]

has a solution and the following formula

\[
u_k = R^k \varphi - i \sum_{j=1}^{k} R^{k-j+1} \left( p + \varphi_j \right) \tau, \quad 1 \leq k \leq N \tag{9}
\]

is satisfied. Applying formula (9) and the boundary condition

\[u_N = \psi, \tag{10}\]

we can write

\[
\psi = R^N \varphi - i \sum_{j=1}^{N} R^{N-j+1} \varphi_j \tau - i \sum_{j=1}^{N} R^{N-j+1} \tau p . \tag{11}
\]

Since

\[
-i \sum_{j=1}^{N} R^{N-j+1} \tau = -i (I - iA)^{-1} (I - R) \sum_{j=1}^{N} R^{N-j} \tau = -i (I - iA)^{-1} \left( I - R^N \right) \tag{12}
\]

we have that

\[
\psi = R^N \varphi - i \sum_{j=1}^{N} R^{N-j+1} \varphi_j \tau - i \sum_{j=1}^{N} R^{N-j+1} \tau p . \tag{13}
\]

By Lemma 1, we get

\[
p = T_\tau \left( \left( I - iA \right) \psi - (I - iA) R^N \varphi \right) - \sum_{j=1}^{N} \left( I - iA \right) R^{N-j+1} \varphi_j \tau . \tag{14}
\]

Using (9) and (14), we get

\[
u_k = R^k \varphi - i \sum_{j=1}^{k} R^{k-j+1} \varphi_j \tau
\]

\[
+ \sum_{j=1}^{k} R^{k-j+1} \tau T_\tau \left( \left( I - iA \right) \psi - (I - iA) R^N \varphi \right) - \sum_{j=1}^{N} \left( I - iA \right) R^{N-j+1} \varphi_j \tau , \tag{15}
\]

\[1 \leq k \leq N.
\]

Since

\[
\sum_{j=1}^{k} R^{k-j+1} \tau = (I - iA)^{-1} (I - R) \sum_{j=1}^{k} R^{k-j} \tau = (I - iA)^{-1} \left( I - R^k \right) , \tag{16}
\]

we have that

\[
u_k = R^k \varphi + \sum_{j=1}^{k} R^{k-j+1} \varphi_j \tau
\]

\[
+ (I - R^k) T_\tau \left( \psi - R^N \varphi - \sum_{j=1}^{N} R^{N-j+1} \varphi_j \tau \right) , \tag{17}
\]

\[1 \leq k \leq N.
\]

Hence, difference scheme (2) is uniquely solvable and for the solution, formulas (14) and (17) hold.
Theorem 2. Suppose that the assumption of Lemma 1 holds. Let \( \varphi, \psi \in D(A) \). Then, for the solution \((u_k^N)_{k=1}^N, p\) of difference scheme (2) in \( C_c(H) \times H \), the estimates

\[
\|p\|_H \leq M \left[ \|A\varphi\|_H + \|A\psi\|_H \\
+ \|\varphi_1\|_H + \max_{2 \leq k \leq N} \left\| \frac{\varphi_k - \varphi_{k-1}}{\tau} \right\|_H \right],
\]

\[
\|u_k^N\|_{C_c(H)} \leq M \left[ \|\varphi\|_H + \|\psi\|_H + \left\| (\varphi_k^N)_{k=1}^N \right\|_{C_c(H)} \right]
\]

hold, where \( M \) is independent of \( \tau, \varphi, \psi, \) and \( (\varphi_k^N)_{k=1}^N \).

Proof. From formulas (9) and (14), it follows that

\[
p = T_\tau \left[ A\psi - R^N A\varphi - \varphi_N + R^N \varphi_1 \\
- \sum_{j=2}^N R^{N-j+1}(\varphi_{j-1} - \varphi_j) \right].
\]

Using this formula, the triangle inequality, and estimate (6), we obtain

\[
\|p\|_H \leq T_\tau \left[ \|A\varphi\|_H + \|A\psi\|_H \\
+ \sum_{j=2}^N \|R^{N-j+1}\|_{H \to H} \|\varphi_j - \varphi_{j-1}\|_H \right]
\]

\[
+ \|\varphi_N\| + \|R^N\|_{H \to H} \|\varphi_1\|_H \leq M \left[ \|A\varphi\|_H + \|A\psi\|_H + \left\| (\varphi_k^N)_{k=1}^N \right\|_{C_c(H)} \right].
\]

Estimate (18) is proved. Using formula (17), the triangle inequality, and estimate (6), we obtain

\[
\|u_k^N\|_{H} \leq \left[ \|R^k\|_{H \to H} \|\varphi\|_H + \sum_{j=1}^k \|R^{k-j+1}\|_{H \to H} \|\varphi_j\|_H \right] T_\tau \left[ \|A\varphi\|_H + \|A\psi\|_H \\
+ \|\varphi_1\|_H + \|\varphi_N\| + \|R^N\|_{H \to H} \|\varphi_1\|_H \right]
\]

\[
\leq M \left[ \|\varphi\|_H + \|\psi\|_H + \left\| (\varphi_k^N)_{k=1}^N \right\|_{C_c(H)} \right]
\]

for any \( k \). From that, it follows estimate (19). This completes the proof of Theorem 2.

3. Applications

Now, we consider the simple applications of main Theorem 2.

First, the boundary value problem for the Schrödinger equation

\[
iu_t - (a(x)u_x)_x + \delta u + iu = p(x) + f(t, x),
\]

\[0 < t < T, \quad 0 < x < 1,\] (23)

\[u(0, x) = \varphi(x), \quad u(T, x) = \psi(x), \quad 0 \leq x \leq 1,\] (24)

\[u(t, 0) = u(t, 1), \quad u_x(t, 0) = u_x(t, 1), \quad 0 \leq t \leq T\] (25)

is considered. Problem (23) has a unique smooth solution \((u(t, x), p(x))\) for the smooth functions \(a(x) \geq a > 0, x \in (0, 1), \delta > 0, a(1) = a(0), \varphi(x), \psi(x) (x \in [0, 1]), \) and \( f(t, x) (t \in (0, T), x \in (0, 1)).\) This allows us to reduce the boundary value problem (23) to the boundary value problem (1) in a Hilbert space \( H = L_2[0, 1] \) with a self-adjoint positive definite operator \( A^* \) defined by formula

\[
A^*u(x) = - (a(x)u_x)_x + \delta u
\]

with domain

\[
D(A^*) = \{u(x) : u(x), u_x(x), (a(x)u_x)_x \} \in L_2[0, 1],
\]

\[u(1) = u(0), \quad u_x(1) = u_x(0).\] (26)

The discretization of problem (23) is carried out in two steps. In the first step, we define the grid space

\[
[0, 1]_h = \{x = x_n : x_n = nh, 0 \leq n \leq M, \ Mh = 1\}.
\] (27)

Let us introduce the Hilbert space \( L_{2h} = L_2([0, 1]_h) \) of the grid functions

\[
\varphi^h(x) = \{\varphi^h_{n-1} : n = 1, \ldots, M\} \]

defined on \([0, 1]_h\), equipped with the norm

\[
\|\varphi^h\|_{L_{2h}} = \left( \sum_{x \in [0, 1]_h} |\varphi(x)|^2 h \right)^{1/2}.
\] (28)

To the differential operator \( A^* \) defined by formula (24), we assign the difference operator \( A^h \) by the formula

\[
A^h \varphi^h(x) = \{- (a(x)\varphi)_x, x, + \delta \varphi^h_{M-1}\}
\]

acting in the space of grid functions \( \varphi^h(x) = \{\varphi^h_{n-1} : n = 1, \ldots, M\} \) satisfying the conditions \( \varphi_0^h = \varphi^h_M, \varphi_1^h - \varphi_0^h = \varphi^h_M - \varphi^h_{M-1}\). It is well known that \( A^h \) is a self-adjoint positive definite operator in \( L_{2h}\). With the help of \( A^h \), we reach the boundary value problem

\[
i \frac{dt}{dt}h(t, x) + A^h u^h(t, x) + iu^h(t, x) = p^h(x) + f^h(t, x),
\]

\[0 < t < T, \quad x \in [0, 1]_h,\] (29)

\[u^h(0, x) = \varphi^h(x), \quad u^h(T, x) = \psi^h(x), \quad x \in [0, 1]_h.\] (30)
In the second step, we replace (30) with the difference scheme (2)
\[
\frac{u^h_k(x) - u^h_{k-1}(x)}{\tau} + A^h_k u^h_k(x) + i u^h_k(x) = p^h_k(x) + f^h_k(x),
\]
\[
f^h_k(x) = f^h(t_k, x), \quad t_k = k\tau, \quad N\tau = T, \quad 1 \leq k \leq N, \quad x \in [0, 1],
\]
\[
u^h(0, x) = \varphi^h(x), \quad u^h(T, x) = \psi^h(x), \quad x \in [0, 1].
\]
(31)

**Theorem 3.** The solution pairs \((u^h_k(x))_0^N, p^h(x))\) of problem (31) satisfy the stability estimates
\[
\left\| p^h \right\|_{L^{2h}} \leq M_1 \left[ \left\| \varphi^h \right\|_{L^{2h}} + \left\| A^x \varphi^h \right\|_{L^{2h}} + \left\| \psi^h \right\|_{L^{2h}} + \frac{\left\| f^h \right\|_{C_T}}{h^2} \right],
\]
\[
\left\| i u^h_k \right\|_{C^0_T} \leq M_2 \left[ \left\| \varphi^h \right\|_{L^{2h}} + \left\| \psi^h \right\|_{L^{2h}} + \frac{\left\| f^h \right\|_{C_T}}{h^2} \right],
\]
(32)

where \(M_1, M_2\) do not depend on \(\varphi^h, \psi^h,\) and \(f^h, 1 \leq k \leq N.\)

Here, \(C^0_T(L_{2h})\) is the grid space of grid functions \(\{f^h\}_1^N\) defined on \([0, T] \times [0, 1] h\) with norm
\[
\left\| i u^h_k \right\|_{C^0_T} = \frac{\left\| f^h \right\|_{C_T}}{h^2} + \sup_{1 \leq k \leq N} \left\| f^h \right\|_{C_T}
\]
(33)

The proof of Theorem 3 is based on formulas for \(p^h(x)\) and \(u^h(x)\), and the symmetry property of operator \(A^x\).

Second, let \(\Omega = \{x = (x_1, \ldots, x_n) : 0 < x_k < 1, k = 1, \ldots, n\}\) be the unit cube in the \(n\)-dimensional Euclidean space \(\mathbb{R}^n\) with boundary \(S, \quad \partial \Omega = \Omega \cup S.\) In \([0, T] \times \Omega,\) the boundary value problem for the multidimensional Schrödinger equation
\[
i \frac{\partial u(t, x)}{\partial t} - \sum_{r=1}^n (a_r(x) u_{x_r})_{x_r} + i u = p(x) + f(t, x), \quad x = (x_1, \ldots, x_n) \in \Omega, \quad 0 < t < T, \quad (34)
\]
\[
u(0, x) = \varphi(x), \quad u(T, x) = \psi(x), \quad x \in \Omega,
\]
\[
u(t, x) = 0, \quad x \in S, \quad 0 \leq t \leq T
\]
is considered. Here, \(a_r(x) \geq a > 0 (x \in \Omega), f(t, x) (t \in (0, T), x \in \Omega),\) and \(\varphi(x), \psi(x) (x \in \Omega)\) are given smooth functions.

We consider the Hilbert space \(L^2(\Omega)\) of all square integrable functions \(u \in \Omega,\) equipped with the norm
\[
\left\| u \right\|_{L^2(\Omega)} = \left( \int_{\Omega} \left| u(x) \right|^2 dx \right)^{1/2}.
\]
(35)

Problem (34) has a unique smooth solution \((u(t, x), p(x))\) for the smooth functions \(\varphi(x), \psi(x), a_r(x),\) and \(f(t, x).\) This allows us to reduce the problem (34) to the boundary value problem (1) in the Hilbert space \(H = L^2(\Omega)\) with a self-adjoint positive definite operator \(A^x\) defined by the formula
\[
A^x u(x) = -\sum_{r=1}^n (a_r(x) u_{x_r})_{x_r},
\]
(36)

with domain
\[
D(A^x) = \left\{ u(x) : u(x), u_{x_r}(x), (a_r(x) u_{x_r})_{x_r} \in L^2(\Omega), 1 \leq r \leq n, u(x) = 0, x \in S \right\}.
\]
(37)

The discretization of problem (34) is carried out in two steps. In the first step, we define the grid space
\[
\Omega_h = \{x = x_r = (h_1 j_1, \ldots, h_n j_n) : j = (j_1, \ldots, j_n), 0 \leq j_r \leq N_r, N_r h_r = 1, r = 1, \ldots, n\},
\]
\[
\Omega_h = \partial \Omega_h, \quad S_h = \partial \Omega_h \cap S
\]
and introduce the Hilbert space \(L^2_{2h} = L^2(\Omega_h)\) of the grid functions
\[
\varphi^h(x) = \{\varphi(h_1 j_1, \ldots, h_n j_n)\}
\]
(39)

defined on \(\Omega_h,\) equipped with the norm
\[
\left\| \varphi^h \right\|_{L^2_{2h}} = \left( \sum_{x \in \Omega_h} \left| \varphi(x) \right|^2 h_1 \cdots h_n \right)^{1/2}.
\]
(40)

To the differential operator \(A^x\) defined by formula (36), we assign the difference operator \(A^x_h\) by the formula
\[
A^x_h u^h = -\sum_{r=1}^n (a_r(x) u^h_{x_r})_{x_r},
\]
(41)

where \(A^x_h\) is known as self-adjoint positive definite operator in \(L^2_{2h}\) acting in the space of grid functions \(u^h(x)\) satisfying the conditions \(u^h(x) = 0\) for all \(x \in S_h.\) With the help of the difference operator \(A^x_h,\) we arrive to the following boundary value problem:
\[
i u^h_t(t, x) + A^x_h u^h(t, x) + i u^h(t, x) = p^h(t, x) + f^h(t, x), \quad 0 < t < T, \quad x \in \Omega_h,
\]
\[
u^h(0, x) = \varphi^h(x), \quad u^h(T, x) = \psi^h(x), \quad x \in \Omega_h
\]
(42)

for an infinite system of ordinary differential equations.
The first order of accuracy difference scheme for the solution of problem (42) is
\[
\frac{u_h^k(x) - u_{h-1}^k(x)}{\tau} + A_{k}^h u_{k}^h(x) + i\delta u_{k}^h(x) = \delta p^h(x) + f_k^h(x),
\]
(43)
\[
f_k^h(x) = f^h(t_k, x), \quad t_k = k\tau, \quad N\tau = T, \quad 1 \leq k \leq N, \quad x \in \Omega_h,
\]
\[
u_h^0(0, x) = \varphi_h^0(x), \quad u_h^T(x) = \psi_h^0(x), \quad x \in \Omega_h.
\]

**Theorem 4.** The solution pairs \((u_h^k(x))_0^N, p^h(x))\) of problem (43) satisfy the stability estimates
\[
\left\|p^h\right\|_{L_2^h} \leq M_1 \left[ \left\|\varphi^h\right\|_{L_2^h} + \left\|A^h\varphi^h\right\|_{L_2^h} + \left\|\psi^h\right\|_{L_2^h} + \left\|A^h\psi^h\right\|_{L_2^h} \right],
\]
\[
\left\|u_h^k\right\|_{C^1(\tau_2^h)} \leq M_2 \left[ \left\|\varphi^h\right\|_{L_2^h} + \left\|\psi^h\right\|_{L_2^h} + \left\|f_k^h\right\|_{C^1(\tau_2^h)} \right],
\]
where \(M_1\) and \(M_2\) do not depend on \(\varphi^h, \psi^h,\) and \(f_k^h, 1 \leq k \leq N.\)

Here, \(C_1^1(\tau_2^h)\) is the grid space of grid functions \(\{f_k^h\}_1^N\) defined on \([0, T]_\tau \times \Omega_h\) with norm
\[
\left\|f_k^h\right\|_{C_1^1(\tau_2^h)} = \left\|\{f_k^h\}_1^N\right\|_{C_1(\tau_2^h)} + \sup_{1 \leq k \leq N} \left\|f_k^h\right\|_{L_2^h},
\]
\[
\left\|\{f_k^h\}_1^N\right\|_{C_1(\tau_2^h)} = \max_{1 \leq k \leq N} \left\|f_k^h\right\|_{L_2^h}.
\]

The proof of Theorem 4 is based on Theorem 3 and the symmetry property of the operator \(A^h\) is defined by formula (34) and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in \(L_2^h.\)

**Theorem 5.** For the solutions of the elliptic difference problem [26]
\[
A^h_u(x) = \omega^h(x), \quad x \in \Omega_h,
\]
\[
u_h^0(x) = 0, \quad x \in S_h,
\]
the following coercivity inequality holds:
\[
\sum_{r=1}^N \left\|u_{r,k}^h\right\|_{L_2^h} \leq M \left\|\omega^h\right\|_{L_2^h},
\]
where \(M\) does not depend on \(h\) and \(\omega^h.\)

Third, in \([0, T] \times \Omega,\) the boundary value problem for the multidimensional Schrödinger equation
\[
i\frac{\partial u(t, x)}{\partial t} - \sum_{r=1}^n \left(a_r(x) u_{x_r}\right)_{x_r} + \delta u + in = p^h(x) + f^h(t, x),
\]
(48)
\[
x = (x_1, \ldots, x_n) \in \Omega, \quad 0 < t < T,
\]
\[
\left\{u(0, x) = \varphi(x), \quad u(T, x) = \psi(x), \quad x \in \Omega,\right.\]
\[
\frac{\partial u(t, x)}{\partial n} = 0, \quad x \in S, \quad 0 \leq t \leq T
\]
with the Neumann condition is considered. Here, \(\bar{n}\) is the normal vector to \(S, \delta > 0,\) and \(a_r(x) \geq a > 0 (x \in \Omega), f(t, x) (t \in (0, T), x \in \Omega),\) and \(\varphi(x), \psi(x) (x \in \Omega)\) are given smooth functions.

Problem (48) has a unique smooth solution \((u(t, x), p(x))\) for the smooth functions \(\varphi(x), \psi(x), a_r(x),\) and \(f(t, x).\) This allows us to reduce the problem (48) to the boundary value problem (1) in the Hilbert space \(H = L_2(\Omega)\) with a self-adjoint positive definite operator \(A^x\) defined by formula
\[
A^x u(x) = -\sum_{r=1}^n \left(a_r(x) u_{x_r}\right)_{x_r} + \delta u
\]
(49)
with domain
\[
D(A^x) = \left\{ u(x) : u(x), u_{x_r}(x), (a_r(x) u_{x_r})_{x_r} \in L_2(\Omega), 1 \leq r \leq n, \frac{\partial u}{\partial n}(x) = 0, x \in S \right\}.
\]
(50)
The discretization of problem (48) is carried out in two steps. In the first step, we define the difference operator \(A^x_h\) by the formula
\[
A^x_h u(x) = -\sum_{r=1}^n \left(a_r(x) u_{x_r}\right)_{x_r} + \delta u^h
\]
(51)
where \(A^x_h\) is known as self-adjoint positive definite operator in \(L_2^h,\) acting in the space of grid functions \(u^h(x)\) satisfying the conditions \(D_h^x u^h(x) = 0\) for all \(x \in S_h.\) Here, \(D_h^x\) is the approximation of the operator \(\partial / \partial n.\) With the help of the difference operator \(A^x_h,\) we arrive to the following boundary value problem:
\[
\left\{u_t^h(t, x) + A^x_h u^h(t, x) + iu^h(t, x) = p^h(t, x) + f^h(t, x), 0 < t < T, x \in \Omega_h, u^h(0, x) = \varphi^h(x), \quad u^h(T, x) = \psi^h(x), \quad x \in \Omega_h\right.\]
(52)
The first order of accuracy difference scheme for the solution of problem (52) is

\[
\frac{u^h_k(x) - u^h_{k-1}(x)}{\tau} + A^h_k u^h_k(x) + i u^h_k(x) = p^h(x) + f^h_k(x),
\]

\[
f^h_k(x) = f^h(t_k, x), \quad t_k = k\tau, \quad N\tau = T, \quad 1 \leq k \leq N, \quad x \in \Omega_h,
\]

\[
u^h(0, x) = \nu^h(x), \quad \nu^h(T, x) = \nu^h(x), \quad x \in \Omega_h.
\]

(61)

**Theorem 6.** The solution pairs \((u^h_k(x))_0^N, \psi^h(x)\) of problem (53) satisfy the stability estimates

\[
\|p^h\|_{L_{2h}} \leq M_1 \left[ \|\phi^h\|_{L_{2h}} + \|A^h_k \phi^h\|_{L_{2h}} + \|\psi^h\|_{L_{2h}} \right] + \|A^h_k \psi^h\|_{L_{2h}} + \left\| f^h_k \right\|_{C(\Omega_{2h})},
\]

\[
\left\| u^h_k \right\|_{C(\Omega_{2h})} \leq M_2 \left[ \|\phi^h\|_{L_{2h}} + \|\psi^h\|_{L_{2h}} + \left\| f^h_k \right\|_{C(\Omega_{2h})} \right],
\]

(54)

where \(M_1\) and \(M_2\) do not depend on \(\phi^h, \psi^h,\) and \(f^h_k, 1 \leq k \leq N.\)

The proof of Theorem 6 is based on Theorem 2 and the symmetry property of the operator \(A^h_k\) is defined by formula (51) and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in \(L_{2h}.\)

**Theorem 7.** For the solution of the elliptic difference problem [26]

\[
A^h_k u^h(x) = \omega^h(x), \quad x \in \Omega_h,
\]

\[
D^h u^h(x) = 0, \quad x \in S_h,
\]

the following coercivity inequality holds:

\[
\sum_{r=1}^{N} \|u^h_k\|_{L_{2h}} \leq M \|\omega^h\|_{L_{2h}},
\]

(56)

where \(M\) does not depend on \(h\) and \(\omega^h.\)

**4. Numerical Results**

In present section, for numerical analysis, the following boundary value problem

\[
\frac{\partial u(t, x)}{\partial t} - \frac{\partial^2 u(t, x)}{\partial x^2} + i u(t, x) = p(t, x) + f(t, x),
\]

\[
x \in (0, \pi), \quad t \in (0, 1),
\]

\[
u(0, x) = \sin x, \quad u(1, x) = e^{-1} \sin x, \quad x \in [0, \pi],
\]

\[
u(t, 0) = u(t, \pi) = 0, \quad t \in [0, 1]
\]

is considered. The exact solution of problem (57) is \(u(t, x) = e^{-t} \sin x\) and \(p(x) = \sin x.\)

The first order of accuracy difference scheme

\[
\frac{u^k_{n} - u^k_{n-1}}{\tau} - \frac{u^k_{n+1} - 2u^k_{n} + u^k_{n-1}}{h^2} + i u^k_{n} = q^k_n + p(x_n),
\]

\[
1 \leq k \leq N, \quad 1 \leq n \leq M - 1,
\]

\[
q^k_n = f(t_k, x_n) = (e^{-t_k} - 1) \sin x_n,
\]

\[
t_k = k\tau, \quad 0 \leq k \leq N, \quad N\tau = 1,
\]

\[
x_n = nh, \quad 1 \leq n \leq M - 1, \quad Mh = \pi,
\]

\[
u^0_0 = \sin(x_n), \quad \nu^N_0 = e^{-1} \sin(x_n), \quad x_n = nh, 0 \leq n \leq M,
\]

\[
u^k_0 = u^k_M = 0, \quad 0 \leq k \leq N
\]

(58)

for the numerical solution of problem (57) is constructed.

For obtaining the values of \(p(x_n)\) at the grid points, we will use the following equation:

\[
p(x_n) = -e^{-1} \sin(x_n+1) - 2 \sin(x_n) + \sin(x_n-1)
\]

\[
+ ie^{-1} \sin(x_n) + \frac{iN^2_n - 2N_n + N^2_n-1 - iv^N_n}{h^2},
\]

(59)

\[
x_n = nh, \quad 1 \leq n \leq M - 1,
\]

where \(v^s_k, s = n \pm 1,\) and \(n\) is the solution of the first order of accuracy difference scheme

\[
\frac{v^k_n - v^k_{n-1}}{\tau} - \frac{v^k_{n+1} - 2v^k_n + v^k_{n-1}}{h^2} + iv^k_n = q^k_n,
\]

\[
q^k_n = f(t_k, x_n), \quad t_k = k\tau, \quad 1 \leq k \leq N, \quad N\tau = 1,
\]

\[
x_n = nh, \quad 1 \leq n \leq M - 1, \quad Mh = \pi,
\]

(60)

\[
v^N_n - v^0_n = (e^{-1} - 1) \sin(x_n), \quad x_n = nh, 0 \leq n \leq M,
\]

\[
v^k_0 = v^k_M = 0, \quad 0 \leq k \leq N
\]

generated by difference scheme (58).

Using the difference scheme (60), we obtain \((N + 1) \times (M + 1)\) system of linear equations and we can write them in the matrix form as

\[
A v_{n+1} + B v_{n} + C v_{n-1} = R q_n, \quad 1 \leq n \leq M - 1,
\]

\[
v_0 = v_M = 0,
\]

(61)
where
\[
C = A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 \\
\end{bmatrix}_{(N+1) \times (N+1)},
\]

\[
B = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
y & z & 0 & 0 & 0 \\
y & z & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & y & z & 0 \\
0 & 0 & 0 & 0 & y & z \\
\end{bmatrix}_{(N+1) \times (N+1)}.
\]

Here,
\[
x = -\frac{1}{h^2}, \quad y = -\frac{i}{\tau}, \quad z = \frac{i}{\tau} + \frac{2}{h^2} + i,
\]

\[
v_s = \begin{bmatrix}
\varphi^0_s \\
\vdots \\
\varphi^N_s \\
\end{bmatrix}_{(N+1) \times 1}
\]

\[
\varphi_n = \begin{bmatrix}
(e^{-1} - 1) \sin x_n \\
q_n^1 \\
\vdots \\
q_n^{N-1} \\
q_n^N \\
\end{bmatrix}_{(N+1) \times 1}.
\]

So, we have the second-order difference equation with respect to \( n \) with matrix coefficients. Using the modified Gauss elimination method, we can obtain \( u^n_k \), \( 0 \leq k \leq N \), \( 0 \leq n \leq M \).

For the solution of the matrix equations, we seek the solution of the form
\[
u_n = \alpha_{n+1} v_{n+1} + \beta_{n+1}, \quad n = M - 1, \ldots, 2, 1,
\]

\[
u_M = 0,
\]

where \( \alpha_j \) and \( \beta_j \), \( j = 1, \ldots, M \), are calculated as
\[
\alpha_{n+1} = -(B + Ca_n)^{-1} (A),
\]

\[
\beta_{n+1} = (B + Ca_n)^{-1} (Dq_n - C\beta_n),
\]

where \( a \) is \((N+1) \times (N+1)\) and \( \beta_1 \) is \((N+1) \times 1\) zero matrix.

Then, using (59), values of \( p(x_n) \) at grid points are obtained. Replacing \( p(x_n) \) in (58), we get \((N+1) \times (M+1)\) system of linear equations and it can be written in the matrix form
\[
A_2 u_{n+1} + B_2 u_n + C_2 u_{n-1} = R \theta_n, \quad 1 \leq n \leq M - 1,
\]

\[
u_0 = u_M = 0,
\]

5. Conclusion

In the present study, the well-posedness of difference problem for the approximate solution of determination of a control
parameter for the Schrödinger equation is established. In practice, the stability inequalities for the solution of difference schemes of the approximate solution of three different types of control parameter problems are obtained. The well-posedness of the boundary value problem (1) is established. The stability inequalities for the solution of difference schemes for three different types of control parameter problems for the Schrödinger equation are obtained. Moreover, applying the result of the monograph [14], the high order of accuracy single-step difference schemes for the numerical solution of the boundary value problem (1) can be presented. Of course, the stability inequalities for the solution of these difference schemes have been established without any assumptions about the grid steps.

Conflict of Interests

The authors declare that they have no conflict of interests.

References
