Research Article

A Note on k -Potence Preservers on Matrix Spaces over Complex Field

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Let $\mathbb{C}$ be the field of all complex numbers, M_n the space of all $n \times n$ matrices over $\mathbb{C}$, and S_n the subspace of M_n consisting of all symmetric matrices. The map $\phi : S_n \rightarrow M_n$ satisfies that $A - \lambda B$ is k -potent in S_n implying that $\phi(A) - \lambda\phi(B)$ is k -potent in M_n , where $\lambda \in \mathbb{C}$, then there exist an invertible matrix $P \in M_n$ and $\epsilon \in \mathbb{C}$ with $\epsilon^k = \epsilon$ such that $\phi(X) = \epsilon P^{-1}(X)P$ for every $X \in S_n$. Moreover, the inductive method used in this paper can be used to characterise similar maps from M_n to M_n .

1. Introduction

Let $\mathbb{C}$ be the field of all complex numbers, M_n the space of all $n \times n$ matrices over $\mathbb{C}$, T_n the subspace of M_n consisting of all triangular matrices, and S_n the subspace of M_n consisting of all symmetric matrices. For fixed integer $k \geq 2$, $A \in M_n$ is called a k -potent matrix if $A^k = A$; especially, A is an idempotent matrix when $k = 2$. The map $\phi : S_n \rightarrow M_n$ satisfies that $A - \lambda B$ is a k -potent matrix in S_n implying that $\phi(A) - \lambda\phi(B)$ is a k -potent matrix in M_n , where $\lambda \in \mathbb{C}$, is a kind of the so-called weak preservers. While replacing “implying that” with “if and only if,” ϕ is called strong preserver. Obviously, a strong preserver must be a weak preserver, while a weak preserver may not be a strong preserver.

The preserver problem in this paper is from LPPs but without linear assumption (more details about LPP in [1–3]). You and Wang characterized the strong k -potence preservers from M_n to M_n in [4]; then Song and Cao extended the result to weak preservers from M_n to M_n in [5]. In [6], Wang and You characterized the strong k -potence preservers from T_n to M_n . In this paper, the authors characterized the weak k -potence preservers from S_n to M_n and proved the following theorem.

Theorem 1. Suppose $\phi : S_n \rightarrow M_n$ satisfy that $A - \lambda B$ is a k -potent matrix in S_n implying that $\phi(A) - \lambda\phi(B)$ is a k -potent matrix in M_n , where $\lambda \in \mathbb{C}$. Then there exist invertible $P \in M_n$ and $\epsilon \in \mathbb{C}$ with $\epsilon^k = \epsilon$ such that $\phi(X) = \epsilon P^{-1}XP$ for every $X \in S_n$.

Furthermore, we can derive the following corollary from Theorem 1.

Corollary 2. Suppose $\phi : S_n \rightarrow S_n$ satisfy that $A - \lambda B$ is a k -potent matrix in S_n implying that $\phi(A) - \lambda\phi(B)$ is a k -potent matrix in S_n , where $\lambda \in \mathbb{C}$. Then there exist invertible $P \in M_n$ and $\epsilon \in \mathbb{C}$ with $\epsilon^k = \epsilon$ such that $\phi(X) = \epsilon P^{-1}XP$ for every $X \in S_n$, where $PP^t = aI_n$ for some nonzero $a \in \mathbb{C}$.

In fact, the proof of Theorem 1 through some adjustments is suitable for the weak k -potence preserver from M_n to M_n , and more details can be seen in remarks.

2. Notations and Lemmas

Γ_n denotes the set of all k -potent matrices in M_n , while $\Sigma\Gamma_n = \Gamma_n \cap S_n$. Λ denotes the set of all complex number ϵ satisfying $\epsilon^{k-1} = 1$, $\Delta = \Lambda \cup \{0\}$. E_{ij} denotes matrices in M_n with 1 in

(i, j) and 0 elsewhere, and I_n denotes the unit matrix in M_n . $\langle n \rangle$ denotes the set of integer s satisfy $1 \leq s \leq n$. GL_n denotes the general linear group consisting of all invertible matrices in M_n . D_n denotes an arbitrary diagonal matrix in M_n . For $A, B \in M_n$, A and B are orthogonal if $AB = BA = 0$. $\mathbb{C}^{n \times 1}$ denotes the space of all $n \times 1$ matrices over $\mathbb{C}$. Φ_n denotes the set of all maps $\phi : S_n \rightarrow M_n$ satisfying that $A - \lambda B$ is a k -potent matrix in S_n implying that $\phi(A) - \lambda \phi(B)$ is a k -potent matrix in M_n , where $\lambda \in \mathbb{C}$.

For an arbitrary matrix $X \in M_n$, we denote by $X[i, j]$ the term in (i, j) position of X , by $X_{\{i_1, \dots, i_s; j_1, \dots, j_t\}}$ the $s \times t$ matrix with the term in its (p, q) position equal to $X[i_p, j_q]$, where $i_1 < \dots < i_s$ and $j_1 < \dots < j_t$. Moreover, we denote by $X_{\{i_1, \dots, i_s; j_1, \dots, j_t\}}$ the $n \times n$ matrix with the term in its (i_p, j_q) position equal to $X[i_p, j_q]$ and terms elsewhere equal to 0. We especially simplify it with $X_{\{i_1, \dots, i_t\}}$ when $s = t$, and $i_l = j_l$ for every $l \in \langle s \rangle$. Naturally, $X_{\{i\}} = X[i, i]E_{ii}$ for every $i \in \langle n \rangle$.

Without fixing X , $X_{\{i_1, \dots, i_s; j_1, \dots, j_t\}}$ also denotes a matrix in M_n with 0 in its (p, q) position, where $p \notin \{i_1, \dots, i_s\}$, $q \notin \{j_1, \dots, j_t\}$, and $1 \leq i_1 < \dots < i_s \leq n$, $1 \leq j_1 < \dots < j_t \leq n$.

At first, we need the following Lemmas 3, 4, 5, and 7, which are about k -potent matrices and orthogonal matrices.

Lemma 3 (see [2]). Suppose $X, Y \in \Gamma_n$, and $X + eY \in \Gamma_n$ for every $e \in \Lambda$; then X and Y are orthogonal.

Lemma 4 ([7, Lemma 1]). Suppose $A_1, A_2, \dots, A_n$ are $n \times n$ mutually orthogonal nonzero k -potent matrices; then there exists $P \in GL_n$ such that $P^{-1}A_iP = c_iE_{ii}$ with $c_i^{k-1} = 1$ for every $i \in \langle n \rangle$.

Lemma 5. Suppose $Z \in M_{n-1}$, $p, q, g, h \in \mathbb{C}^{(n-1) \times 1}$ with $gh^t \neq 0$, $\delta \in \mathbb{C}$, for arbitrary nonzero $\alpha \in \mathbb{C}$ with $h^t g + \alpha^2 \neq 0$ and $\tau = (\alpha^{-1}h^t g + \alpha)^{-1}$, $\tau \begin{bmatrix} Z & p \\ q^t & \delta \end{bmatrix} + \tau \begin{bmatrix} \alpha^{-1}gh^t & g \\ h^t & \alpha \end{bmatrix} \in \Gamma_n$. Then $Z = 0$, $\delta = 0$, and there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ with $(\lambda_1 + 1)(\lambda_2 + 1) = 1$ such that $p = \lambda_1 g$ and $q = \lambda_2 h$.

Proof. By the assumption of α and τ , $\tau \begin{bmatrix} \alpha^{-1}gh^t & g \\ h^t & \alpha \end{bmatrix}$ is idempotent. Denote this matrix by X , and then we can get the following equation:

$$\begin{bmatrix} I_{n-1} & -\alpha^{-1}g \\ \tau h^t & 1 - \tau \alpha^{-1}h^t g \end{bmatrix} X \begin{bmatrix} I_{n-1} - \tau \alpha^{-1}gh^t & \alpha^{-1}g \\ -\tau h^t & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (1)$$

Since the matrices on both sides of X satisfy the following equation:

$$\begin{bmatrix} I_{n-1} & -\alpha^{-1}g \\ \tau h^t & 1 - \tau \alpha^{-1}h^t g \end{bmatrix} \begin{bmatrix} I_{n-1} - \tau \alpha^{-1}gh^t & \alpha^{-1}g \\ -\tau h^t & 1 \end{bmatrix} = I_n \quad (2)$$

then the following matrix is k -potent by the assumption of lemma:

$$\begin{bmatrix} I_{n-1} & -\alpha^{-1}g \\ \tau h^t & 1 - \tau \alpha^{-1}h^t g \end{bmatrix} \left(\tau \begin{bmatrix} Z & p \\ q^t & \delta \end{bmatrix} + X \right) \times \begin{bmatrix} I_{n-1} - \tau \alpha^{-1}gh^t & \alpha^{-1}g \\ -\tau h^t & 1 \end{bmatrix}. \quad (3)$$

We denote by A the following matrix:

$$\begin{bmatrix} I_{n-1} & -\alpha^{-1}g \\ \tau h^t & 1 - \tau \alpha^{-1}h^t g \end{bmatrix} \begin{bmatrix} Z & p \\ q^t & \delta \end{bmatrix} \begin{bmatrix} I_{n-1} - \tau \alpha^{-1}gh^t & \alpha^{-1}g \\ -\tau h^t & 1 \end{bmatrix}; \quad (4)$$

then the following equation is obvious:

$$\left(\tau A + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)^k = \tau A + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (5)$$

Unfolding it, we get $\tau^k A^k + \tau^{k-1}(\dots)_{k-1} + \dots + \tau(\dots)_1 + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \tau A + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$; that is, $\tau^k A^k + \tau^{k-1}(\dots)_{k-1} + \dots + \tau(\dots)_1 - \tau A = 0$, where $(\dots)_i$ is the coefficient matrix of τ^i for every $i \in \langle k-1 \rangle$.

Let $A = \begin{bmatrix} Z_1 & p_1 \\ q_1^t & \delta_1 \end{bmatrix}$, then we calculate it and get the following equations:

$$\begin{aligned} Z_1 &= (Z - \alpha^{-1}gq^t)(I_{n-1} - \tau \alpha^{-1}gh^t) - (p - \delta \alpha^{-1}g)\tau h^t, \\ p_1 &= (Z - \alpha^{-1}gq^t)\alpha^{-1}g + p - \delta \alpha^{-1}g, \\ q_1^t &= (\tau h^t Z + q^t - \tau \alpha^{-1}q^t h^t g)(I_{n-1} - \tau \alpha^{-1}gh^t) \\ &\quad - (\tau h^t p + \delta - \delta \tau \alpha^{-1}h^t g)\tau h^t, \\ \delta_1 &= (\tau h^t Z + q^t - \tau \alpha^{-1}q^t h^t g)\alpha^{-1}g \\ &\quad + \tau h^t p + \delta - \delta \tau \alpha^{-1}h^t g. \end{aligned} \quad (6)$$

It is easy to get $\tau(\dots)_1 = \tau \begin{bmatrix} 0 & p_1 \\ q_1^t & k\delta_1 \end{bmatrix}$ and the following equation:

$$\begin{aligned} &\tau^k A^k + \tau^{k-1}(\dots)_{k-1} + \dots + \tau^2(\dots)_2 \\ &\quad + \tau \begin{bmatrix} -Z_1 & 0 \\ 0 & (k-1)\delta_1 \end{bmatrix} = 0. \end{aligned} \quad (7)$$

Note that the highest degree of α in $\tau^{-2}A$ is 2; then the highest degree of α in $\tau^{-3k+i}(\dots)_{k-i}$ is less or equal to $3k - i$ for every i with $2 \leq i \leq k-1$, and the highest degree of α in $\tau^{-3k+1} \begin{bmatrix} -Z_1 & 0 \\ 0 & (k-1)\delta_1 \end{bmatrix}$ is $3k-1$, where Z is the coefficient matrix of α^{3k-1} in Z_1 and δ is the coefficient of α^{3k-1} in δ_1 .

By the assumption of α , we have $Z = 0$ and $\delta = 0$. Then the following equations are true:

$$\begin{aligned} Z_1 &= -\alpha^{-1}gq^t(I_{n-1} - \tau \alpha^{-1}gh^t) - p\tau h^t, \\ p_1 &= -\alpha^{-1}gq^t\alpha^{-1}g + p, \\ q_1^t &= (q^t - \tau \alpha^{-1}q^t h^t g)(I_{n-1} - \tau \alpha^{-1}gh^t) - \tau h^t p\tau h^t, \\ \delta_1 &= (q^t - \tau \alpha^{-1}q^t h^t g)\alpha^{-1}g + \tau h^t p \end{aligned} \quad (8)$$

and $\tau^{-3k+1}Z_1 = \tau^{-3k+1}[-\alpha^{-1}gq^t + \alpha^{-1}gq^t\tau \alpha^{-1}gh^t - p\tau h^t] = \tau^{-3k+2}[-\tau^{-1}\alpha^{-1}gq^t + \alpha^{-2}gq^tgh^t - ph^t]$, where the highest degree of α is $3k-2$ and $-gq^t - ph^t$ is the coefficient matrix of α^{3k-2} .

Now, we calculate the upper left part of $\tau^{-3k+2}(\dots)_2$.

When $k = 2$, $\tau^{-3k+2}(\dots)_2 = \tau^{-4}A^2$, of which the upper left part is $\tau^{-4}[pq^t(I_{n-1} - \tau\alpha^{-1}gh^t) - q^t p\alpha^{-1}gth^t] = \tau^{-4}[pq^t - \tau\alpha^{-1}pq^tgh^t - \tau\alpha^{-1}q^t pgh^t]$. Then in the upper left part of $\tau^{-4}A^2 + \tau^{-5}\begin{bmatrix} -Z_1 & 0 \\ 0 & (k-1)\delta_1 \end{bmatrix}$, the highest degree of α is 4, and the coefficient matrix is $pq^t + gq^t + ph^t$.

When $k > 2$, if $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ appears in the left (or right) end of an additive item of $\tau^{-3k+2}(\dots)_2$, then the upper left part of this item is 0. So, the upper left part of $\tau^{-3k+2}(\dots)_2$ is equal to the upper left part of $\tau^{-3k+2}A\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^{k-2}A$; that is, the upper left part is $\tau^{-3k+2}p_1q_1^t = \tau^{-3k+4}[\tau^{-1}\alpha pq^t - (q^t g + h^t p)ph^t - \tau^{-1}\alpha^{-1}gq^t gq^t + \alpha^{-2}q^t g(q^t g + h^t p)gh^t]$, and the highest degree of α is $3k-2$ with pq^t as the coefficient matrix of α^{3k-2} .

By the assumption of α , we have $pq^t + gq^t + ph^t = 0$.

By $gh^t \neq 0$, we have $g \neq 0$, $h \neq 0$, and $p = 0$ if and only if $q = 0$. When $p \neq 0$, we can get $p = \lambda_1 g$ by $p(q^t + h^t) + gq^t = 0$, and $q = \lambda_2 h$ by $(p + g)q^t + ph^t = 0$, where λ_1 and λ_2 satisfy $\lambda_1 \lambda_2 gh^t + \lambda_2 gh^t + \lambda_1 gh^t = 0$; that is, $\lambda_1 \lambda_2 + \lambda_2 + \lambda_1 = 0$ by $gh^t \neq 0$, which is equivalent to $(\lambda_1 + 1)(\lambda_2 + 1) = 1$. When $p = q = 0$, $\lambda_1 = \lambda_2 = 0$. $\square$

Remark 6. Replacing $gh^t \neq 0$ with $gh^t = 0$ in Lemma 5, we have $g = 0$ implies $p = 0$ or $q + h = 0$, and $h = 0$ implies $q = 0$ or $p + g = 0$. These cases will not appear in the proof of Theorem 1, but are necessary for the weak preservers from M_n to M_n .

Lemma 7. Suppose $A = \begin{bmatrix} 0 & (\lambda(a) - \lambda(b))/(a-b) \\ (\lambda(a)^{-1} - \lambda(b)^{-1})/(a-b) & 1 \end{bmatrix} \in \Gamma_2$ for arbitrary $a, b \in \mathbb{C}$ with $a \neq b$, where $\lambda : \mathbb{C} \rightarrow \mathbb{C}$ is a map satisfying $\lambda(x) \neq 0$ for every $x \in \mathbb{C}$. Then there exists nonzero $\lambda_0 \in \mathbb{C}$ such that $\lambda(x) = \lambda_0$ for every $x \in \mathbb{C}$.

Proof. Since the trace of A is equal to 1, then $(\lambda(a) - \lambda(b))(\lambda^{-1}(a) - \lambda^{-1}(b))/(a-b)^2 = 0$, or -1 , especially, when equal to -1 , $k-1 = 6p$ with $p \in \mathbb{Z}^+$. Denote $\lambda(a)/\lambda(b)$ by y , and $a-b$ by c ; then we have $(2-y-y^{-1})/c^2 = 0$ or -1 .

- (1) If $(2-y-y^{-1})/c^2 = 0$, then $y = 1$, that is, $\lambda(a) = \lambda(b)$;
- (2) if $(2-y-y^{-1})/c^2 = -1$, then $y = (2+c^2 \pm \sqrt{4c^2 + c^4})/2$. When $c = 1$, $\lambda(b+1)/\lambda(b) = (3 \pm \sqrt{5})/2$; when $c = 2$, $\lambda(b+2)/\lambda(b) = (6 \pm \sqrt{32})/2 = 3 \pm 2\sqrt{2}$. But $\lambda(b+1)/\lambda(b) = (3 \pm \sqrt{5})/2$ implies $\lambda(b+2)/\lambda(b) = (\lambda(b+2)/\lambda(b+1))(\lambda(b+1)/\lambda(b)) = 1$, or $(7 \pm 3\sqrt{5})/2$. It is a contradiction! So it is impossible that $(2-y-y^{-1})/c^2 = -1$.

Hence, there exists nonzero $\lambda_0 \in \mathbb{C}$ such that $\lambda(x) = \lambda_0$ for every $x \in \mathbb{C}$. $\square$

We can prove the following Lemmas 8 and 9 similar as Lemmas 4 and 5 in [4].

Lemma 8 (see [4], Lemma 4). Suppose $\phi \in \Phi_n$, A and B are $n \times n$ orthogonal k -potent matrices; then $\phi(A)$ and $\phi(B)$ are orthogonal.

Lemma 9 (see [4], Lemma 5). Suppose $\phi \in \Phi_n$; then ϕ are homogeneous; that is, $\phi(\lambda X) = \lambda\phi(X)$ for every $X \in S_n$ and every $\lambda \in \mathbb{C}$.

Corollary 10. Suppose $\phi \in \Phi_n$, $A + B, C \in S\Gamma_n$, and for every $\epsilon \in \Lambda$, $A + B + \epsilon C \in S\Gamma_n$, $\phi(B + \epsilon C) = \phi(B) + \phi(\epsilon C)$. Then $\phi(A) + \phi(B)$ and $\phi(C)$ are orthogonal.

Proof. By the assumption and Lemma 9, we have $\phi(A) + \phi(B) \in \Gamma_n$, $\phi(C) \in \Gamma_n$, $\phi(A) + \phi(B + \epsilon C) = \phi(A) + \phi(B) + \epsilon\phi(C) \in \Gamma_n$. By Lemma 3, $\phi(A) + \phi(B)$ and $\phi(C)$ are orthogonal. $\square$

Corollary 11. Suppose $\phi \in \Phi_n$ and $\phi(D_n) = D_n$ for arbitrary diagonal matrix $D_n \in M_n$. Then for every $i, j \in \langle n \rangle$ with $i \neq j$, $\phi(E_{ij} + E_{ji} + D_n) = \lambda_{ij}E_{ij} + \lambda_{ij}^{-1}E_{ji} + D_n$, where $\lambda_{ij} \in \mathbb{C}$ is only decided by i and j .

Proof. Let $A = (1/2)(E_{ij} + E_{ji} + D_n)$, $B = (1/2)(E_{ii} + E_{jj} - D_n)$, and $C = \sum_{l \neq i, j} E_{ll}$; then A, B and C satisfy the assumption of Corollary 10, and $\phi(A) + \phi(B)$ and $\phi(C)$ are orthogonal; that is, $\phi((E_{ij} + E_{ji} + D_n)) = \alpha_{ii}E_{ii} + \beta_{ij}E_{ij} + \gamma_{ji}E_{ji} + \delta_{jj}E_{jj} + D_n$ for some $\alpha_{ii}, \beta_{ij}, \gamma_{ji}$, and $\delta_{jj} \in \mathbb{C}$.

Since $(\eta^{-1} + \eta)^{-1}[(E_{ij} + E_{ji} + D_n) - (D_n - \eta^{-1}E_{ii} - \eta E_{jj})] = (\eta^{-1} + \eta)^{-1}(\eta^{-1}E_{ii} + E_{ij} + E_{ji} + \eta E_{jj}) \in S\Gamma_n$ for arbitrary nonzero $\eta \in \mathbb{C}$ with $1 + \eta^2 \neq 0$, after applying ϕ , we have $(\eta^{-1} + \eta)^{-1}[\alpha_{ii}E_{ii} + \beta_{ij}E_{ij} + \gamma_{ji}E_{ji} + \delta_{jj}E_{jj} + \eta^{-1}E_{ii} + \eta E_{jj}] = (\eta^{-1} + \eta)^{-1}[\alpha_{ii}E_{ii} + (\beta_{ij} - 1)E_{ij} + (\gamma_{ji} - 1)E_{ji} + \delta_{jj}E_{jj}] + (\eta^{-1} + \eta)^{-1}(\eta^{-1}E_{ii} + E_{ij} + E_{ji} + \eta E_{jj}) \in \Gamma_n$. By Lemma 5, $\alpha_{ii} = \delta_{jj} = 0$, $\beta_{ij}\gamma_{ji} = 1$.

Let $D_n = \sum_{l=1}^n x_l E_{ll}$, where $x_l \in \mathbb{C}$ for every $l \in \langle n \rangle$; then β_{ij} is the function of i, j , and x_l and denote by $\beta_{ij}(D_n)$ the value of β_{ij} on $x_1, \dots, x_n, i$, and j .

Fix i, j , and D_n and add a free variable x to x_i for some $l \in \langle n \rangle$; then $\beta_{ij}(D_n + xE_{ii})$ becomes into a map of x . Since $(1/(a-b))(E_{ij} + E_{ji} + D_n + aE_{jj}) - (1/(a-b))(E_{ij} + E_{ji} + D_n + bE_{jj}) \in S\Gamma_n$ for arbitrary a and $b \in \mathbb{C}$ with $a-b \neq 0$, then by $\phi(E_{ij} + E_{ji} + D_n + aE_{jj}) = \beta_{ij}(D_n + aE_{jj})E_{ij} + \beta_{ij}^{-1}(D_n + aE_{jj})E_{ji} + D_n + aE_{jj}$ and $\phi(E_{ij} + E_{ji} + D_n + bE_{jj}) = \beta_{ij}(D_n + bE_{jj})E_{ij} + \beta_{ij}^{-1}(D_n + bE_{jj})E_{ji} + D_n + bE_{jj}$, we can derive that $((\beta_{ij}(D_n + aE_{jj}) - \beta_{ij}(D_n + bE_{jj}))/((a-b))E_{ij} + ((\beta_{ij}^{-1}(D_n + aE_{jj}) - \beta_{ij}^{-1}(D_n + bE_{jj}))/((a-b))E_{ji} + E_{jj}) \in \Gamma_n$. By Lemma 7, $\beta_{ij}(D_n + aE_{jj}) = \beta_{ij}(D_n + bE_{jj})$ for fixed i, j , and D_n ; that is, $\beta_{ij}(D_n + xE_{jj}) = \beta_{ij}(D_n)$ for arbitrary $x \in \mathbb{C}$. Similarly, we can prove $\beta_{ij}(D_n + xE_{ii}) = \beta_{ij}(D_n)$ for arbitrary $x \in \mathbb{C}$.

In fact, we have proved that $\beta_{ij}(D_n + xE_{ii}) = \beta_{ij}(D_n)$ and $\beta_{ij}(D_n + yE_{jj}) = \beta_{ij}(D_n)$ for arbitrary $x, y \in \mathbb{C}$ and arbitrary D_n ; then $\beta_{ij}(D_n + xE_{ii} + yE_{jj}) = \beta_{ij}(D_n + xE_{ii}) (= \beta_{ij}(D_n + yE_{jj})) = \beta_{ij}(D_n)$ follows.

Since $\beta_{ij}(D_n + xE_{jj} + yE_{ii}) = \beta_{ij}(D_n + yE_{ii})$ for fixed i, j , and l with $l \neq i, j$, and arbitrary $x, y \in \mathbb{C}$, then $(1/(a-b))(E_{ij} + E_{ji} + D_n + (a-b)E_{jj} + aE_{ii}) - (1/(a-b))(E_{ij} + E_{ji} + D_n + bE_{ii}) \in S\Gamma_n$ implies $((\beta_{ij}(D_n + aE_{ii}) - \beta_{ij}(D_n + bE_{ii}))/((a-b))E_{ij} + ((\beta_{ij}^{-1}(D_n + aE_{ii}) - \beta_{ij}^{-1}(D_n + bE_{ii}))/((a-b))E_{ji} + E_{jj} + E_{ll}) \in \Gamma_n$. By Lemma 7, we can get $\beta_{ij}(D_n + aE_{ii}) = \beta_{ij}(D_n + bE_{ii})$ for arbitrary a and

$b \in \mathbb{C}$ with $a - b \neq 0$; that is, $\beta_{ij}(D_n + xE_{ll}) = \beta_{ij}(D_n)$ for arbitrary $x \in \mathbb{C}$.

Until now, we have proved that $\beta_{ij}(D_n) = \beta_{ij}(\sum_{l=1}^n x_l E_{ll}) = \beta_{ij}(\sum_{l=1}^{n-1} x_l E_{ll}) = \dots = \beta_{ij}(x_1 E_{11}) = \beta_{ij}(0)$ for arbitrary D_n ; that is, β_{ij} is only decided by i and j . $\square$

Remark 12. The proof of Corollary 11 presents the basic procedure of proof of Theorem 1. In order to decide the image of matrix A , we use Corollary 10 and the images of B and C , which usually are diagonal matrices or some matrices with images already decided.

If ϕ is a weak preserver from M_n to M_n , then Corollary 11 is also true. Let $A = E_{ij} + D_n$, $B = -(E_{ij} + E_{ji} + D_n) + E_{ii}$, and $C = \sum_{l \neq i, j} E_{ll}$; then we can prove $\phi(A) = a_{ii}E_{ii} + a_{ij}E_{ij} + a_{ji}E_{ji} + a_{jj}E_{jj} + D_n$ similarly as proving $\phi((E_{ij} + E_{ji} + D_n)) = \alpha_{ii}E_{ii} + \beta_{ij}E_{ij} + \gamma_{ji}E_{ji} + \delta_{jj}E_{jj} + D_n$, and $(a_{ii} + 1)E_{ii} + (a_{ij} - \lambda_{ij})E_{ij} + (a_{ji} - \lambda_{ji}^{-1})E_{ji} + a_{jj}E_{jj} \in \Gamma_n$. Since $\alpha^{-1}A + \alpha^{-1}(-(E_{ij} + E_{ji} + D_n) + \alpha E_{ii}) = -\alpha^{-1}E_{ji} + E_{ii} \in \Gamma_n$ for arbitrary nonzero α , then the following matrix is k -potent:

$$\alpha^{-1} \begin{bmatrix} a_{ii} & a_{ij} - \lambda_{ij} \\ a_{ji} - \lambda_{ji}^{-1} & a_{jj} \end{bmatrix} + \alpha^{-1} \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}. \quad (9)$$

Remark 6 tells us that $a_{ii} = a_{jj} = 0$, $a_{ij} - \lambda_{ij} = 0$, or $a_{ji} - \lambda_{ji}^{-1} = 0$; that is, $\phi(A) = \lambda_{ij}E_{ij} + D_n$, or $\phi(A) = \lambda_{ij}^{-1}E_{ji} + D_n$, $\phi(A) = \lambda_{ij}E_{ij} + \lambda_{ij}^{-1}E_{ji} + D_n$. Similarly, we can prove $\phi(E_{ji} + D_n) = \lambda_{ij}^{-1}E_{ji} + D_n$, $\phi(E_{ji} + D_n) = \lambda_{ij}E_{ij} + D_n$, or $\phi(E_{ji} + D_n) = \lambda_{ij}E_{ij} + \lambda_{ij}^{-1}E_{ji} + D_n$. Since D_n is arbitrary, we set $D_n = 0$ for convenience.

If $\phi(E_{ij}) = \lambda_{ij}E_{ij} + \lambda_{ij}^{-1}E_{ji}$; then $(1/3)E_{ij} + (1/3)(E_{ij} + E_{ji} + 2E_{ii} + E_{jj}) = (1/3)(2E_{ij} + E_{ji} + 2E_{ii} + E_{jj}) \in \Gamma_n$ implies $(1/3)(\lambda_{ij}E_{ij} + \lambda_{ij}^{-1}E_{ji}) + (1/3)(\lambda_{ij}E_{ij} + \lambda_{ij}^{-1}E_{ji} + 2E_{ii} + E_{jj}) = (1/3)(2\lambda_{ij}E_{ij} + 2\lambda_{ij}^{-1}E_{ji} + 2E_{ii} + E_{jj}) \in \Gamma_n$; that is, $-2/9 \in \Delta$, which is a contradiction. Hence, we proved that it is impossible $\phi(E_{ij}) = \lambda_{ij}E_{ij} + \lambda_{ij}^{-1}E_{ji}$ or $\phi(E_{ji}) = \lambda_{ij}E_{ij} + \lambda_{ij}^{-1}E_{ji}$.

If $\phi(E_{ij}) = \lambda_{ij}E_{ij}$ and $\phi(E_{ji}) = \lambda_{ij}E_{ij}$, then $(1/2)(E_{ii} + E_{ij} + E_{ji} + E_{jj}) \in \Gamma_n$ implies $(1/2)(\phi(E_{ij}) + \phi(E_{ii} + E_{ji} + E_{jj})) \in \Gamma_n$; that is, $(1/2)(E_{ii} + 2\lambda_{ij}E_{ij} + E_{jj}) \in \Gamma_n$, which is a contradiction. Hence, we proved that $\phi(E_{ij}) = \lambda_{ij}E_{ij}$ and $\phi(E_{ji}) = \lambda_{ij}^{-1}E_{ji}$, or $\phi(E_{ij}) = \lambda_{ij}^{-1}E_{ji}$ and $\phi(E_{ji}) = \lambda_{ij}E_{ij}$.

3. Proof of Theorem 1

Suppose $\phi \in \Phi_n$, then we can derive Theorem 1 from Propositions 13, 14, and 16.

Proposition 13. Suppose $i, j \in \langle n \rangle$ with $i \neq j$; then $\phi(E_{ii}) = 0$ if and only if $\phi(E_{jj}) = 0$.

Proof. Suppose $\phi(E_{ii}) = 0$ and $\phi(E_{jj}) \neq 0$ for some $i, j \in \langle n \rangle$ with $i \neq j$. At first, we prove that $\phi(aE_{ii} + E_{jj}) = \phi(E_{jj})$ for arbitrary $a \in \mathbb{C}$. Since the equation is already true when $a = 0$, then we assume $a \neq 0$ in the following proof.

Let $A = a^{-1}(aE_{ii} + E_{jj})$, $B = -a^{-1}E_{jj}$, and $C = E_{jj}$; then it is easy to verify A, B , and C satisfying the assumption of Corollary 10. So $\phi(a^{-1}(aE_{ii} + E_{jj})) + \phi(-a^{-1}E_{jj})$ and $\phi(E_{jj})$

are orthogonal. Moreover, we can derive $\phi(aE_{ii} + E_{jj}) \in \Gamma_n$ from $(aE_{ii} + E_{jj}) - aE_{ii} \in \Gamma_n$ and $\phi(E_{ii}) = 0$. Let $a^{-1}(\phi(aE_{ii} + E_{jj}) - \phi(E_{jj})) = D$, then D and $\phi(E_{jj})$ are orthogonal k -potent matrices. While $\phi(aE_{ii} + E_{jj}) \in \Gamma_n$ implies $aD + \phi(E_{jj}) \in \Gamma_n$; then $aD \in \Gamma_n$. There are two cases on a .

- (1) If $a \notin \Lambda$, then $D = 0$; that is, $\phi(aE_{ii} + E_{jj}) = \phi(E_{jj})$;
- (2) if $a \in \Lambda$, we can derive that $(1/3)\phi(aE_{ii} + E_{jj}) - (1/3)\phi((a - 3)E_{ii} + E_{jj}) \in \Gamma_n$ from $(1/3)(aE_{ii} + E_{jj}) - (1/3)((a - 3)E_{ii} + E_{jj}) \in \Gamma_n$. Note that $a - 3 \notin \Lambda$, so it is true that $\phi((a - 3)E_{ii} + E_{jj}) = \phi(E_{jj})$; that is, $(1/3)\phi(aE_{ii} + E_{jj}) - (1/3)\phi(E_{jj}) = (a/3)D \in \Gamma_n$. Finally, we can derive $D = 0$ from $a/3 \notin \Lambda$ and $D \in \Gamma_n$. At the same time, $\phi(aE_{ii} + E_{jj}) = \phi(E_{jj})$.

Anyway, $\phi(aE_{ii} + E_{jj}) = \phi(E_{jj})$ for arbitrary $a \in \mathbb{C}$.

Since $(b^{-1} + b)^{-1}(b^{-1}E_{ii} + E_{ij} + E_{ji} + bE_{jj}) \in \Gamma_n$ for every nonzero $b \in \mathbb{C}$ with $1 + b^2 \neq 0$, then $(b^{-1} + b)^{-1}[\phi(E_{ij} + E_{ji}) + \phi(b^{-1}E_{ii} + bE_{jj})] \in \Gamma_n$, and $(b^{-1} + b)^{-1}[\phi(E_{ij} + E_{ji}) + b\phi(E_{jj})] \in \Gamma_n$ by $\phi(b^{-1}E_{ii} + bE_{jj}) = b\phi(E_{jj})$. While the equation $(b^{-1} + b)^{-k}[\phi(E_{ij} + E_{ji}) + b\phi(E_{jj})]^k = (b^{-1} + b)^{-1}[\phi(E_{ij} + E_{ji}) + b\phi(E_{jj})]$ is equivalent to $b^{k-1}[\phi(E_{ij} + E_{ji}) + b\phi(E_{jj})]^k = (1 + b^2)^{k-1}[\phi(E_{ij} + E_{ji}) + b\phi(E_{jj})]$. Note that $\phi(E_{ij} + E_{ji})$ is the constant term of the equation; then $\phi(E_{ij} + E_{ji}) = 0$ by the infinite property of b , and $(b^{-1} + b)^{-1}b\phi(E_{jj}) \in \Gamma_n$ follows. Then we can derive $\phi(E_{jj}) = 0$ which is a contradiction to the assumption. $\square$

Proposition 14. Suppose $\phi(E_{ii}) = 0$ for every $i \in \langle n \rangle$; then $\phi(X) = 0$ for arbitrary $X \in S_n$.

Proof. The proof will be completed by induction on the following equation for arbitrary $X \in S_n$ with $X[i, i] = x_i$ for every $i \in \langle n \rangle$:

$$\phi \left(X_{\{1, \dots, m\}} + \sum_{i=m+1}^n x_i E_{ii} \right) = 0, \quad (10)$$

where $1 \leq m \leq n - 1$.

When $m = 1$, (10) is equivalent to $\phi(\sum_{i=1}^n a_i E_{ii}) = 0$ for arbitrary $D_n = \sum_{i=1}^n a_i E_{ii} \in S_n$.

At first, by the assumption, it is already true that $\phi(E_{ii}) = 0$ for every $i \in \langle n \rangle$.

Suppose $\phi(\sum_{j=1}^s a_{i_j} E_{i_j i_j}) = 0$ for every $s \in \langle n - 1 \rangle$ with $1 \leq i_1 < \dots < i_s \leq n$; then by the homogeneity of ϕ , we just need to prove the following equation for i_{s+1} with $i_s < i_{s+1} \leq n$:

$$\phi \left(\sum_{j=1}^s a_{i_j} E_{i_j i_j} + E_{i_{s+1} i_{s+1}} \right) = 0. \quad (11)$$

There are two cases on $B_s = \sum_{j=1}^s a_{i_j} E_{i_j i_j}$.

- (1) If $B_s \notin \Gamma_n$, then there exists $l \in \langle s \rangle$ such that $a_{i_l} \notin \Lambda$, and the following statements are true:

$$(B_s + E_{i_{s+1} i_{s+1}}) - B_s = E_{i_{s+1} i_{s+1}} \in \Gamma_n,$$

$$a_{i_l}^{-1} (B_s + E_{i_{s+1} i_{s+1}}) - a_{i_l}^{-1} (B_s + E_{i_{s+1} i_{s+1}} - a_{i_l} E_{i_l i_l}) = E_{i_l i_l} \in \Gamma_n. \quad (12)$$

Note that $\phi(B_s) = 0$ and $\phi(B_s + E_{i_{s+1}i_{s+1}} - a_{i_i}E_{i_i i_i}) = 0$ by the assumption; then the following statements are true:

$$\begin{aligned} \phi(B_s + E_{i_{s+1}i_{s+1}}) &\in \Gamma_n, \\ a_{i_i}^{-1}\phi(B_s + E_{i_{s+1}i_{s+1}}) &\in \Gamma_n. \end{aligned} \quad (13)$$

Since $a_{i_i} \notin \Delta$, then $a_{i_i}^{-1} \notin \Delta$, and $\phi(B_s + E_{i_{s+1}i_{s+1}}) = 0$ follows.

(2) If $B_s \in \Sigma\Gamma_n$, then we have the following statements:

$$\begin{aligned} B_s + E_{i_{s+1}i_{s+1}} &\in \Sigma\Gamma_n, \\ \frac{1}{3}(B_s + E_{i_{s+1}i_{s+1}}) - \frac{1}{3}(-3E_{i_i i_i} + B_s + E_{i_{s+1}i_{s+1}}) &= E_{i_i i_i} \in \Sigma\Gamma_n. \end{aligned} \quad (14)$$

Since $a_{i_i} - 3 \notin \Delta$; then $\phi(-3E_{i_i i_i} + B_s + E_{i_{s+1}i_{s+1}}) = 0$ by case 1, and $(1/3)\phi(B_s + E_{i_{s+1}i_{s+1}}) \in \Gamma_n$ follows. While $\phi(B_s + E_{i_{s+1}i_{s+1}}) \in \Gamma_n$, hence we get $\phi(B_s + E_{i_{s+1}i_{s+1}}) = 0$.

Anyway, we prove $\phi(\sum_{j=1}^s a_{i_j}E_{i_j i_j} + E_{i_{s+1}i_{s+1}}) = 0$; then by the induction, (10) is true for $m = 1$.

Suppose (10) is true for $m \in \langle n-1 \rangle$, then we prove the case on $m+1$.

Let $X_m = X_{[1, \dots, m; 1, \dots, m]}$, $g = X_{[1, \dots, m; m+1]}$, $A_{n-m} = \sum_{i=1}^{n-m} x_{i+m}E_{ii} \in M_{n-m}$; then we have $g^t = X_{[m+1; 1, \dots, m]}$ and the following equation:

$$\phi\left(\begin{bmatrix} X_m & 0 \\ 0 & A_{n-m} \end{bmatrix}\right) = 0. \quad (15)$$

We will prove the following equation which is equivalent to (10) on $m+1$:

$$\phi\left(\begin{bmatrix} X_m & g & 0 \\ g^t & x_{m+1} & 0 \\ 0 & 0 & A_{n-m-1} \end{bmatrix}\right) = 0 \quad (16)$$

For arbitrary nonzero $\alpha \in \mathbb{C}$ with $g^t g + \alpha^2 \neq 0$, the following $n \times n$ matrix B is idempotent:

$$B = \tau \begin{bmatrix} \alpha^{-1} g g^t & g & 0 \\ g^t & \alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (17)$$

where $\tau = (\alpha^{-1} g^t g + \alpha)^{-1}$.

Note that $X_{m+1} = \begin{bmatrix} X_m & g \\ g^t & x_{m+1} \end{bmatrix}$ and A_{n-m-1} satisfy the following equation:

$$\begin{aligned} \tau \begin{bmatrix} X_m & g & 0 \\ g^t & x_{m+1} & 0 \\ 0 & 0 & A_{n-m-1} \end{bmatrix} \\ - \tau \begin{bmatrix} X_m - \alpha^{-1} g g^t & 0 & 0 \\ 0 & x_{m+1} - \alpha & 0 \\ 0 & 0 & A_{n-m-1} \end{bmatrix} &= B. \end{aligned} \quad (18)$$

After applying ϕ on the above matrices, we have $\tau\phi(X_{m+1} \oplus A_{n-m-1}) \in \Gamma_n$ by the inductive assumption. Then $\phi(X_{m+1} \oplus A_{n-m-1}) = 0$ because of the assumption of α ; that is, (10) holds for $m+1$.

Finally, we prove that $\phi(X) = 0$ for every $X \in S_n$ by the induction. $\square$

Remark 15. If ϕ is a weak k -potence preserver from M_n to M_n ; then Propositions 13 and 14 (replacing g^t with h^t for arbitrary $X \in M_n$ in the proof of Proposition 14) hold since Corollary 10 is true under this assumption.

Proposition 16. Suppose $\phi(E_{ii}) \neq 0$ for every $i \in \langle n \rangle$, then there exist $P \in GL_n$ and $c \in \Lambda$ such that $\phi(X) = cP^{-1}XP$ for every $X \in S_n$.

Proof. The proof will be completed in the following 4 steps.

Step 1. $\phi(E_{ii}) = c_i E_{ii}$, where $c_i \in \Lambda$ for every $i \in \langle n \rangle$.

Since $\phi(E_{ii})$ is nonzero k -potent, then we can derive from Lemma 4 that there exists $P_1 \in GL_n$ such that $P_1^{-1}\phi(E_{ii})P_1 = c_i E_{ii}$ for every $i \in \langle n \rangle$, where $c_i \in \Lambda$. It is obvious that the following map $\varphi \in \Phi_n$ and $\varphi(E_{ii}) = c_i E_{ii}$ for every $i \in \langle n \rangle$.

$$\varphi(X) = P_1^{-1}\phi(X)P_1. \quad (19)$$

Without loss of generality, we can assume $\phi(E_{ii}) = c_i E_{ii}$.

Step 2. $\phi(\sum_{i=1}^n a_i E_{ii}) = \sum_{i=1}^n a_i \phi(E_{ii})$, for arbitrary diagonal matrix $\sum_{i=1}^n a_i E_{ii}$.

The proof of this step can be seen in Step 3, Section 3 in [5].

Step 3. $c_i = c \in \Lambda$ for every $i \in \langle n \rangle$.

Let $A = (1/2)(E_{ij} + E_{ji})$, $B = (1/2)(E_{ii} + E_{jj})$, and $C = \sum_{l \in \langle n \rangle \setminus \{i, j\}} E_{ll}$, we can derive the following equation from Step 2 and Corollary 10:

$$\phi(E_{ij} + E_{ji}) = \alpha_0 E_{ii} + \beta_0 E_{ij} + \gamma_0 E_{ji} + \delta_0 E_{jj}, \quad (20)$$

where $\alpha_0, \beta_0, \gamma_0, \delta_0 \in \mathbb{C}$, $i, j \in \langle n \rangle$ with $i \neq j$.

Note that $pE_{ii} + q(E_{ij} + E_{ji}) + (1-p)E_{jj} \in \Sigma\Gamma_n$ for $p, q \in \mathbb{C}$ with $q^2 = p(1-p)$. In fact, 0 and 1 are all the eigenvalues of this matrix. Applying ϕ on the matrix $q(E_{ij} + E_{ji}) + [pE_{ii} + (1-p)E_{jj}]$, we have $H(p) = q(\alpha_0 E_{ii} + \beta_0 E_{ij} + \gamma_0 E_{ji} + \delta_0 E_{jj}) + pc_i E_{ii} + (1-p)c_j E_{jj} = (pc_i + q\alpha_0)E_{ii} + q\beta_0 E_{ij} + q\gamma_0 E_{ji} + ((1-p)c_j + q\delta_0)E_{jj} \in \Gamma_n$.

Since k is fixed, then Δ is the finite set which contains all of eigenvalues of $H(p)$, and there exists $w \in \{c+d \mid c, d \in \Delta\}$ such that the trace of $H(p)$ is w for infinite choices of p ; that is, there exist (p_1, p_2) with $p_1 \neq p_2$ such that the traces of $H(p_1)$ and $H(p_2)$ are all equal to w ; then we have the following equation:

$$\begin{aligned} (p_1 c_i + q_1 \alpha_0) + ((1-p_1)c_j + q_1 \delta_0) \\ = (p_2 c_i + q_2 \alpha_0) + ((1-p_2)c_j + q_2 \delta_0) \end{aligned} \quad (21)$$

which is equivalent to

$$(q_1 - q_2)(\alpha_0 + \delta_0) = (p_2 - p_1)(c_i - c_j), \quad (22)$$

where $q_s^2 = p_s(1 - p_s)$, for $s = 1, 2$.

Naturally, there are infinite choices of p_2 for fixed p_1 such that the above equation is true. If $(q_1 - q_2)/(p_2 - p_1)$ is equal to some $a \in \mathbb{C}$, where $p_2 \neq p_1$, p_1 and q_1 are fixed, then we can derive from the following equation:

$$(a^2 + 1)p_2^2 - (2aq_1 + 2a^2p_1 + 1)p_2 + (q_1 + ap_1)^2 = 0 \quad (23)$$

that there are infinite choices of p_2 for constant $(q_1 - q_2)/(p_2 - p_1)$ if and only if $a^2 + 1 = 2aq_1 + 2a^2p_1 + 1 = (q_1 + ap_1)^2 = 0$. While $a^2 + 1 = (q_1 + ap_1)^2 = 0$ and $q_1^2 = p_1(1 - p_1)$ imply $p_1 = q_1 = 0$, which is a contradiction to $2aq_1 + 2a^2p_1 + 1 = 0$, hence $(q_1 - q_2)/(p_2 - p_1)$ varies with p_2 .

Since $\alpha_0 + \delta_0$ and $c_i - c_j$ are all fixed numbers for fixed ϕ , then $\alpha_0 + \delta_0 \neq 0$ implies that there are at least two different values of $c_i - c_j = (q_1 - q_2)/(p_2 - p_1)(\alpha_0 + \delta_0)$ for fixed p_1 and infinite choices of p_2 ; it is a contradiction. So $\alpha_0 + \delta_0 = 0$ and $c_i = c_j$ follows. Hence $c_i = c$ for every $i \in \langle n \rangle$.

Step 4. $\phi(X) = X$ for every $X \in S_n$.

After the discussion in Steps 1, 2, and 3, we already have the following equation:

$$\phi\left(\sum_{i=1}^n a_i E_{ii}\right) = c \sum_{i=1}^n a_i E_{ii}, \quad (24)$$

where $c \in \Lambda$, $a_i \in \mathbb{C}$ for every $i \in \langle n \rangle$. Since the map $c^{-1}\phi \in \Phi_n$, then we can assume $\phi(\sum_{i=1}^n a_i E_{ii}) = \sum_{i=1}^n a_i E_{ii}$ without loss of generality.

The proof in this step will be completed by induction on the following equation for arbitrary $X \in S_n$ with $X[i, i] = x_i$ for every $i \in \langle n \rangle$:

$$\begin{aligned} & \phi\left(X_{\{i_1, \dots, i_m\}} + \sum_{j \in \langle n \rangle \setminus \{i_1, \dots, i_m\}} x_j E_{jj}\right) \\ &= X_{\{i_1, \dots, i_m\}} + \sum_{j \in \langle n \rangle \setminus \{i_1, \dots, i_m\}} x_j E_{jj}, \end{aligned} \quad (25)$$

where $1 \leq i_1 < \dots < i_m \leq n$ with $2 \leq m \leq n - 1$.

When $m = 2$, (25) is equivalent to $\phi(E_{ij} + E_{ji} + D_n) = E_{ij} + E_{ji} + D_n$ for arbitrary diagonal matrix $D_n \in S_n$ and $i, j \in \langle n \rangle$ with $i < j$, since ϕ is homogeneous. The proof will be completed in the following (1) and (2).

$$(1) \phi(E_{ii+1} + E_{i+1i} + D_n) = E_{ii+1} + E_{i+1i} + D_n \text{ for every } i \in \langle n-1 \rangle.$$

We already derive from Corollary 11 that $\phi(E_{ii+1} + E_{i+1i} + D_n) = \lambda_i E_{ii+1} + \lambda_i^{-1} E_{i+1i} + D_n$ for every $i \in \langle n-1 \rangle$, where $\lambda_i \in \mathbb{C}$ is only decided by i .

Suppose the map $\rho : S_n \rightarrow M_n$ satisfies the following equation for every $X \in S_n$,

$$\begin{aligned} \rho(X) &= \text{diag}\left(1, \lambda_1, \lambda_1 \lambda_2, \dots, \prod_{i=1}^{n-1} \lambda_i\right) \phi(X) \\ &\quad \times \text{diag}\left(1, \lambda_1^{-1}, \lambda_1^{-1} \lambda_2^{-1}, \dots, \prod_{i=1}^{n-1} \lambda_i^{-1}\right); \end{aligned} \quad (26)$$

then $\rho \in \Phi_n$, and for arbitrary diagonal matrix D_n and every $i \in \langle n-1 \rangle$, $\rho(D_n) = D_n$ and $\rho(E_{ii+1} + E_{i+1i} + D_n) = E_{ii+1} + E_{i+1i} + D_n$.

Without loss of generality, we can assume $\phi(E_{ii+1} + E_{i+1i} + D_n) = E_{ii+1} + E_{i+1i} + D_n$ for every $i \in \langle n-1 \rangle$ and arbitrary D_n .

$$(2) \text{ Suppose } \phi(E_{ij} + E_{ji} + D_n) = E_{ij} + E_{ji} + D_n \text{ for every } i, j \text{ with } 1 \leq j - i < s < n - 1; \text{ then } \phi(E_{ij} + E_{ji} + D_n) = E_{ij} + E_{ji} + D_n \text{ for every } i, j \text{ with } j - i = s.$$

At first, we have to prove that $\phi(x_{ii+1}(E_{ii+1} + E_{i+1i}) + x_{i+1i+m}(E_{i+1i+m} + E_{i+mi+1}) + D_n) = x_{ii+1}(E_{ii+1} + E_{i+1i}) + x_{i+1i+m}(E_{i+1i+m} + E_{i+mi+1}) + D_n$ for arbitrary nonzero x_{ii+1} and $x_{i+1i+m} \in \mathbb{C}$.

By the assumption, we already have the following equations:

$$\begin{aligned} \phi(x_{ii+1}(E_{ii+1} + E_{i+1i}) + D_n) &= x_{ii+1}(E_{ii+1} + E_{i+1i}) + D_n, \\ \phi(x_{i+1i+m}(E_{i+1i+m} + E_{i+mi+1}) + D_n) &= x_{i+1i+m}(E_{i+1i+m} + E_{i+mi+1}) + D_n. \end{aligned} \quad (27)$$

Let $X_1 = x_{ii+1}(E_{ii+1} + E_{i+1i}) + x_{i+1i+m}(E_{i+1i+m} + E_{i+mi+1}) + D_n$, $X_2 = x_{ii+1}(E_{ii+1} + E_{i+1i}) + D_n$, and $X_3 = x_{i+1i+m}(E_{i+1i+m} + E_{i+mi+1}) + D_n$. Then the following statements are true

$$X_1 - (X_2 - a_{i+1}E_{i+1i+1} - a_{i+m}E_{i+mi+m}) \in \Sigma_n,$$

$$X_1 - (X_2 - a_{i+1}E_{i+1i+1} - a_{i+m}E_{i+mi+m}) + \epsilon \sum_{l \neq i+1, i+m} E_{ll} \in \Sigma_n,$$

$$X_1 - (X_3 - b_i E_{ii} - b_{i+1} E_{i+1i+1}) \in \Sigma_n,$$

$$X_1 - (X_3 - b_i E_{ii} - b_{i+1} E_{i+1i+1}) + \epsilon \sum_{l \neq i, i+1} E_{ll} \in \Sigma_n, \quad (28)$$

where $x_{i+1i+m}(E_{i+1i+m} + E_{i+mi+1}) + a_{i+1}E_{i+1i+1} + a_{i+m}E_{i+mi+m}$ and $x_{ii+1}(E_{ii+1} + E_{i+1i}) + b_i E_{ii} + b_{i+1} E_{i+1i+1}$ are k -potent.

Let $A = X_1$, $B = -(X_2 - a_{i+1}E_{i+1i+1} - a_{i+m}E_{i+mi+m})$, and $C = \sum_{l \neq i+1, i+m} E_{ll}$, then A , B , and C satisfy the assumption of Corollary 10. Hence we get $\phi(A) + \phi(B)$ and $\phi(C)$ are orthogonal; that is,

$$\begin{aligned} \phi(X_1) &= X_2 + y_{i+1}E_{i+1i+1} + y_{i+m}E_{i+mi+m} \\ &\quad + y_{i+1i+m}E_{i+1i+m} + y_{i+mi+1}E_{i+mi+1}. \end{aligned} \quad (29)$$

Similarly, we can derive the following equation from Corollary 10:

$$\begin{aligned} \phi(X_1) &= X_3 + z_i E_{ii} + z_{i+1} E_{i+1i+1} \\ &\quad + z_{ii+1} E_{ii+1} + z_{i+1i} E_{i+1i}. \end{aligned} \quad (30)$$

Comparing the above two equations, we have $z_i = y_{i+m} = 0$, $z_{i+1} = y_{i+1}$, $z_{ii+1} = z_{i+1i} = x_{ii+1}$, and $y_{i+1i+m} = y_{i+mi+1} = x_{i+1i+m}$, that is, $\phi(X_1) = X_1 + y_{i+1}E_{i+1i+1}$.

We will prove $z_{i+1} = y_{i+1} = 0$. For arbitrary nonzero α with $x_{i+1i+m}^2 + \alpha^2 \neq 0$, let $\tau = (\alpha^{-1}x_{i+1i+m}^2 + \alpha)^{-1}$, and $X_4 =$

$-X_2 + \alpha^{-1}x_{i+1+m}^2 E_{i+1+m} + \alpha E_{i+mi+m}$; then $\tau X_1 + \tau X_4 \in S\Gamma_n$ implies $\tau\phi(X_1) + \tau\phi(X_4) \in \Gamma_n$; that is, the following matrix is k -potent since $\phi(X_4) = X_4$ by the assumption

$$\tau \begin{bmatrix} y_{i+1} & 0 \\ 0 & 0 \end{bmatrix} + \tau \begin{bmatrix} \alpha^{-1}x_{i+1+m}^2 & x_{i+1+m} \\ x_{i+1+m} & \alpha \end{bmatrix} \quad (31)$$

by Lemma 5, $y_{i+1} = 0$. Hence we prove $\phi(X_1) = X_1$.

Now we prove $\phi(E_{ii+m} + E_{i+mi} + D_n) = E_{ii+m} + E_{i+mi} + D_n$.

By Corollary 11, we already have $\phi(E_{ii+m} + E_{i+mi} + D_n) = \lambda_{ii+m} E_{ii+m} + \lambda_{ii+m}^{-1} E_{i+mi} + D_n$.

For arbitrary nonzero α with $2 + \alpha^2 \neq 0$, $(2\alpha^{-1} + \alpha)^{-1}(E_{ii+m} + E_{i+mi} + D_n) - (2\alpha^{-1} + \alpha)^{-1}(-\alpha^{-1}(E_{ii} + E_{ii+1} + E_{i+1i} + E_{i+1i+1}) - E_{i+1i+m} - E_{i+mi+1} - \alpha E_{i+mi+m} + D_n) = (2\alpha^{-1} + \alpha)^{-1}(\alpha^{-1}(E_{ii} + E_{ii+1} + E_{i+1i} + E_{i+1i+1}) + (E_{ii+m} + E_{i+1i+m}) + (E_{i+mi} + E_{i+mi+1}) + \alpha E_{i+mi+m})$ is idempotent.

After applying ϕ on the above matrices, we have $(2\alpha^{-1} + \alpha)^{-1}\phi(E_{ii+m} + E_{i+mi} + D_n) - (2\alpha^{-1} + \alpha)^{-1}\phi(-\alpha^{-1}(E_{ii} + E_{ii+1} + E_{i+1i} + E_{i+1i+1}) - E_{i+1i+m} - E_{i+mi+1} - \alpha E_{i+mi+m} + D_n) = (2\alpha^{-1} + \alpha)^{-1}(\alpha^{-1}(E_{ii} + E_{ii+1} + E_{i+1i} + E_{i+1i+1}) + (E_{ii+m} + E_{i+1i+m}) + (E_{i+mi} + E_{i+mi+1}) + \alpha E_{i+mi+m}) + (2\alpha^{-1} + \alpha)^{-1}((\lambda_{ii+m} - 1)E_{ii+m} + (\lambda_{ii+m}^{-1} - 1)E_{i+mi}) \in \Gamma_n$.

Then $\lambda_{ii+m} = 1$ by Lemma 5.

By the induction, we prove $\phi(E_{ij} + E_{ji} + D_n) = E_{ij} + E_{ji} + D_n$ for every i, j with $1 \leq i < j \leq n$.

(3) Suppose (25) is true for every s with $2 \leq s < m \leq n$; then we prove it holds on m .

For arbitrary $X \in S_n$ with $X[i, i] = x_i$ for every $i \in \langle n \rangle$, let A, B, U, V, y_{i_m} , and τ satisfy the following equations:

$$\begin{aligned} A &= X_{\{i_1, \dots, i_m\}} + \sum_{j \in \langle n \rangle \setminus \{i_1, \dots, i_m\}} x_j E_{jj}, \\ B &= X_{\{i_1, \dots, i_{m-1}\}} + \sum_{j \in \langle n \rangle \setminus \{i_1, \dots, i_{m-1}\}} x_j E_{jj}, \\ U &= X_{\{i_1, \dots, i_{m-1}, i_m\}}, \\ V &= X_{\{i_m, i_1, \dots, i_{m-1}\}}, \\ y_{i_m} &= (X_{\{i_m, i_1, \dots, i_{m-1}\}} X_{\{i_1, \dots, i_{m-1}, i_m\}}) [i_m, i_m], \\ \tau &= (\alpha^{-1} y_{i_m} + \alpha)^{-1}. \end{aligned} \quad (32)$$

Then $\tau A + \tau(-B + \alpha^{-1}UV + \alpha E_{i_m i_m})$ is idempotent for arbitrary nonzero α with $y_{i_m} + \alpha^2 \neq 0$. Applying ϕ on it, we have $\tau\phi(A) + \tau\phi(-B + \alpha^{-1}UV + \alpha E_{i_m i_m}) \in \Gamma_n$. Let $C = -B + \alpha^{-1}UV + \alpha E_{i_m i_m}$; then by $\tau A + \tau C + \epsilon \sum_{j \in \langle n \rangle \setminus \{i_1, \dots, i_m\}} E_{jj} \in S\Gamma_n$ for every $\epsilon \in \Lambda$, we have $\tau\phi(A) + \phi(\tau C + \epsilon \sum_{j \in \langle n \rangle \setminus \{i_1, \dots, i_m\}} E_{jj}) \in \Gamma_n$.

Note that $\phi(\tau C + \epsilon \sum_{j \in \langle n \rangle \setminus \{i_1, \dots, i_m\}} E_{jj}) = \tau C + \epsilon \sum_{j \in \langle n \rangle \setminus \{i_1, \dots, i_m\}} E_{jj}$ and $\phi(C) = C$ by the assumption; then $\tau\phi(A) + \tau\phi(C)$ and $\sum_{j \in \langle n \rangle \setminus \{i_1, \dots, i_m\}} E_{jj}$ are orthogonal by Corollary 10; that is, $\phi(A) = Y_{\{i_1, \dots, i_m\}} + \sum_{j \in \langle n \rangle \setminus \{i_1, \dots, i_m\}} x_j E_{jj}$ for some $Y \in M_n$.

On the other hand, $C = -(X_{\{i_1, \dots, i_{m-1}\}} + \sum_{j \in \langle n \rangle \setminus \{i_1, \dots, i_{m-1}\}} x_j E_{jj}) + \alpha^{-1}UV + \alpha E_{i_m i_m} = -(X_{\{i_1, \dots, i_m\}} +$

$\sum_{j \in \langle n \rangle \setminus \{i_1, \dots, i_m\}} x_j E_{jj}) + \alpha^{-1}UV + \alpha E_{i_m i_m} + U + V$ implies $\tau(Y_{\{i_1, \dots, i_m\}} - X_{\{i_1, \dots, i_m\}} + \alpha^{-1}UV + \alpha E_{i_m i_m} + U + V) = \tau(Y_{\{i_1, \dots, i_m\}} - X_{\{i_1, \dots, i_m\}}) + \tau(\alpha^{-1}UV + \alpha E_{i_m i_m} + U + V) \in \Gamma_n$ by $\tau\phi(A) + \tau\phi(C) \in \Gamma_n$. By Lemma 5, we can derive the following equations:

$$\begin{aligned} Y_{\{i_1, \dots, i_{m-1}\}} &= X_{\{i_1, \dots, i_{m-1}\}}, \\ Y[i_m, i_m] &= X[i_m, i_m], \\ Y_{\{i_1, \dots, i_{m-1}\}, \{i_m\}} &= \eta U, \\ Y_{\{i_m\}, \{i_1, \dots, i_{m-1}\}} &= \eta^{-1} V \end{aligned} \quad (33)$$

that is, $\phi(A) = X_{\{i_1, \dots, i_{m-1}\}} + \eta U + \eta^{-1} V + \sum_{j \in \langle n \rangle \setminus \{i_1, \dots, i_{m-1}\}} x_j E_{jj}$.

Let B_1 and B_2 satisfy the following equations:

$$\begin{aligned} B_1 &= X_{\{i_2, \dots, i_m\}} + \sum_{j \in \langle n \rangle \setminus \{i_2, \dots, i_m\}} x_j E_{jj}, \\ B_2 &= X_{\{i_1, \dots, i_{m-2}, i_m\}} + \sum_{j \in \langle n \rangle \setminus \{i_1, \dots, i_{m-2}, i_m\}} x_j E_{jj}; \end{aligned} \quad (34)$$

then we can prove

$$\begin{aligned} Y_{\{i_2, \dots, i_m\}} &= X_{\{i_2, \dots, i_m\}}, \\ Y[i_1, i_1] &= X[i_1, i_1], \\ Y_{\{i_2, \dots, i_m\}, \{i_1\}} &= \beta X_{\{i_2, \dots, i_m\}, \{i_1\}}, \\ Y_{\{i_1\}, \{i_2, \dots, i_m\}} &= \beta^{-1} X_{\{i_1\}, \{i_2, \dots, i_m\}}, \\ Y_{\{i_1, \dots, i_{m-2}, i_m\}} &= X_{\{i_1, \dots, i_{m-2}, i_m\}}, \\ Y[i_{m-1}, i_{m-1}] &= X[i_{m-1}, i_{m-1}], \\ Y_{\{i_1, \dots, i_{m-2}, i_m\}, \{i_{m-1}\}} &= \gamma X_{\{i_1, \dots, i_{m-2}, i_m\}, \{i_{m-1}\}}, \\ Y_{\{i_{m-1}\}, \{i_1, \dots, i_{m-2}, i_m\}} &= \gamma^{-1} X_{\{i_{m-1}\}, \{i_1, \dots, i_{m-2}, i_m\}}. \end{aligned} \quad (35)$$

Comparing the above three sets of equations, we can get $\phi(A) = A$, which is equivalent to (25) on m .

By the induction, we prove that $\phi(X) = X$ for arbitrary $X \in S_n$. $\square$

Remark 17. If ϕ is a weak k -potence preserver from M_n to M_n , then the proof in Steps 1, 2, and 3 of Proposition 16 holds, and we prove $\phi(X) = X$ or $\phi(X) = X^t$ in Step 4. We omit the detailed proof since the case on X^t is totally the same after changing relevant notations.

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