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Research Article

Numerical Solution of the Fractional Partial Differential Equations by the Two-Dimensional Fractional-Order Legendre Functions

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A numerical method is presented to obtain the approximate solutions of the fractional partial differential equations (FPDEs). The basic idea of this method is to achieve the approximate solutions in a generalized expansion form of two-dimensional fractional-order Legendre functions (2D-FLFs). The operational matrices of integration and derivative for 2D-FLFs are first derived. Then, by these matrices, a system of algebraic equations is obtained from FPDEs. Hence, by solving this system, the unknown 2D-FLFs coefficients can be computed. Three examples are discussed to demonstrate the validity and applicability of the proposed method.

1. Introduction

Fractional partial differential equations play a significant role in modeling physical and engineering processes. Therefore, there is an urgent need to develop efficient and fast convergent methods for FPDEs. Recently, several different techniques, including Adomian's decomposition method (ADM) [1, 2], homotopy perturbation method (HPM) [3–5], variational iteration method (VIM) [6–8], spectral methods [9–13], orthogonal polynomials method [14–17], and wavelets method [18–21] have been presented and applied to solve FPDEs.

The method based on the orthogonal functions is a wonderful and powerful tool for solving the FDEs and has enjoyed many successes in this realm. The operational matrix of fractional integration has been determined for some types of orthogonal polynomials, such as Chebyshev polynomials [16], Legendre polynomials [22], Laguerre polynomials [23–25], and Jacobi polynomials [26]. Moreover, the operational matrix of fractional derivative for Chebyshev polynomials [9] and Legendre polynomials [9, 14] also has been derived. However, since these polynomials using integer power series to approximate fractional ones, it cannot accurately represent properties of fractional calculus. Recently, Rida and Yousef [27] presented a fractional extension of the classical Legendre polynomials by replacing the integer order derivative in Rodrigues formula with fractional order derivatives. The defect is that the complexity of these functions made them unsuitable for solving FDEs. Subsequently, Kazem et al. [28] presented the orthogonal fractional order Legendre functions based on shifted Legendre polynomials to find the numerical solution of FDEs and drew a conclusion that their method is accurate, effective, and easy to implement.

Benefiting from their "exponential-convergence" property when smooth solutions are involved, spectral methods have been widely and effectively used for the numerical solution of partial differential equations. The basic idea of spectral methods is to expand a function into sets of smooth global functions, called the trial functions. Because of their special properties, the orthogonal polynomials are usually chosen to be trial functions. Spectral methods can obtain very accurate approximations for a smooth solution while only need a few degrees of freedom. Recently, Chebyshev spectral method [9], Legendre spectral method [10], and adaptive pseudospectral method [11] were proposed for solving fractional boundary value problems. Moreover, generalized Laguerre spectral

algorithms and Legendre spectral Galerkin method were developed by Baleanu et al. [12] and Bhrawy and Alghamdi [13] for fractional initial value problems, respectively.

Motivated and inspired by the ongoing research in orthogonal polynomials methods and spectral methods, we construct two-dimensional fractional-order Legendre functions and derive the operational matrices of integration and derivative for the solution of FPDEs. To the best of the authors' knowledge, such approach has not been employed for solving FPDEs.

The rest of the paper is organized as follows. In Section 2, we introduce some mathematical preliminaries of the fractional calculus theory and fractional-order Legendre functions. In Section 3, a basis of 2D-FLFs is defined and some properties are given. Section 4 is devoted to the operational matrices of fractional derivative and integration for 2D-FLFs. Some numerical examples are presented in Section 5. Finally, we conclude the paper with some remarks.

2. Preliminaries and Notations

2.1. Fractional Calculus Theory. Some necessary definitions and Lemma of the fractional calculus theory [29, 30] are listed here for our subsequent development.

Definition 1. A real function $h(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in R$, if there exists a real number $p > \mu$, such that $h(t) = t^p h_1(t)$, where $h_1(t) \in C(0, \infty)$, and it is said to be in the space C_μ^n if and only if $h^{(n)} \in C_\mu$, $n \in N$.

Definition 2. Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$ is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad t > 0, \qquad J^0 f(t) = f(t), \tag{1}$$

where $\Gamma(\alpha)$ is the well-known Gamma function. Some properties of the operator J^α can be found, for example, in [29, 30].

Definition 3. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$(D^\alpha f)(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \times \displaystyle\int_0^x \frac{f^{(m)}(\xi)}{(x-\xi)^{\alpha-m+1}} d\xi, & (\alpha > 0,\ m-1 < \alpha < m), \\ \dfrac{d^m f(x)}{dx^m}, & \alpha = m, \end{cases} \tag{2}$$

where $f : R \to R$, $x \to f(x)$ denotes a continuous (but not necessarily differentiable) function.

Lemma 4. *Let $n-1 < \alpha \leq n$, $n \in N$, $t > 0$, $h \in C_\mu^n$, $\mu \geq -1$. Then*

$$(J^\alpha D^\alpha) h(t) = h(t) - \sum_{k=0}^{n-1} h^{(k)}(0^+) \frac{t^k}{k!}. \tag{3}$$

2.2. Fractional-Order Legendre Functions. In this section, we introduce the fractional-order Legendre functions which were first proposed by Kazem et al. [28]. The normalized eigenfunctions problem for FLFs is

$$\left(\left(x - x^{1+\alpha}\right) L_i'^{\alpha}(x)\right)' + \alpha^2 i(i+1) x^{\alpha-1} L_i^\alpha(x) = 0, \quad x \in (0,1), \tag{4}$$

which is a singular Sturm-Liouville problem. The fractional-order Legendre polynomials, denoted by $\mathrm{FL}_i^\alpha(x)$, are defined on the interval $[0,1]$ and can be determined with the aid of following recurrence formulae:

$$\begin{aligned} &\mathrm{FL}_0^\alpha(x) = 1, \qquad \mathrm{FL}_1^\alpha(x) = 2x^\alpha - 1, \\ &\mathrm{FL}_{i+1}^\alpha(x) = \frac{(2i+1)(2x^\alpha - 1)}{i+1} \mathrm{FL}_i^\alpha(x) \\ &\qquad - \frac{i}{i+1} \mathrm{FL}_{i-1}^\alpha(x), \quad i = 1, 2, \ldots, \end{aligned} \tag{5}$$

and the analytic form of $\mathrm{FL}_i^\alpha(x)$ of degree i is given by

$$\mathrm{FL}_i^\alpha(x) = \sum_{s=0}^{i} b_{s,i} x^{s\alpha}, \qquad b_{s,i} = \frac{(-1)^{i+s}(i+s)!}{(i-s)!(s!)^2}, \tag{6}$$

where $\mathrm{FL}_i^\alpha(0) = (-1)^i$ and $\mathrm{FL}_i^\alpha(1) = 1$. The orthogonality condition is

$$\int_0^1 \mathrm{FL}_n^\alpha(x) \mathrm{FL}_m^\alpha(x) \omega(x)\, dx = \frac{1}{(2n+1)\alpha} \delta_{nm}, \tag{7}$$

where $\omega(x) = x^{\alpha-1}$ is the weight function and δ is the Kronecker delta. For more details, please see [28].

3. 2D-FLFs

In this section, the definitions and theorems of 2D-FLFs are given by Liu's method described in [31].

3.1. Definitions and Properties of the 2D-FLFs

Definition 5. Let $\{\mathrm{FL}_n^\alpha(x)\}_{n=0}^\infty$ be the fractional Legendre polynomials on $[0,1]$; we call $\{\mathrm{FL}_i^\alpha(x)\mathrm{FL}_j^\beta(y)\}_{i,j=0}^\infty$ the two-dimensional fractional Legendre polynomials on $[0,1] \times [0,1]$.

Theorem 6. *The basis $\{FL_i^\alpha(x) FL_j^\beta(y)\}_{i,j=0}^\infty$ is orthogonal on $[0,1] \times [0,1]$ with the weight function $\omega(x,y) = \omega(x)\omega(y) = x^{\alpha-1} y^{\beta-1}$.*

Proof. Let $i \neq m$ or $j \neq n$

$$\begin{aligned}\int_0^1\int_0^1 \omega(x,y)\,\mathrm{FL}_i^\alpha(x)\,\mathrm{FL}_j^\beta(y)\,\mathrm{FL}_m^\alpha(x)\,\mathrm{FL}_n^\beta(y)\,dx\,dy &= \int_0^1 \omega(x)\,\mathrm{FL}_i^\alpha(x)\,\mathrm{FL}_m^\alpha(x)\,dx \\ &\quad\times \int_0^1 \omega(y)\,\mathrm{FL}_j^\beta(y)\,\mathrm{FL}_n^\beta(y)\,dy = 0.\end{aligned} \tag{8}$$

□

Theorem 7. *Consider*

$$\begin{aligned}&\int_0^1\int_0^1 \omega(x,y)\left[FL_i^\alpha(x)\,FL_j^\beta(y)\right]^2 dx\,dy \\ &\quad= \frac{1}{(2i+1)\,\alpha}\frac{1}{(2j+1)\,\beta}, \\ &\int_0^1\int_0^1 \omega(x,y)\left[FL_i^\alpha(x)\,FL_j^\beta(y)\right]^2 dx\,dy \\ &\quad= \int_0^1 \omega(x)\left[FL_i^\alpha(x)\right]^2 dx \int_0^1 \omega(y)\left[FL_j^\beta(y)\right]^2 dy \\ &\quad= \frac{1}{(2i+1)\,\alpha}\frac{1}{(2j+1)\,\beta}.\end{aligned} \tag{9}$$

3.2. 2D-FLFs Expansion

Definition 8. A function of two independent variables $f(x,y)$ which is integrable in square $[0,1]\times[0,1]$ can be expanded as

$$f(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\mathrm{FL}_i^\alpha(x)\,\mathrm{FL}_j^\beta(y), \tag{10}$$

where

$$\begin{aligned}a_{ij} &= (2i+1)(2j+1)\,\alpha\beta \\ &\quad\times \int_0^1\int_0^1 f(x,y)\,\omega(x,y)\,\mathrm{FL}_i^\alpha(x)\,\mathrm{FL}_j^\beta(y)\,dx\,dy.\end{aligned} \tag{11}$$

Theorem 9. *If the series $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}FL_i^\alpha(x)FL_j^\beta(y)$ converges uniformly to $f(x,y)$ on the square $[0,1]\times[0,1]$, then we have*

$$\begin{aligned}a_{ij} &= (2i+1)(2j+1)\,\alpha\beta \\ &\quad\times \int_0^1\int_0^1 f(x,y)\,\omega(x,y)\,FL_i^\alpha(x)\,FL_j^\beta(y)\,dx\,dy.\end{aligned} \tag{12}$$

Proof. By multiplying $\omega(x,y)\mathrm{FL}_n^\alpha(x)\mathrm{FL}_m^\beta(y)$ on both sides of (10), where n and m are fixed and integrating termwise with regard to x and y on $[0,1]\times[0,1]$, then

$$\begin{aligned}&\int_0^1\int_0^1 f(x,y)\,\omega(x,y)\,\mathrm{FL}_n^\alpha(x)\,\mathrm{FL}_m^\beta(y)\,dx\,dy \\ &\quad= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\int_0^1\int_0^1 \omega(x,y)\,\mathrm{FL}_i^\alpha(x)\,\mathrm{FL}_j^\beta(y) \\ &\qquad\qquad\times \mathrm{FL}_n^\alpha(x)\,\mathrm{FL}_m^\beta(y)\,dx\,dy \\ &\quad= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\int_0^1 \omega(x)\,\mathrm{FL}_i^\alpha(x)\,\mathrm{FL}_n^\alpha(x)\,dx \\ &\qquad\qquad\times \int_0^1 \omega(y)\,\mathrm{FL}_j^\beta(y)\,\mathrm{FL}_m^\beta(y)\,dy \\ &\quad= a_{nm}\int_0^1 \omega(x)\left[\mathrm{FL}_n^\alpha(x)\right]^2 dx \int_0^1 \omega(y)\left[\mathrm{FL}_m^\beta(y)\right]^2 dy \\ &\quad= a_{nm}\frac{1}{(2n+1)\,\alpha}\frac{1}{(2m+1)\,\beta}.\end{aligned} \tag{13}$$

Finally one can get (11). □

If the infinite series in (10) is truncated, then it can be written as

$$f(x,y) \approx \sum_{i=0}^{m}\sum_{j=0}^{m'} a_{ij}\mathrm{FL}_i^\alpha(x)\,\mathrm{FL}_j^\beta(y) = C^T\Psi(x^\alpha, y^\alpha), \tag{14}$$

where C and $\Psi(x^\alpha, y^\beta)$ are given by

$$\begin{aligned}C = [&c_{0,0}, c_{0,1}, \ldots, c_{0,m'-1}, c_{1,0}, c_{1,1}, \ldots, \\ &c_{1,m'-1}, \ldots, c_{m-1,0}, c_{m-1,1}, \ldots, c_{m-1,m'-1}]^T,\end{aligned} \tag{15}$$

$$\begin{aligned}\Psi(x^\alpha, y^\beta) = [&\psi_{0,0}, \psi_{0,1}, \ldots, \psi_{0,m'-1}, \psi_{1,0}, \psi_{1,1}, \ldots, \\ &\psi_{1,m'-1}, \ldots, \psi_{m-1,0}, \psi_{m-1,1}, \ldots, \psi_{m-1,m'-1}]^T,\end{aligned} \tag{16}$$

where $\psi_{ij} = \mathrm{FL}_i^\alpha(x)\mathrm{FL}_j^\beta(y)$, $i = 0,1,\ldots,m$, and $j = 0,1,\ldots,m'$.

According to the definition of FLFs, one can find that fractional Legendre polynomials are identical to Legendre polynomials shifted to $[0,1]$ when using the transform $x^\alpha \to x$, $y^\beta \to y$. Therefore, in a similar method described in [31], we can easily get the convergence and stability theorems of proposed method.

Lemma 10. *If the function $f(x,y)$ is a continuous function on $[0,1]\times[0,1]$ and the series $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}FL_i^\alpha(x)FL_j^\beta(y)$ converges uniformly to $f(x,y)$, then $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}FL_i^\alpha(x)FL_j^\beta(y)$ is the 2D-FLFs expansion of $f(x,y)$.*

Proof (by contradiction). Let

$$f(x,y)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}b_{ij}\mathrm{FL}_i^{\alpha}(x)\,\mathrm{FL}_j^{\beta}(y),$$
$$f(x,y)\sim\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{ij}\mathrm{FL}_i^{\alpha}(x)\,\mathrm{FL}_j^{\beta}(y). \tag{17}$$

Then there is at least one coefficient such that $a_{nm}\neq b_{nm}$. However,

$$\begin{aligned} b_{nm} &= (2n+1)(2m+1)\alpha\beta \\ &\quad\times\int_0^1\int_0^1 f(x,y)\,\omega(x,y)\,\mathrm{FL}_n^{\alpha}(x)\,\mathrm{FL}_m^{\beta}(y)\,dx\,dy \\ &= a_{nm}. \end{aligned} \tag{18}$$

□

Lemma 11. *If two continuous functions defined on* $[0,1]\times[0,1]$ *have the identical 2D-FLFs expansions, then these two functions are identical.*

Proof. Suppose that $f(x,y)$ and $g(x,y)$ can be expanded by 2D-FLFs as follows:

$$f(x,y)\sim\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{ij}\mathrm{FL}_i^{\alpha}(x)\,\mathrm{FL}_j^{\beta}(y),$$
$$g(x,y)\sim\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{ij}\mathrm{FL}_i^{\alpha}(x)\,\mathrm{FL}_j^{\beta}(y). \tag{19}$$

By subtracting the above two equations with each other, one has

$$\begin{aligned} f(x,y)-g(x,y) &\sim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(a_{ij}-a_{ij}\right)\mathrm{FL}_i^{\alpha}(x)\,\mathrm{FL}_j^{\beta}(y) \\ &= 0=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}0\times\mathrm{FL}_i^{\alpha}(x)\,\mathrm{FL}_j^{\beta}(y). \end{aligned} \tag{20}$$

Then Lemma 11 can be proved. □

Theorem 12. *If the 2D-FLFs expansion of a continuous function* $f(x,y)$ *converges uniformly, then the 2D-FLFs expansion converges to the function* $f(x,y)$.

Proof. Theorem 12 can be proved by Theorems 7 and 9. □

Theorem 13. *If the sum of the absolute values of the 2D-FLFs coefficients of a continuous function* $f(x,y)$ *forms a convergent series, then the 2D-FLFs expansion is absolutely uniformly convergent, and converges to the function* $f(x,y)$.

Proof. Consider

$$\begin{aligned} \left|\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{ij}\mathrm{FL}_i^{\alpha}(x)\,\mathrm{FL}_j^{\beta}(y)\right| &\le \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left|a_{ij}\right|\left|\mathrm{FL}_i^{\alpha}(x)\right|\left|\mathrm{FL}_j^{\beta}(y)\right| \\ &\le \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left|a_{ij}\right|. \end{aligned} \tag{21}$$

Then $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{ij}\mathrm{FL}_i^{\alpha}(x)\mathrm{FL}_j^{\beta}(y)$ converges uniformly to the function $f(x,y)$. □

Theorem 14. *If a continuous function* $f(x,y)$*, defined on* $[0,1]\times[0,1]$*, has bounded mixed partial derivative* $D_x^{2\alpha}D_y^{2\beta}f(x,y)$*, then the 2D-FLFs expansion of the function converges uniformly to the function.*

Proof. Let $f(x,y)$ be a function defined on $[0,1]\times[0,1]$ such that

$$\left|D_x^{2\alpha}D_y^{2\beta}f(x,y)\right|\le M, \tag{22}$$

where M is a positive constant and

$$\begin{aligned} a_{ij} &= (2i+1)(2j+1)\alpha\beta \\ &\quad\times\int_0^1\int_0^1 f(x,y)\,\omega(x,y)\,\mathrm{FL}_i^{\alpha}(x)\,\mathrm{FL}_j^{\beta}(y)\,dx\,dy. \end{aligned} \tag{23}$$

By employing the transform $X=2x^{\alpha}-1$ and $Y=2y^{\beta}-1$, one can obtain

$$a_{ij}=\frac{2i+1}{2}\frac{2j+1}{2}\int_{-1}^{1}\int_{-1}^{1}f(X,Y)\,p_i(X)\,p_j(Y)\,dX\,dY. \tag{24}$$

Consequently, in a similar method described in [31], Theorem 14 can be proved. □

4. Operational Matrices of 2D-FLFs

4.1. Integration Operational Matrices of 2D-FLFs

Lemma 15. *The Riemann-Liouville fractional integration of order* $\gamma>0$ *of the 2D-FLFs* ψ_{ij} *can be obtained in the form of*

$$J_x^{\gamma}\left\{\psi_{ij}\left(x^{\alpha},y^{\beta}\right)\right\}=FL_j^{\beta}(y)\sum_{s=0}^{i}b_{si}\frac{\Gamma(1+s\alpha)}{\Gamma(1+s\alpha+\gamma)}x^{s\alpha+\gamma}. \tag{25}$$

Proof. Consider

$$\begin{aligned} J_x^{\gamma}\left\{\psi_{ij}\left(x^{\alpha},y^{\beta}\right)\right\} &= J_x^{\gamma}\left\{\mathrm{FL}_i^{\alpha}(x)\,\mathrm{FL}_j^{\beta}(y)\right\} \\ &= J_x^{\gamma}\left\{\mathrm{FL}_i^{\alpha}(x)\right\}\mathrm{FL}_j^{\beta}(y) \\ &= J_x^{\gamma}\left\{\sum_{s=0}^{i}b_{si}x^{s\alpha}\right\}\mathrm{FL}_j^{\beta}(y) \\ &= \mathrm{FL}_j^{\beta}(y)\sum_{s=0}^{i}b_{si}\frac{\Gamma(1+s\alpha)}{\Gamma(1+s\alpha+\gamma)}x^{s\alpha+\gamma}. \end{aligned} \tag{26}$$

□

Lemma 16. *Let $\gamma > 0$; then one has*

$$\int_0^1\int_0^1 J_x^\gamma\left\{\psi_{ij}\right\}\psi_{i'j'}\omega(x,y)\,dx\,dy = \begin{cases}\displaystyle\sum_{s=0}^{i}\sum_{s'=0}^{i'}\frac{b_{si}b_{s'i'}}{(s+s'+1)\alpha+\gamma} \\ \qquad\times\dfrac{\Gamma(1+s\alpha)}{\Gamma(1+s\alpha+\gamma)}\dfrac{1}{(2j+1)\beta}, & j=j' \\ 0, & j\neq j'.\end{cases} \tag{27}$$

Proof. Using previous Lemma 15 and (6), one can have

$$\begin{aligned}\int_0^1\int_0^1 J_x^\gamma\left\{\psi_{ij}\left(x^\alpha,y^\beta\right)\right\}\psi_{i'j'}\left(x^\alpha,y^\beta\right)\omega(x,y)\,dx\,dy &= \int_0^1\int_0^1\omega(x,y)\,\mathrm{FL}_{i'}^{\alpha}\left(x^\alpha\right)\mathrm{FL}_{j'}^{\beta}\left(y^\beta\right)\mathrm{FL}_{j}^{\beta}\left(y^\beta\right) \\ &\quad\times\sum_{s=0}^{i}b_{si}\frac{\Gamma(1+s\alpha)}{\Gamma(1+s\alpha+\gamma)}x^{s\alpha+\gamma}dx\,dy \\ &= \int_0^1\int_0^1\omega(y)\,\mathrm{FL}_{j}^{\beta}\left(y^\beta\right)\mathrm{FL}_{j'}^{\beta}\left(y^\beta\right) \\ &\quad\times\sum_{s=0}^{i}\sum_{s'=0}^{i'}b_{si}b_{s'i'}\frac{\Gamma(1+s\alpha)}{\Gamma(1+s\alpha+\gamma)} \\ &\quad\times x^{(s+s'+1)\alpha+\gamma-1}dx\,dy \\ &= \int_0^1\omega(y)\,\mathrm{FL}_{j}^{\beta}\left(y^\beta\right)\mathrm{FL}_{j'}^{\beta}\left(y^\beta\right) \\ &\quad\times\left(\int_0^1\sum_{s=0}^{i}\sum_{s'=0}^{i'}b_{si}b_{s'i'}\frac{\Gamma(1+s\alpha)}{\Gamma(1+s\alpha+\gamma)}\right. \\ &\quad\left.\times x^{(s+s'+1)\alpha+\gamma-1}dx\right)dy \\ &= \sum_{s=0}^{i}\sum_{s'=0}^{i'}\frac{b_{si}b_{s'i'}}{(s+s'+1)\alpha+\gamma}\frac{\Gamma(1+s\alpha)}{\Gamma(1+s\alpha+\gamma)} \\ &\quad\times\int_0^1\omega(y)\,\mathrm{FL}_{j}^{\beta}\left(y^\beta\right)\mathrm{FL}_{j'}^{\beta}\left(y^\beta\right)dy \\ &= \begin{cases}\displaystyle\sum_{s=0}^{i}\sum_{s'=0}^{i'}\frac{b_{si}b_{s'i'}}{(s+s'+1)\alpha+\gamma} \\ \qquad\times\dfrac{\Gamma(1+s\alpha)}{\Gamma(1+s\alpha+\gamma)}\dfrac{1}{(2j+1)\beta}, & j=j' \\ 0, & j\neq j'.\end{cases}\end{aligned} \tag{28}$$

□

Theorem 17. *Let $\Psi(x^\alpha,y^\beta)$ be the 2D-FLFs vector defined in* (16)*; then one has*

$$J_x^\gamma\Psi\left(x^\alpha,y^\beta\right)\simeq\mathbf{P}_x^\gamma\Psi\left(x^\alpha,y^\beta\right), \tag{29}$$

where $\mathbf{P}_x^\gamma$ is the $mm'\times mm'$ operational matrix of Riemann-Liouville fractional integration of order $\gamma>0$, and has the form as follows:

$$\mathbf{P}_x^\gamma = \begin{bmatrix} E_{0,0} & E_{0,1} & \cdots & E_{0,m-1} \\ E_{1,0} & E_{1,1} & \cdots & E_{1,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ E_{m-1,0} & E_{m-1,1} & \cdots & E_{m-1,m-1}\end{bmatrix}, \tag{30}$$

in which $E_{i,i'}$ is $m'\times m'$ matrix and the elements are defined as follows:

$$E_{i,i'} = I\sum_{s=0}^{i}\sum_{s'=0}^{i'}\frac{b_{si}b_{s'i'}\left(2i'+1\right)\alpha}{(s+s'+1)\alpha+\gamma}\frac{\Gamma(1+s\alpha)}{\Gamma(1+s\alpha+\gamma)}, \qquad i,i'=0,1,\ldots,m-1, \tag{31}$$

and I is $m'\times m'$ identity matrix.

Proof. Using (29) and orthogonality property of FLFs, one can get

$$\mathbf{P}_x^\gamma = \left\langle J_x^\gamma\Psi\left(x^\alpha,y^\beta\right),\Psi^T\left(x^\alpha,y^\beta\right)\right\rangle H^{-1}, \tag{32}$$

where $\langle J_x^\gamma\Psi(x^\alpha,y^\beta),\Psi^T(x^\alpha,y^\beta)\rangle$ and H^{-1} are two $m'\times m'$ matrices defined as

$$\begin{aligned}&\left\langle J_x^\gamma\Psi\left(x^\alpha,y^\beta\right),\Psi^T\left(x^\alpha,y^\beta\right)\right\rangle \\ &\quad= \left\{\int_0^1\int_0^1 J_x^\gamma\left\{\Psi_k\left(x^\alpha,y^\beta\right)\right\}\right. \\ &\qquad\left.\times\Psi_{k'}\left(x^\alpha,y^\beta\right)\omega(x,y)\,dx\,dy\right\}_{k,k'}^{mm'} \\ &\quad= \left\{\sum_{s=0}^{i}\sum_{s'=0}^{i'}b_{si}b_{s'i'}\frac{\Gamma(1+s\alpha)}{\Gamma(1+s\alpha+\gamma)}\right. \\ &\qquad\left.\times\frac{1}{(s+s'+1)\alpha+\gamma}\frac{1}{(2j+1)\beta}\right\}_{i,i';j=j'}^{m;m'}, \\ &\quad H^{-1} = \left\{\left(2i'+1\right)(2j+1)\alpha\beta\right\}_{i,i';j=j'}^{m;m'}.\end{aligned} \tag{33}$$

Now by substituting above equations in (32), Theorem 12 can be proved. □

In a similar way as previous, one can obtain the operational matrix of Riemann-Liouville fractional integration with respect to variable y.

Theorem 18. *Let $\Psi(x^\alpha,y^\beta)$ be the 2D-FLFs vector defined in* (16)*; one has*

$$J_y^\gamma\Psi\left(x^\alpha,y^\beta\right)\simeq\mathbf{P}_y^\gamma\Psi\left(x^\alpha,y^\beta\right), \tag{34}$$

where $\mathbf{P}_y^\gamma$ is the $mm' \times mm'$ operational matrix of Riemann-Liouville fractional integration of order $\gamma > 0$, and has the form as follows:

$$\mathbf{P}_y^\gamma = \begin{bmatrix} E & O & \cdots & O \\ O & E & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & E \end{bmatrix}, \tag{35}$$

in which E is $m' \times m'$ matrix and the elements are defined as follows:

$$E_{j,j'} = \sum_{r=0}^{j}\sum_{r'=0}^{j'} \frac{b_{rj}b_{r'j'}\left(2j'+1\right)\beta}{\left(r+r'+1\right)\beta+\gamma}\frac{\Gamma\left(1+r\beta\right)}{\Gamma\left(1+r\beta+\gamma\right)}, \quad j, j' = 0, 1, \ldots, m'-1. \tag{36}$$

4.2. Derivative Operational Matrices of 2D-FLFs

Lemma 19. *The FLFs Caputo fractional derivative of $\gamma > 0$ can be obtained in the form of*

$$D_x^\gamma\left\{\psi_{ij}\left(x^\alpha, y^\beta\right)\right\} = FL_j^\beta\left(y^\beta\right)\sum_{s=0}^{i} b'_{si}\frac{\Gamma\left(1+s\alpha\right)}{\Gamma\left(1+s\alpha-\gamma\right)}x^{s\alpha-\gamma}, \tag{37}$$

where $b'_{s,i} = 0$ when $s\alpha \in N_0$ and $s\alpha < \gamma$ in other case $b'_{s,i} = b_{s,i}$.

Proof. Consider

$$\begin{aligned} D_x^\gamma\left\{\psi_{ij}\left(x^\alpha, y^\beta\right)\right\} &= D_x^\gamma\left\{\mathrm{FL}_i^\alpha\left(x^\alpha\right)\mathrm{FL}_j^\beta\left(y^\beta\right)\right\} \\ &= \mathrm{FL}_j^\beta\left(y^\beta\right)D_x^\gamma\left\{\mathrm{FL}_i^\alpha\left(x^\alpha\right)\right\} \\ &= D_x^\gamma\left\{\sum_{s=0}^{i} b_{si}x^{s\alpha}\right\}\mathrm{FL}_j^\beta\left(y^\beta\right) \\ &= \mathrm{FL}_j^\beta\left(y^\beta\right)\sum_{s=0}^{i} b'_{si}\frac{\Gamma\left(1+s\alpha\right)}{\Gamma\left(1+s\alpha-\gamma\right)}x^{s\alpha-\gamma}. \end{aligned} \tag{38}$$

□

Lemma 20. *Let $\gamma > 0$, $\alpha \notin N$; then one has*

$$\begin{aligned} &\int_0^1\int_0^1 D_x^\gamma\left\{\psi_{ij}\right\}\psi_{i'j'}\omega\left(x,y\right)dx\,dy \\ &= \begin{cases} \displaystyle\sum_{s=0}^{i}\sum_{s'=0}^{i'}\frac{b_{si}b_{s'i'}}{\left(s+s'+1\right)\alpha-\gamma} \\ \qquad \displaystyle\times\frac{\Gamma\left(1+s\alpha\right)}{\Gamma\left(1+s\alpha-\gamma\right)}\frac{1}{\left(2j+1\right)\beta}, & j = j' \\ 0, & j \neq j'. \end{cases} \end{aligned} \tag{39}$$

Proof. Using previous Lemma 19 and (6), one can have

$$\begin{aligned} &\int_0^1\int_0^1 D_x^\gamma\left\{\psi_{ij}\left(x^\alpha, y^\beta\right)\right\}\psi_{i'j'}\left(x^\alpha, y^\beta\right)\omega\left(x,y\right)dx\,dy \\ &= \int_0^1\int_0^1 \omega\left(x,y\right)\mathrm{FL}_{i'}^\alpha\left(x^\alpha\right)\mathrm{FL}_{j'}^\beta\left(y^\beta\right)\mathrm{FL}_j^\beta\left(y^\beta\right) \\ &\qquad \times\sum_{s=0}^{i} b'_{si}\frac{\Gamma\left(1+s\alpha\right)}{\Gamma\left(1+s\alpha-\gamma\right)}x^{s\alpha-\gamma}dx\,dy \\ &= \int_0^1\int_0^1 \omega\left(y\right)\mathrm{FL}_j^\beta\left(y^\beta\right)\mathrm{FL}_{j'}^\beta\left(y^\beta\right) \\ &\qquad \times\sum_{s=0}^{i}\sum_{s'=0}^{i'} b'_{si}b_{s'i'}\frac{\Gamma\left(1+s\alpha\right)}{\Gamma\left(1+s\alpha-\gamma\right)} \\ &\qquad \times x^{(s+s'+1)\alpha-\gamma-1}dx\,dy \\ &= \int_0^1 \omega\left(y\right)\mathrm{FL}_j^\beta\left(y^\beta\right)\mathrm{FL}_{j'}^\beta\left(y^\beta\right) \\ &\qquad \times\left(\int_0^1\sum_{s=0}^{i}\sum_{s'=0}^{i'} b'_{si}b_{s'i'}\frac{\Gamma\left(1+s\alpha\right)}{\Gamma\left(1+s\alpha-\gamma\right)}\right. \\ &\qquad \left.\times x^{(s+s'+1)\alpha-\gamma-1}dx\right)dy \\ &= \sum_{s=0}^{i}\sum_{s'=0}^{i'}\frac{b'_{si}b_{s'i'}}{\left(s+s'+1\right)\alpha-\gamma}\frac{\Gamma\left(1+s\alpha\right)}{\Gamma\left(1+s\alpha-\gamma\right)} \\ &\qquad \times\int_0^1 \omega\left(y\right)\mathrm{FL}_j^\beta\left(y^\beta\right)\mathrm{FL}_{j'}^\beta\left(y^\beta\right)dy \\ &= \begin{cases} \displaystyle\sum_{s=0}^{i}\sum_{s'=0}^{i'}\frac{b'_{si}b_{s'i'}}{\left(s+s'+1\right)\alpha-\gamma} \\ \qquad \displaystyle\times\frac{\Gamma\left(1+s\alpha\right)}{\Gamma\left(1+s\alpha-\gamma\right)}\frac{1}{\left(2j+1\right)\beta}, & j = j' \\ 0, & j \neq j'. \end{cases} \end{aligned} \tag{40}$$

□

Theorem 21. *Let $\Psi(x^\alpha, y^\beta)$ be the 2D-FLFs vector defined in (16); one has*

$$D_x^\gamma\Psi\left(x^\alpha, y^\beta\right) \simeq \mathbf{D}_x^\gamma\Psi\left(x^\alpha, y^\beta\right), \tag{41}$$

where $\mathbf{D}_x^\gamma$ is the $mm' \times mm'$ operational matrix of Caputo fractional derivative of order $\gamma > 0$, and has the form as follows:

$$\mathbf{D}_x^\gamma = \begin{bmatrix} O & O & \cdots & O \\ F_{1,0} & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ F_{m-1,0} & F_{m-1,1} & \cdots & O \end{bmatrix} \tag{42}$$

in which $F_{i,i'}$ is $m' \times m'$ matrix and the elements are defined as follows:

$$F_{i,i'} = I \sum_{s=0}^{i} \sum_{s'=0}^{i'} \frac{b'_{si} b_{s'i'} \left(2i'+1\right) \alpha}{\left(s+s'+1\right) \alpha - \gamma} \frac{\Gamma\left(1+s\alpha\right)}{\Gamma\left(1+s\alpha-\gamma\right)}, \quad i, i' = 0, 1, \ldots, m-1, \tag{43}$$

and I is a $m' \times m'$ identity matrix.

Proof. Using (41) and the orthogonality property of FLFs, one can have

$$\mathbf{D}_x^\gamma = \left\langle D_x^\gamma \Psi\left(x^\alpha, y^\beta\right), \Psi^T\left(x^\alpha, y^\beta\right)\right\rangle H^{-1}, \tag{44}$$

where $\langle D_x^\gamma \Psi(x^\alpha, y^\beta), \Psi^T(x^\alpha, y^\beta)\rangle$ and H^{-1} are two $mm' \times mm'$ matrices defined as

$$\begin{aligned}
&\left\langle D_x^\gamma \Psi\left(x^\alpha, y^\beta\right), \Psi^T\left(x^\alpha, y^\beta\right)\right\rangle \\
&\quad = \left\{\int_0^1 \int_0^1 D_x^\gamma \left\{\Psi_k\left(x^\alpha, y^\beta\right)\right\} \times \Psi_{k'}\left(x^\alpha, y^\beta\right) \omega\left(x, y\right) dx\, dy\right\}_{k,k'}^{mm'} \\
&\quad = \left\{\sum_{s=0}^{i} \sum_{s'=0}^{i'} \frac{b'_{si} b_{s'i'}}{\left(s+s'+1\right)\alpha - \gamma} \times \frac{\Gamma\left(1+s\alpha\right)}{\Gamma\left(1+s\alpha-\gamma\right)} \frac{1}{\left(2j+1\right)\beta}\right\}_{i,i';j=j'}^{m;m'} \\
&\quad H^{-1} = \left\{\left(2i'+1\right)\left(2j+1\right)\alpha\beta\right\}_{i,i';j=j'}^{m;m'}.
\end{aligned} \tag{45}$$

Now by substituting above equations in (44), Theorem 21 can be proved. □

In a similar way as above, one can get Caputo fractional derivative of order $\gamma > 0$ with respect to variable y.

Theorem 22. *Let $\Psi(x^\alpha, y^\beta)$ be the 2D-FLFs vector defined in* (16)*; one can have*

$$D_y^\gamma \Psi\left(x^\alpha, y^\beta\right) \simeq \mathbf{D}_y^\gamma \Psi\left(x^\alpha, y^\beta\right), \tag{46}$$

where $\mathbf{D}_y^\gamma$ is the $mm' \times mm'$ operational matrix of Caputo fractional derivative of order $\gamma > 0$, and has the form as follows:

$$\mathbf{D}_y^\gamma = \begin{bmatrix} F & O & \cdots & O \\ O & F & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & F \end{bmatrix}, \tag{47}$$

in which F is $m' \times m'$ matrix and the elements are defined as follows:

$$F_{j,j'} = \sum_{r=0}^{j} \sum_{r'=0}^{j'} \frac{b'_{rj} b_{r'j'} \left(2j'+1\right)\beta}{\left(r+r'+1\right)\beta + \gamma} \frac{\Gamma\left(1+r\beta\right)}{\Gamma\left(1+r\beta+\gamma\right)}, \quad j, j' = 0, 1, \ldots, m'-1. \tag{48}$$

5. Applications and Results

Consider the following FPDEs:

$$\begin{aligned}
&D_x^\alpha u(x,t) + D_t^\beta u(x,t) \\
&\quad + N\left[u(x,t)\right] + L\left[u(x,t)\right] = g(x,t), \quad \alpha, \beta \in (0,1],
\end{aligned} \tag{49}$$

where L and N are linear operator and nonlinear operator; respectively. D^α and D^β are the Caputo fractional derivatives of order α and β, respectively; g is a known analytic function.

By employing operator J_t^β on both sides of (49) and then using the Lemma 4, one can have

$$\begin{aligned}
&u(x,t) + J_t^\beta \left\{D_x^\alpha u(x,t) + Nu(x,t) + Lu(x,t)\right\} \\
&\quad - \sum_{k=0}^{m-1} u^{(k)}(x,0) \frac{x^k}{k!} - J_t^\beta g(x,t) = 0.
\end{aligned} \tag{50}$$

We first express unknown function $u(x,t)$ and derivative term $D_x^\alpha u(x,t)$ as

$$u(x,t) = C^T \Psi\left(x^\alpha, t^\beta\right), \qquad D_x^\alpha u(x,t) = C^T \mathbf{D}_x^\alpha \Psi\left(x^\alpha, t^\beta\right). \tag{51}$$

Now for the nonlinear part, by employing the nonlinear term approximation method described in [32] and then by using transform $x \to x^\alpha$, $t \to t^\beta$, one can get the 2D-FLFs expansion of nonlinear term as

$$Nu(x,t) = N^T \Psi\left(x^\alpha, t^\beta\right), \tag{52}$$

where N^T is coefficient matrix of nonlinear term which must be computed and its order is $mm' \times mm'$.

For the linear part, we have

$$Lu(x,t) = L^T \Psi\left(x^\alpha, t^\beta\right), \tag{53}$$

where L is a matrix of order $mm' \times mm'$.

After substituting (51)–(53) into (50), one can obtain

$$C^T + \left(C^T \mathbf{D}_x^\alpha + N^T + L^T\right) \mathbf{P}_y^\beta - C_{\text{guess}}^T = 0. \tag{54}$$

According to the Wu's [33] technology for determining the initial iteration value, the initial iteration value is chosen as $u_{\text{guess}} = \sum_{k=0}^{m-1} u^{(k)}(x,0)(x^k/k!) + J_t^\beta\{g(x,t)\} = C_{\text{guess}}^T \Psi(x^\alpha, t^\beta)$. The coefficient matrix C^T can be computed by using the MATLAB function fsolve( ) or the method described in [34].

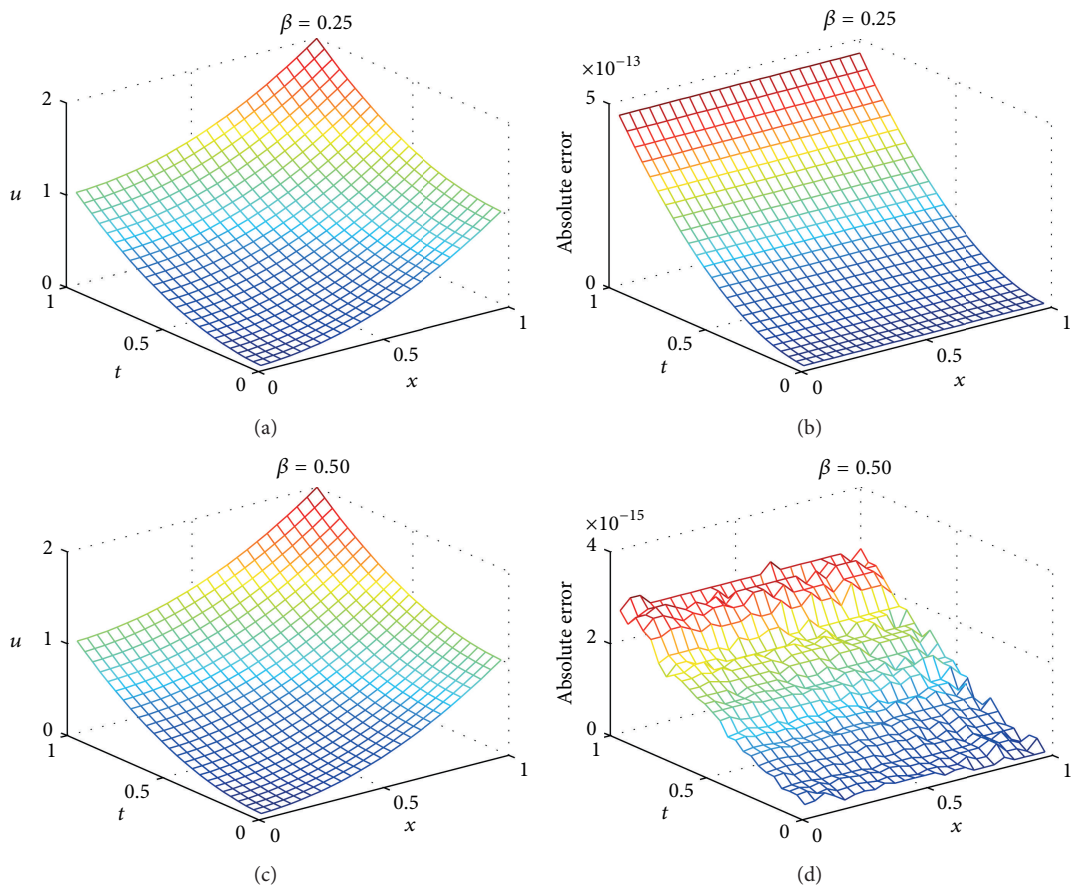

Figure 1: Numerical results for Example 23 when $\beta = 0.25, 0.50$.

Now, the present method is applied to solve the linear and nonlinear FPDEs, and their results are compared with the solution of other methods. The accuracy of our approach is estimated by the following error functions:

$$e_j = (u_{\text{exact}})_j - (u_{\text{approx}})_j, \qquad e = u_{\text{exact}} - u_{\text{approx}},$$
$$\|e\|_{L_\infty} = \max_{1\le j\le N} |e_j|, \qquad \|e\|_{L_2} = \sqrt{\sum_{j=1}^{N} \left|(e_j)^2\right|}, \tag{55}$$
$$\|e\|_{\text{RMS}} = \sqrt{\frac{1}{N}\sum_{j=1}^{N} \left|(e_j)^2\right|}.$$

Example 23. Consider the one-dimensional linear inhomogeneous fractional Burger's equation [35]:

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} + \frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{2t^{2-\beta}}{\Gamma(3-\beta)} + 2x - 2, \quad 0 < \beta \le 1, \tag{56}$$

with the initial condition $u(x,0) = x^2$ and the exact solution being $u(x,t) = x^2 + t^2$.

By employing 2D-FLFs method, one can get

$$C^T \left[I + \left(\mathbf{D}_x^\alpha - (\mathbf{D}_x^\alpha)^2\right)\mathbf{P}_t^\beta\right] = C_{\text{guess}}^T, \tag{57}$$

where $\alpha = 1$. Then we can get $C^T = C_{\text{guess}}^T \text{inv}(I + (\mathbf{D}_x^\alpha - (\mathbf{D}_x^\alpha)^2)\mathbf{P}_t^\beta)$.

Figures 1(a) and 1(b) show the numerical results for $\beta = 0.25$ with $m = 3$, $m' = 9$ and $\beta = 0.5$ with $m = 3$, $m' = 5$, respectively. It should be found that the accuracy of 2D-FLFs method is very high while only a small number of 2D-FLFs are needed.

Example 24. Consider nonlinear fractional Klein-Gordon equation [36, 37]:

$$D_t^\beta u(x,t) - D_x^\alpha u(x) + u^3(x) = g(x,t), \quad x \ge 0, \quad t > 0, \quad \alpha, \beta \in (1,2], \tag{58}$$

subject to the initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = 0, \tag{59}$$

and $g(x,t) = \Gamma(\beta+1)x^\alpha - \Gamma(\alpha+1)t^\beta + x^{3\alpha}t^{3\beta}$. The exact solution of (58) is $u(x,t) = x^\alpha t^\beta$.

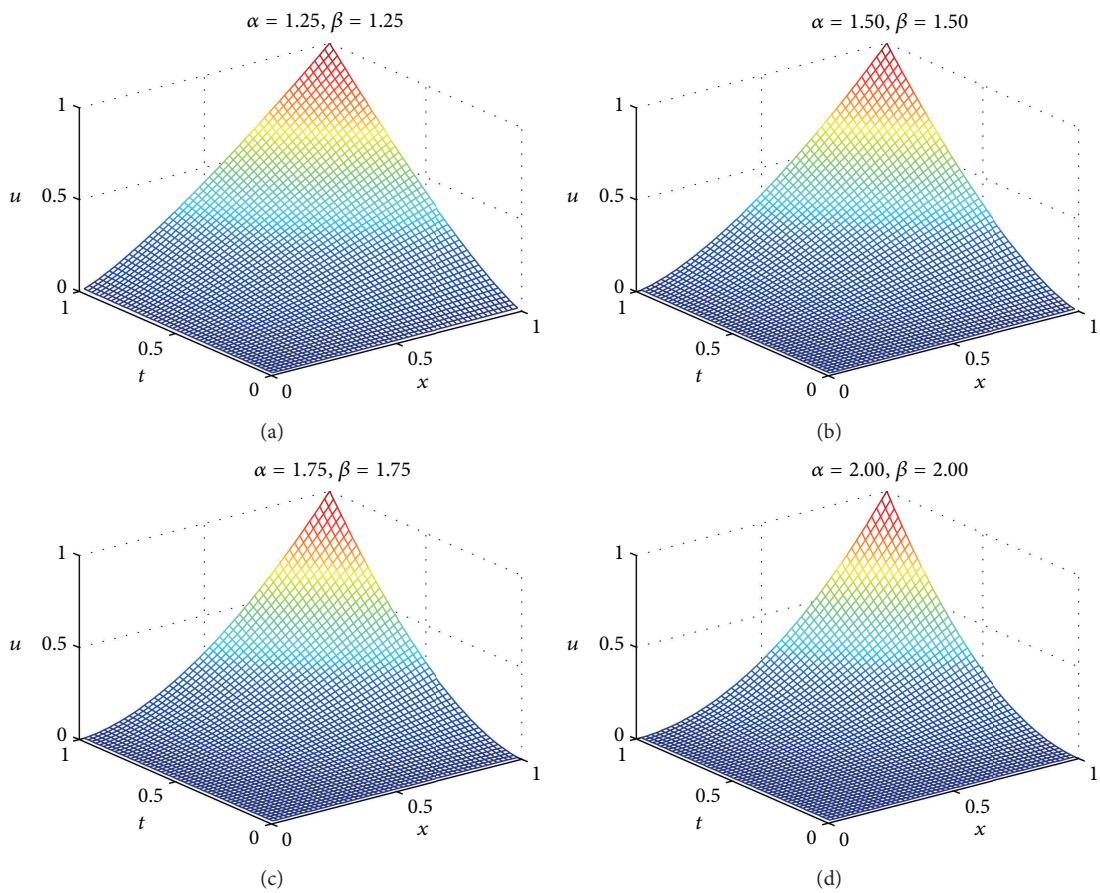

Figure 2: Numerical results of Example 24 for different values of α and β.

By employing 2D-FLFs method with $m = 3$ and $m' = 3$, one can have

$$C^T + \left(-C^T \mathbf{D}_x^\alpha + N^T\right) \mathbf{P}_t^\beta - C_{\text{guess}}^T = 0. \tag{60}$$

The numerical results of Example 24 for different values of α and β are shown in Figure 2. In addition, L_2 and L_∞ errors are presented in Table 1. From Table 1, one can conclude that the solutions of 2D-FLFs method are in good agreement with the exact results. Compared with homotopy analysis method (HAM) [36] and homotopy perturbation method (HPM) [37], 2D-FLFs method can get high accuracy solution while only need a few terms of 2D-FLFs.

Example 25. Consider the nonlinear time-fractional advection partial differential equation [37–39]

$$\begin{gathered} D_t^\beta u(x,t) + u(x,t)\, u_x(x,t) = x + xt^2, \\ t > 0, \quad x \in R, \quad 0 < \beta \le 1, \end{gathered} \tag{61}$$

subject to the initial condition

$$u(x,0) = 0. \tag{62}$$

Figure 3 gives the approximation solutions of (61) for $\beta = 0.50$ with $m = 4$, $m' = 5$ and $\beta = 0.75$ with $m = 4$, $m' = 9$. Moreover, Table 2 shows the approximate solutions for (61) obtained for different values of β using the fractional variational iteration method (FVIM) [39] and 2D-FLFs method. The values of $\beta = 1$ are the only case for which we know the exact solution $u(x,t) = xt$. It should be noted that only the fourth-order term of the FVIM was used in evaluating the approximate solutions for Table 2. From Table 2, it clearly appears that 2D-FLFs method is more accurate than FVIM and the obtained results are in good agreement with exact solution.

Example 26. We finally consider the linear time-fractional wave equation:

$$\frac{\partial^{2\beta} u}{\partial t^{2\beta}} = \frac{1}{2} x^2 \frac{\partial^2 u}{\partial x^2}, \quad t > 0,\ x \in R,\ 0.5 < \beta \le 1, \tag{63}$$

subject to the initial conditions

$$u(x,0) = x, \qquad \frac{\partial u(x,0)}{\partial t} = x^2. \tag{64}$$

Table 3 gives a comparison of the approximate solutions at different values of β using the FVIM [39] and 2D-FLFs method. Figure 4 shows the numerical solutions of 2D-FLFs method for (63) at different values of β with $m = 3$, $m' = 9$. The values of $\beta = 1$ are the only case for which we know

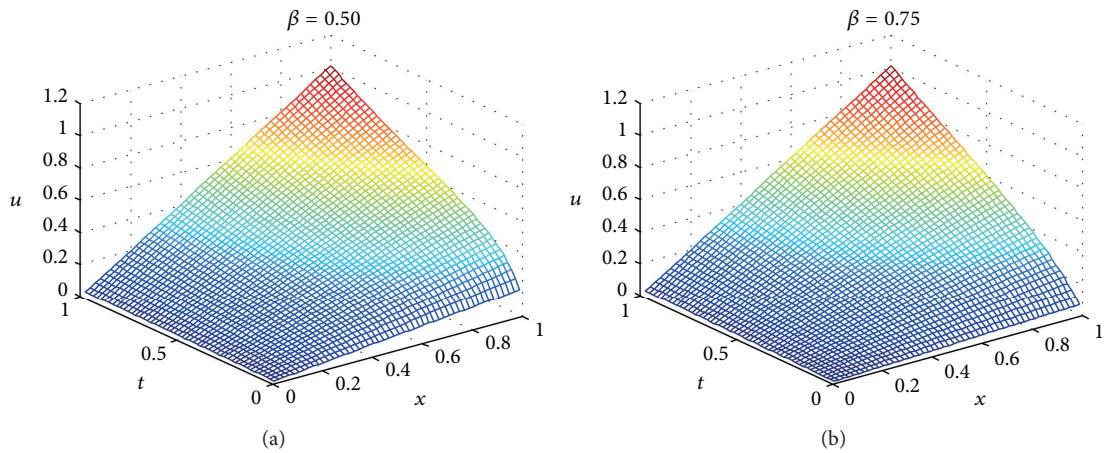

Figure 3: Numerical results of Example 25 for different value of β.

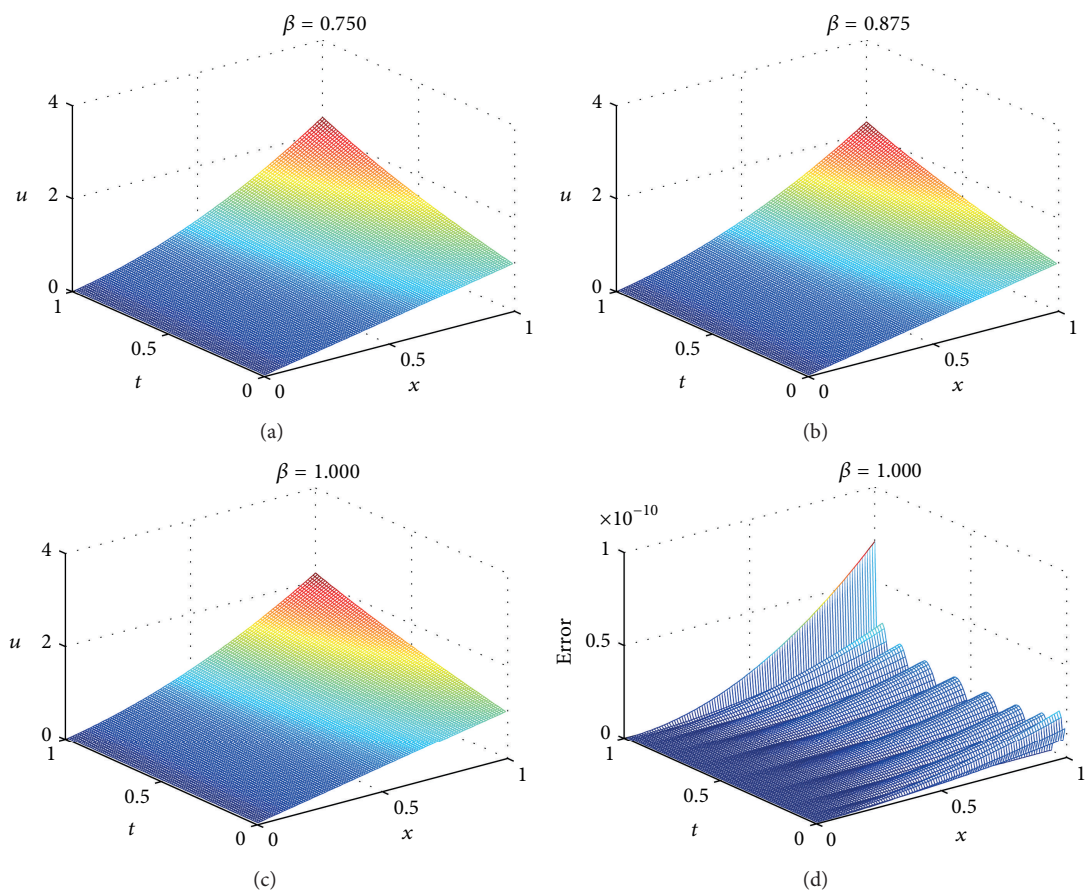

Figure 4: Numerical results of Example 26 for different value of β.

Table 1: Errors of Example 24 for different values of α and β with $M = M' = 4$.

Error	$\alpha = \beta = 1.25$	$\alpha = \beta = 1.50$	$\alpha = \beta = 1.75$	$\alpha = \beta = 2.00$
L_2	$5.6437e-015$	$1.2075e-015$	$3.4584e-015$	$8.9917e-016$
L_∞	$4.4409e-016$	$1.1102e-016$	$3.3307e-016$	$1.1102e-016$

Table 2: Numerical values when $\beta = 0.50$, 0.75, and 1.0 for (61).

t	x	$\beta = 0.50$		$\beta = 0.75$		$\beta = 1.00$		
		FVIM	2D-FLFs	FVIM	2D-FLFs	FVIM	2D-FLFs	Exact
0.25	0.25	0.12422501	0.12225461	0.09230374	0.09224583	0.06250058	0.062500	0.062500
	0.50	0.24845002	0.24450922	0.18460748	0.18449165	0.12500117	0.125000	0.125000
	0.75	0.37267504	0.36676383	0.27691122	0.27673748	0.18750175	0.187500	0.187500
	1.00	0.49690005	0.48901844	0.36921496	0.36898331	0.25000234	0.250000	0.250000
0.50	0.25	0.18377520	0.16584130	0.15148283	0.14985508	0.12507592	0.125000	0.125000
	0.50	0.36755040	0.33168259	0.30296566	0.29971016	0.25015184	0.250000	0.250000
	0.75	0.55132559	0.49752389	0.45444848	0.44956524	0.37522776	0.375000	0.375000
	1.00	0.73510079	0.66336518	0.60593131	0.59942032	0.50030368	0.500000	0.500000
0.75	0.25	0.27227270	0.20678964	0.21407798	0.20119503	0.18881843	0.187500	0.187500
	0.50	0.54454540	0.41357929	0.42815596	0.40239005	0.37763687	0.375000	0.375000
	0.75	0.81681810	0.62036893	0.64223394	0.60358508	0.56645530	0.562500	0.562500
	1.00	1.08909080	0.82715857	0.85631192	0.80478011	0.75527373	0.750000	0.750000

Table 3: Numerical values when $\beta = 0.750$, 0.875, and 1.000 for (63).

t	x	$\beta = 0.750$		$\beta = 0.875$		$\beta = 1.000$	
		FVIM	2D-FLFs	FVIM	2D-FLFs	FVIM	Exact
0.25	0.25	0.26622298	0.26622021	0.26593959	0.26594005	0.26578827	0.26578827
	0.50	0.56489190	0.56488083	0.56375836	0.56376020	0.56315308	0.56315308
	0.75	0.89600678	0.89598187	0.89345630	0.89346046	0.89209443	0.89209443
	1.00	1.25956762	1.25952332	1.25503343	1.25504082	1.25261232	1.25261232
0.50	0.25	0.28474208	0.28474415	0.28340402	0.28340659	0.28256846	0.28256846
	0.50	0.63896831	0.63897662	0.63361610	0.63362636	0.63027383	0.63027383
	0.75	1.06267869	1.06269739	1.05063622	1.05065931	1.04311611	1.04311611
	1.00	1.55587323	1.55590647	1.53446439	1.53450544	1.52109530	1.52109531
0.75	0.25	0.30690489	0.30690747	0.30361709	0.30361656	0.30139478	0.30139480
	0.50	0.72761955	0.72762986	0.71446834	0.71446625	0.70557913	0.70557918
	0.75	1.26214400	1.26216719	1.23255378	1.23254905	1.21255304	1.21255316
	1.00	1.91047821	1.91051944	1.85787338	1.85786498	1.82231652	1.82231673

the exact solution $u(x,t) = x + x^2 \sinh(t)$. As previous, only the fourth-order term of the FVIM was used in evaluating the numerical solutions for Table 3. In the case of $\beta = 1$, it can be found that absolute error of 2D-FLFs is not bigger than $1.0e-10$ which is very small compared with that obtained by FVIM.

6. Conclusion

We define a basis of 2D-FLFs and derived its operational matrices of fractional derivative and integration, which are used to approximate the numerical solution of FPDEs. Compared with other numerical methods, 2D-FLFs method can accurately represent properties of fractional calculus. Moreover, only a small number of 2D-FLFs are needed to obtain a satisfactory result. The obtained results demonstrate the validity and applicability of proposed method for solving the FPFEs.

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