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Research Article

Measure Functional Differential Equations in the Space of Functions of Bounded Variation

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We establish general conditions for the unique solvability of nonlinear measure functional differential equations in terms of properties of suitable linear majorants.

1. Introduction, Motivation, and Problem Setting

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^n \ni x = (x_k)_{k=1}^n \mapsto \|x\| := \max_{1\le k\le n}|x_k|$ be the norm in $\mathbb{R}^n$, and let $\mathrm{BV}([a,b],\mathbb{R}^n)$ be the Banach space of functions of bounded variation with the standard norm $\mathrm{BV}([a,b],\mathbb{R}^n) \ni u \mapsto \|u\|_{\mathrm{BV}} := |u(a)| + \mathrm{Var}_{[a,b]}u$, where $-\infty < a < b < \infty$.

Our aim is to examine the solvability of the equation

$$u(t) = \varphi(u) + \int_a^t (fu)(s)\,dg(s), \quad t \in [a,b]; \tag{1}$$

$f : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$ is a, generally speaking, nonlinear operator and $\varphi : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ is a nonlinear vector functional. The integral on the right-hand side of (1) is the Kurzweil-Stieltjes integral with respect to a nondecreasing function $g : [a,b] \to \mathbb{R}$. We refer to [1–5] for the definition and properties of this kind of an integral, recalling only that (1) is a particular case of a generalised differential equation [2, 6]. It is important to note that, for any $u \in \mathrm{BV}([a,b],\mathbb{R}^n)$, the Kurzweil-Stieltjes integral in (1) exists (see, e.g., [4, 7]) and, therefore, the equation itself makes sense.

By a *solution* of (1), we mean a vector function $u : [a,b] \to \mathbb{R}^n$ which has bounded variation and satisfies (1) on the interval $[a,b]$.

Equation (1) is an extension of a measure differential equation studied systematically, for example, in [2, 8–10]. It is a fairly general object which includes many other types of equations such as differential equations with impulses [11] or functional dynamic equations on time scales [12] (see, e.g., [13, 14]). In particular, if $g(s) = s$, $s \in [a,b]$, (1) takes the form

$$u(t) = \varphi(u) + \int_a^t (fu)(s)\,ds, \quad t \in [a,b], \tag{2}$$

and, thus, in the absolutely continuous case, reduces to the nonlocal boundary value problem for a functional differential equation

$$u'(t) = (fu)(t), \quad t \in [a,b], \qquad u(a) = \varphi(u), \tag{3}$$

whose various particular types are the object of investigation of many authors (see, e.g., [15–19]). A more general choice of g in (1) allows one to cover further cases where solutions lose their absolute continuity at some points. For example, consider the impulsive functional differential equation [16, page 191]

$$\begin{aligned} u'(t) &= (fu)(t), \quad t \in [a,b] \setminus \{t_1, t_2, \ldots, t_m\}, \\ \Delta u(t) &= I_i(u(t)) \quad \text{for } t = t_i,\ i \in \{1, 2, \ldots, m\}, \end{aligned} \tag{4}$$

where $\Delta u(t) := u(t+) - u(t-)$ for any function u from $\mathrm{BV}([a,b],\mathbb{R}^n)$ (in fact, $\Delta u(t) = u(t+) - u(t)$ if, as is

customary [11] in that context, a solution is assumed to be left continuous). Here, $f = (f_k)_{k=1}^n : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, the jumps may occur at the preassigned times $t_1, t_2, \ldots, t_m$, and their action is described by the operators $I_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \ldots, m$. By the usual integration argument [11], one can represent (4) alternatively in the form

$$u(t) = u(a) + \int_a^t (fu)(s)\, ds + \sum_{k:a<t_k<t,} I_k(u(t_k)), \quad t \in [a,b]. \tag{5}$$

It follows, in particular, from [14, Lemma 2.4] that (5) is equivalent to the measure functional differential equation

$$u(t) = u(a) + \int_a^t (\tilde{f}u)(s)\, dg(s), \quad t \in [a,b], \tag{6}$$

with $g(s) = s + \sum_{i=1}^m \chi_{(t_i,b]}(s)$, $s \in [a,b]$, and $\tilde{f} : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$ defined by the relation

$$(\tilde{f}u)(s) = \begin{cases} (fu)(s) & \text{if } s \in [a,b] \setminus \{t_1, t_2, \ldots, t_m\}, \\ I_i(u(s)) & \text{if } s \in \{t_1, t_2, \ldots, t_m\}. \end{cases} \tag{7}$$

Thus, system (4) can be considered as a particular case of (1). Likewise, an appropriate construction [13, 20] allows one to regard differential equations on time scales [12] as measure differential equations. The same is true for equations involving functional components; in the case of a differential equation on a time scale with retarded argument, by choosing g suitably [13], one arrives at the equation

$$u(t) = u(a) + \int_a^t h(u_s, s)\, dg(s), \quad t \in [a,b], \qquad u_a = \phi. \tag{8}$$

In (8), $h : C([-r,0],\mathbb{R}^n) \times [a,b] \to \mathbb{R}^n$ is a functional in the first variable, ϕ is from the space $C([-r,0],\mathbb{R}^n)$ of continuous functions on $[-r,0]$, and the Krasovsky notation $u_t : [-r,0] \ni s \mapsto u(t+s)$, $r > 0$, is used [21, Chapter VI]. Finally, eliminating the initial function ϕ from (8) in a standard way by transforming it to a forcing term (see [15]), we conclude that the resulting equation falls into the class of equations of form (1).

Note that, by measure functional differential equations, the Volterra type equations of form (8) are usually meant in the existing bibliography on the subject (see, e.g., [8, 13, 22]), whereas equations with more general types of argument deviation are rather scarce (we can cite, perhaps, only [4, page 217]). Comparing (8) with (1), we find that the latter includes non-Volterra cases as well.

This list of examples can be continued. It is interesting to observe that solutions of problems of type (3) studied in the literature up to now are always assumed, at least locally, to be absolutely continuous [16], or even continuously differentiable [23]. In contrast to this, the gauge integral involved in (1) allows one to deal with a considerably wider class of solutions, which are, in fact, assumed to be of bounded variation only. A possible noteworthy consequence for systems with impulses may be that the unpleasant effect of the so-called *pulsation* phenomenon [11, page 5] might be more natural to be dealt with in the framework of the space $\mathrm{BV}([a,b],\mathbb{R}^n)$. Our interest in (1), originally motivated by a relation to problems of type (3), has strengthened still further due to the last observation.

The general character of the object represented by (1) suggests a natural idea to examine its solvability by comparing it to simpler linear equations with suitable properties. Here, we show that such statements can indeed be obtained rather easily by analogy to [24–26]. The key assumption is that certain linear operators associated with the nonlinear operator f appearing in (1) possess the following property.

Definition 1. Let $h : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ be a linear mapping. One says that a linear operator $p : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$ belongs to the set $\mathcal{S}_h([a,b],\mathbb{R}^n)$ if the equation

$$u(t) = h(u) + \int_a^t (pu)(s)\, dg(s) + r(t), \quad t \in [a,b] \tag{9}$$

has a unique solution u for any r from $\mathrm{BV}([a,b],\mathbb{R}^n)$, and, moreover, the solution u is nonnegative for any nonnegative r.

The property described by Definition 1, in fact, means that the linear operator associated with (9) is positively invertible on $\mathrm{BV}([a,b],\mathbb{R}^n)$, and thus it corresponds to the existence and positivity of Green's operator for a boundary value problem [15].

Remark 2. The inclusion $p \in \mathcal{S}_h([a,b],\mathbb{R}^n)$, generally speaking, does not imply that $\lambda p \in \mathcal{S}_h([a,b],\mathbb{R}^n)$ for $\lambda \neq 1$!

The question on the unique solvability of (1) is thus reduced to estimating the nonlinearities suitably, so that the appropriate majorants generate linear equations with a unique solution depending monotonously on forcing terms. The problem of finding such majorants is a separate topic not discussed here. We only note that, in a number of cases, the existing results on differential inequalities can be applied (see, e.g., [17–19]).

Note that, due to the nature of the techniques used, statements of this kind available in the literature on problems of type (3), as a rule, are established separately in every concrete case (see, e.g., [27–29]). Here, we provide a simple unified proof, which is, in a sense, independent on the character of the equation and also allows one to gain a considerable degree of generality. The results may be useful in studies of the solvability of various measure functional differential equations and, in particular, of problem (3) and its generalisations (note that, e.g., rather complicated neutral-type functional differential equations [23] can be formulated in form (1); see also [4, 30]).

2. Unique Solvability Conditions

We are going to show that the knowledge of the property $p \in \mathcal{S}_h([a,b],\mathbb{R}^n)$ for certain *linear* operators p and h associated with (1) allows one to guarantee its unique solvability.

2.1. Nonlinear Equations. The following statement is true.

Theorem 3. *Assume that there exist certain linear operators* $p_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $i = 1,2$, *such that*

$$\begin{aligned} p_2(u-v)(t) &\le (fu)(t) - (fv)(t) \\ &\le p_1(u-v)(t), \quad t \in [a,b], \end{aligned} \tag{10}$$

for arbitrary functions $u : [a,b] \to \mathbb{R}^n$, $v : [a,b] \to \mathbb{R}^n$ *with the property*

$$u(t) \ge v(t) \quad \forall t \in [a,b]. \tag{11}$$

Furthermore, let the inclusions

$$\begin{gathered} p_1 \in \mathcal{S}_{h_1}([a,b],\mathbb{R}^n), \\ \frac{1}{2}(p_1+p_2) \in \mathcal{S}_{(1/2)(h_1+h_2)}([a,b],\mathbb{R}^n) \end{gathered} \tag{12}$$

be fulfilled with some linear functionals $h_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}$, $i = 1,2$. *Then* (1) *has a unique solution for an arbitrary* φ *such that*

$$h_2(u-v) \le \varphi(u) - \varphi(v) \le h_1(u-v) \tag{13}$$

whenever (11) *holds.*

The inequality sign and modulus for vectors in (10), (11), (13), and similar relations below are understood componentwise. The theorem as well as the other statements formulated below will be proved later.

Theorem 4. *Let there exist certain linear operators* $l_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $i = 1,2$, *and linear functionals* $h_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}$, $i = 1,2$ *satisfying the inclusions*

$$\begin{gathered} l_1 + l_2 \in \mathcal{S}_{h_1}([a,b],\mathbb{R}^n), \\ l_1 \in \mathcal{S}_{(1/2)(h_1+h_2)}([a,b],\mathbb{R}^n), \end{gathered} \tag{14}$$

and such that (13) *and the inequality*

$$\begin{aligned} &\left|(fu)(t) - (fv)(t) - l_1(u-v)(t)\right| \\ &\quad \le l_2(u-v)(t), \quad t \in [a,b], \end{aligned} \tag{15}$$

is true for arbitrary functions u and v of bounded variation with property (11). *Then* (1) *is uniquely solvable.*

Theorem 4 is, in fact, an alternative form of Theorem 3, where the estimate of a "linear part" is more visible.

In other statements, we need the following natural notion of positivity of a linear operator in the space $\mathrm{BV}([a,b],\mathbb{R}^n)$.

Definition 5. A linear operator $q : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$ will be called *positive* if qu is a nonnegative function for an arbitrary nonnegative u from $\mathrm{BV}([a,b],\mathbb{R}^n)$.

Note that no monotonicity assumptions are imposed on l_1 in Theorem 4. In the cases where the positivity of certain linear majorants is known, the following statement may be of use.

Corollary 6. *Assume that there exist some positive linear operators* $q_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $i = 1,2$, *such that the inequalities*

$$\left|(fu)(t) - (fv)(t) + q_2(u-v)(t)\right| \le q_1(u-v)(t) \tag{16}$$

hold on $[a,b]$ *for any u and v from* $\mathrm{BV}([a,b],\mathbb{R}^n)$ *with property* (11). *Moreover, let one can specify linear functionals* $h_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}$, $i = 1,2$, *satisfying* (13), *and such that the inclusions*

$$\begin{gathered} q_1 + (1-2\theta)q_2 \in \mathcal{S}_{h_1}([a,b],\mathbb{R}^n), \\ -\theta q_2 \in \mathcal{S}_{(1/2)(h_1+h_2)}([a,b],\mathbb{R}^n) \end{gathered} \tag{17}$$

hold for a certain $\theta \in (0,1)$. *Then* (1) *has a unique solution.*

Corollary 6 allows one to obtain, in particular, the following statements.

Corollary 7. *Assume that, for arbitrary u and v from* $\mathrm{BV}([a,b],\mathbb{R}^n)$ *with property* (11), *f and* φ *satisfy estimates* (13) *and* (16) *with some linear functionals* $h_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}$, $i = 1,2$ *and positive linear operators* $q_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $i = 1,2$. *Then the inclusions*

$$q_1 \in \mathcal{S}_{h_1}([a,b],\mathbb{R}^n), \qquad -\frac{1}{2}q_2 \in \mathcal{S}_{(1/2)(h_1+h_2)}([a,b],\mathbb{R}^n) \tag{18}$$

guarantee that (1) *is uniquely solvable.*

Corollary 8. *The assertion of Corollary 7 is true with* (18) *replaced by the condition*

$$\begin{gathered} q_1 + \frac{1}{2}q_2 \in \mathcal{S}_{h_1}([a,b],\mathbb{R}^n), \\ -\frac{1}{4}q_2 \in \mathcal{S}_{(1/2)(h_1+h_2)}([a,b],\mathbb{R}^n). \end{gathered} \tag{19}$$

The statements formulated above express fairly general properties of (1) and extend, in particular, the corresponding results of [25, 27, 29, 31].

2.2. Linear Equations. Let us now assume that $f : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$ in (1) is an affine mapping, and, therefore, (1) has the form

$$u(t) = h(u) + \int_a^t (lu)(s)\,dg(s) + y(t), \quad t \in [a,b], \tag{20}$$

where $l : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$ and $h : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ are linear, and $y \in \mathrm{BV}([a,b],\mathbb{R}^n)$ is a given function.

Corollary 9. *Assume that there exist certain linear operators* $p_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $i = 0,1$, *and a linear mapping* $h : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ *such that the inclusions*

$$p_1 \in \mathcal{S}_h([a,b],\mathbb{R}^n), \qquad p_0 + p_1 \in \mathcal{S}_h([a,b],\mathbb{R}^n) \tag{21}$$

hold, and the estimate

$$|(lu)(t) - (p_1 u)(t)| \le (p_0 u)(t), \quad t \in [a,b] \tag{22}$$

is satisfied for any nonnegative $u \in \mathrm{BV}([a,b],\mathbb{R}^n)$. *Then* (20) *has a unique solution.*

We also have the following.

Corollary 10. *Let there exist positive linear operators* $q_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $i = 0,1$, *satisfying the inclusions*

$$q_1 \in \mathcal{S}_h([a,b],\mathbb{R}^n), \qquad -\frac{1}{2}q_2 \in \mathcal{S}_h([a,b],\mathbb{R}^n), \tag{23}$$

and such that the inequalities

$$|(lu)(t) + (q_2 u)(t)| \le (q_1 u)(t), \quad t \in [a,b] \tag{24}$$

are true for an arbitrary nonnegative function $u : [a,b] \to \mathbb{R}^n$ *of bounded variation. Then* (20) *has a unique solution for any* $y \in \mathrm{BV}([a,b],\mathbb{R}^n)$.

We conclude this note by considering the case where l in (20) is a linear mapping admitting a decomposition into the sum of its positive and negative parts; that is,

$$l = l_0 - l_1, \tag{25}$$

where $l_k : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $k = 0,1$, are linear and positive. In that case, for the equation of the form

$$u(t) = h(u) + \int_a^t [(l_0 u)(s) - (l_1 u)(s)]\, dg(s) + y(t), \quad t \in [a,b], \tag{26}$$

where $h : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ is linear and $y \in \mathrm{BV}([a,b],\mathbb{R}^n)$, the following result is obtained.

Corollary 11. *Let the linear vector functional* $h : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ *and the linear positive operators* $l_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $i = 1,2$, *be such that the inclusions*

$$l_0 \in \mathcal{S}_h([a,b],\mathbb{R}^n), \qquad \frac{1}{2}(l_0 - l_1) \in \mathcal{S}_h([a,b],\mathbb{R}^n) \tag{27}$$

are satisfied. Then (26) *has a unique solution for any* $y \in \mathrm{BV}([a,b],\mathbb{R}^n)$.

It is interesting to observe the second condition in (27); it thus turns out that property $\mathcal{S}_h([a,b],\mathbb{R}^n)$ for *one half* of the operator under the integral sign in (26) ensures the unique solvability of the *original* equation (26).

3. Proofs

Let $\langle E, \|\cdot\|\rangle$ be real Banach space, let $z \in E$ be a given vector, and let $F : E \to E$ be a mapping. Let $K_i \subset E$, $i = 1,2$, be closed cones inducing the corresponding partial orderings $\leqq_{K_i}$, so that $x \leqq_{K_i} y$ if and only if $y - x \in K_i$. The following statement [32, 33] on the abstract equation

$$Fu = z \tag{28}$$

will be used below.

Theorem 12 (see [33], Theorem 49.4). *Let the cone* K_2 *be normal and generating. Furthermore, let* $B_k : E \to E$, $k = 1,2$, *be linear operators such that* B_1^{-1} *and* $(B_1 + B_2)^{-1}$ *exist and possess the properties*

$$B_1^{-1}(K_2) \subset K_1, \qquad (B_1 + B_2)^{-1}(K_2) \subset K_1, \tag{29}$$

and, furthermore, let the order relation

$$B_1(x - y) \leqq_{K_2} Fx - Fy \leqq_{K_2} B_2(x - y) \tag{30}$$

be satisfied for any pair (x, y) *such that* $y \leqq_{K_1} x$. *Then* (28) *has a unique solution for an arbitrary element* $z \in E$.

Recall that K_2 is normal if all the sets order bounded with respect to $\leqq_{K_2}$ are also norm bounded and that K_1 is generating if and only if $\{u - v \mid \{u, v\} \subset K_1\} = E$ (see, e.g., [33, 34]).

Let $\mathrm{BV}^+([a,b],\mathbb{R}^n)$ (resp., $\mathrm{BV}^{++}([a,b],\mathbb{R}^n)$) be the set of all the nonnegative (resp., nonnegative and nondecreasing) functions from $\mathrm{BV}([a,b],\mathbb{R}^n)$.

Lemma 13. *(1) The set* $\mathrm{BV}^+([a,b],\mathbb{R}^n)$ *is a cone in the space* $\mathrm{BV}([a,b],\mathbb{R}^n)$.

(2) The set $\mathrm{BV}^{++}([a,b],\mathbb{R}^n)$ *is a normal and generating cone in* $\mathrm{BV}([a,b],\mathbb{R}^n)$.

Proof. The first assertion of the lemma being obvious, only the second one should be verified.

It follows directly from the definition of the set $\mathrm{BV}^{++}([a,b],\mathbb{R}^n)$ that it is a cone in $\mathrm{BV}([a,b],\mathbb{R}^n)$, which is also generating due to the Jordan decomposition of a function of bounded variation (see, e.g., [3]). In order to verify its normality, it will be sufficient to show [32, Theorem 4.1] that the set

$$A(\alpha,\beta) := \{x \in \mathrm{BV}([a,b],\mathbb{R}^n) : \{x - \alpha, \beta - x\} \subset \mathrm{BV}^{++}([a,b],\mathbb{R}^n)\} \tag{31}$$

is bounded for any $\{\alpha,\beta\} \subset \mathrm{BV}([a,b],\mathbb{R}^n)$. Indeed, if $x \in A(\alpha,\beta)$, then the functions $x - \alpha$ and $\beta - x$ are both nonnegative and nondecreasing. Therefore,

$$\operatorname*{Var}_{[a,b]}(x - \alpha) = \alpha(a) - \alpha(b) + x(b) - x(a), \tag{32}$$

and, hence,

$$\begin{aligned}\|x\|_{\mathrm{BV}} &\le \|\alpha\|_{\mathrm{BV}} + \|x-\alpha\|_{\mathrm{BV}} \\ &= \|\alpha\|_{\mathrm{BV}} + |x(a)-\alpha(a)| + \operatorname*{Var}_{[a,b]}(x-\alpha) \\ &= \|\alpha\|_{\mathrm{BV}} + x(b)-\alpha(b) \le \|\alpha\|_{\mathrm{BV}} + \beta(b)-\alpha(b).\end{aligned} \tag{33}$$

The last estimate shows that the norms of all such x are uniformly bounded. □

Let $p : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$ be a linear operator and $h : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}$ a linear functional. Let us put

$$V_{p,h}u := u - \int_a^{\cdot} (pu)(\xi)\, dg(\xi) - h(u) \tag{34}$$

for any u from $\mathrm{BV}([a,b],\mathbb{R}^n)$. It follows immediately from Definition 1 that the linear operator $V_{p,h} : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$ defined by (34) has the following property.

Lemma 14. *If p is a linear operator such that*

$$p \in \mathcal{S}_h([a,b],\mathbb{R}^n), \tag{35}$$

then $V_{p,h}$ is invertible and, moreover, its inverse $V_{p,h}^{-1}$ satisfies the inclusion

$$V_{p,h}^{-1}(\mathrm{BV}^{++}([a,b],\mathbb{R}^n)) \subset \mathrm{BV}^{+}([a,b],\mathbb{R}^n). \tag{36}$$

We will also use the obvious identity

$$V_{p_1,h_1} + V_{p_2,h_2} = 2V_{(1/2)(p_1+p_2),(1/2)(h_1+h_2)}, \tag{37}$$

which is valid for any linear $p_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $i = 1,2$.

3.1. Proof of Theorem 3. Let us set $E = \mathrm{BV}([a,b],\mathbb{R}^n)$ and put

$$(Fu)(t) := u(t) - \int_a^t (fu)(s)\, dg(s) - \varphi(u), \quad t \in [a,b], \tag{38}$$

for any u from $\mathrm{BV}([a,b],\mathbb{R}^n)$. Then (1) takes the form of (28) with $z = 0$. Since fu and g are both from $\mathrm{BV}([a,b],\mathbb{R}^n)$, it follows (see, e.g., [30]) that the function

$$[a,b] \ni t \longmapsto \int_a^t (fu)(s)\, dg(s) \tag{39}$$

also belongs to $\mathrm{BV}([a,b],\mathbb{R}^n)$. Therefore, F given by (38) is an operator acting in E.

Note that relation (10) is equivalent to inequalities

$$-p_1(u-v)(t) \le -(fu)(t) + (fv)(t) \le -p_2(u-v)(t), \tag{40}$$

for any $t \in [a,b]$ and $\{u,v\}$ from $\mathrm{BV}([a,b],\mathbb{R}^n)$ with properties (11). Integrating (40) with respect to g, we obtain

$$\begin{aligned}-\int_a^t p_1(u-v)(s)\, dg(s) &\le -\int_a^t (fu)(s)\, dg(s) \\ &\quad + \int_a^t (fv)(s)\, dg(s) \\ &\le -\int_a^t p_2(u-v)(s)\, dg(s),\end{aligned} \tag{41}$$

and, therefore, according to (38),

$$\begin{aligned}&u(t) - v(t) - \int_a^t p_1(u-v)(s)\, dg(s) - \varphi(u) + \varphi(v) \\ &\quad \le (Fu)(t) - (Fv)(t) \\ &\quad \le u(t) - v(t) - \int_a^t p_2(u-v)(s)\, dg(s) - \varphi(u) + \varphi(v),\end{aligned} \tag{42}$$

for all $t \in [a,b]$. Taking assumption (13) into account and using notation (34), we get

$$V_{p_1,h_1}(u-v)(t) \le (Fu)(t) - (Fv)(t) \le V_{p_2,h_2}(u-v)(t), \tag{43}$$

for all $t \in [a,b]$ and u and v from $\mathrm{BV}([a,b],\mathbb{R}^n)$ with properties (11). Furthermore, it follows immediately from (34) and (38) that, for any $t \in [a,b]$,

$$\begin{aligned}&(Fu)(t) - (Fv)(t) - V_{p_1,h_1}(u-v)(t) \\ &\quad = \varphi(v) - \varphi(u) \\ &\qquad + \int_a^t [p_1(u-v)(s) - (fu)(s) + (fv)(s)]\, dg(s).\end{aligned} \tag{44}$$

Therefore, by virtue of inequality (43) and assumption (10), the function $Fu - Fv - V_{p_1,h_1}(u-v)$ is nonnegative and nondecreasing and, hence,

$$Fu - Fv - V_{p_1,h_1}(u-v) \in \mathrm{BV}^{++}([a,b],\mathbb{R}^n). \tag{45}$$

In the same manner, one shows that

$$V_{p_2,h_2}(u-v) - Fu + Fv \in \mathrm{BV}^{++}([a,b],\mathbb{R}^n). \tag{46}$$

Considering (45) and (46), we conclude that F satisfies condition (30) with

$$B_i = V_{p_i,h_i}, \tag{47}$$

$i = 1,2$, and

$$\begin{aligned}K_1 &= \mathrm{BV}^{+}([a,b],\mathbb{R}^n), \\ K_2 &= \mathrm{BV}^{++}([a,b],\mathbb{R}^n).\end{aligned} \tag{48}$$

By virtue of Lemma 13, K_2 is a normal and generating cone in $\mathrm{BV}([a,b],\mathbb{R}^n)$.

Since, by assumption (12), $p_1 \in \mathcal{S}_{h_1}$, it follows that V_{p_1,h_1} is invertible and the inclusion

$$V_{p_1,h_1}^{-1}(K_2) \subset K_1 \tag{49}$$

holds. Furthermore, by (12) and Lemma 14, the operator $(1/2)V_{(1/2)(p_1+p_2),(1/2)(h_1+h_2)}^{-1}$ exists and coincides with the inverse operator to $V_{p_1,h_1} + V_{p_2,h_2}$. It is moreover positive in the sense that

$$\left(V_{p_1,h_1} + V_{p_2,h_2}\right)^{-1}(K_2) \subset K_1. \tag{50}$$

Combining (49) and (50), we see that the inverse operators B^{-1} and $(B_1 + B_2)^{-1}$ exist and possess properties (29) with respect to cones (48). Applying now Theorem 12, we prove the unique solvability of (28) and, hence, that of (1).

3.2. Proof of Theorem 4. Rewriting relations (15) in the form

$$\begin{aligned} &l_1(u-v)(t) - l_2(u-v)(t) \\ &\quad \le (fu)(t) - (fv)(t) \\ &\quad \le l_2(u-v)(t) + l_1(u-v)(t), \quad t \in [a,b], \end{aligned} \tag{51}$$

and putting

$$p_i := l_1 - (-1)^i l_2, \quad i = 1,2, \tag{52}$$

we find that f admits estimate (10) with p_1 and p_2 defined by (52). Therefore, it remains only to note that assumption (14) ensures the validity of inclusions (12), and to apply Theorem 3.

3.3. Proof of Corollary 6. It turns out that, under assumptions (16) and (17), the operators $l_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $i = 1,2$, defined by the formulae

$$l_1 := -\theta q_2, \qquad l_2 := q_1 + (1-\theta) q_2 \tag{53}$$

with $\theta \in (0,1)$, satisfy conditions (14) and (15) of Theorem 4. Indeed, estimate (16) and the positivity of the operator q_2 imply that, for any u and v with properties (11) and all $t \in [a,b]$, the relations

$$\begin{aligned} &\left|(fu)(t) - (fv)(t) + \theta q_2(u-v)(t)\right| \\ &\quad = \left|(fu)(t) - (fv)(t) + q_2(u-v)(t) \right. \\ &\qquad \left. - (1-\theta) q_2(u-v)(t)\right| \\ &\quad \le q_1(u-v)(t) + \left|(1-\theta) q_2(u-v)(t)\right| \\ &\quad = q_1(u-v)(t) + (1-\theta) q_2(u-v)(t) \end{aligned} \tag{54}$$

are true. This means that f admits estimate (15) with the operators l_1 and l_2 of form (53). It is easy to verify that assumption (17) ensures the validity of inclusions (14) for operators (53), and, therefore, Theorem 4 can be applied.

3.4. Proof of Corollaries 7 and 8. The results follow directly from Corollary 6 if one puts $\theta = (1/2)$ and $\theta = (1/4)$, respectively.

3.5. Proof of Corollary 9. If $y = 0$, one should apply Theorem 4 with $f = l$, $l_1 = p_1$, $l_2 = p_0$, and $h_1 = h$, $h_2 = h$. For a nonzero $y \in \mathrm{BV}([a,b],\mathbb{R}^n)$, one can modify the theorem slightly by incorporating the forcing term y directly into (1) similarly to (20). Then we find that the argument of Section 3.1 remains almost unchanged.

3.6. Proof of Corollary 10. Corollary 7 with $f = l$, $h_1 = h$, and $h_2 = h$ is applied.

3.7. Proof of Corollary 11. It is sufficient to note that, under these assumptions, the linear operators $p_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $i = 1,2$, defined by the formulae

$$p_0 := \frac{1}{2}(l_0 + l_1), \qquad p_1 := \frac{1}{2}(l_0 - l_1), \tag{55}$$

satisfy conditions (21) and (22) of Corollary 9.

4. Comments

The following can be pointed out in relation to the above said.

4.1. Remark on Constants. The conditions presented in Sections 2.1 and 2.2 are, in a sense, optimal and cannot be improved. For example, it follows from [26] that assumption (14) of Corollary 7 can be replaced neither by the condition

$$(1-\varepsilon) l_1 \in \mathcal{S}_h([a,b],\mathbb{R}^n), \qquad l_0 + l_1 \in \mathcal{S}_h([a,b],\mathbb{R}^n) \tag{56}$$

nor by the condition

$$l_1 \in \mathcal{S}_h([a,b],\mathbb{R}^n), \qquad (1-\varepsilon)(l_0 + l_1) \in \mathcal{S}_h([a,b],\mathbb{R}^n), \tag{57}$$

no matter how small $\varepsilon \in (0,\infty)$ may be. Likewise, counterexamples show that the assertion of Corollary 11 is not true any more if condition (27) is replaced by either of its weaker versions

$$\begin{aligned} &(1-\varepsilon) l_0 \in \mathcal{S}_h([a,b],\mathbb{R}^n), \\ &\frac{1}{2}(l_0 - l_1) \in \mathcal{S}_h([a,b],\mathbb{R}^n), \end{aligned} \tag{58}$$

and

$$\begin{aligned} &l_0 \in \mathcal{S}_h([a,b],\mathbb{R}^n), \\ &\frac{1}{2+\varepsilon}(l_0 - l_1) \in \mathcal{S}_h([a,b],\mathbb{R}^n) \end{aligned} \tag{59}$$

with a positive ε. The same holds for the other inequalities and constants.

4.2. Equations with Matrix-Valued Functions. It is clear from the proofs given above that similar statements can also be obtained in the case where the integrals of matrix-valued functions are considered in (1), as described, for example, in [3, 4].

4.3. The Case of a Nonmonotone Measure. Results similar to those stated above can also be formulated in the case where the function g involved in (1) is of bounded variation only and not necessarily nondecreasing. For this purpose, one should use the representation

$$g = g_1 - g_2, \tag{60}$$

where g_k, $k = 1, 2$, are nondecreasing functions, and modify the definition of the set $\mathcal{S}_h([a,b],\mathbb{R}^n)$ in the following way.

Definition 15. A pair of operators (q_1, q_2) is said to belong to $\mathcal{S}_h([a,b],\mathbb{R}^n)$ if the equation

$$\begin{aligned} u(t) = h(u) &+ \int_a^t (q_1 u)(s)\, dg_1(s) \\ &- \int_a^t (q_2 u)(s)\, dg_2(s) + r(t), \quad t \in [a,b], \end{aligned} \tag{61}$$

has a unique solution u for any r from $\mathrm{BV}([a,b],\mathbb{R}^n)$ and, moreover, the solution u is nonnegative for nonnegative r.

In that case, an analogue of the assertion of Theorem 3 is obtained if assumption (12) is replaced by the pair of conditions

$$\begin{gathered} (p_1, p_2) \in \mathcal{S}_{h_1}([a,b],\mathbb{R}^n), \\ \left(\frac{1}{2}(p_1+p_2), \frac{1}{2}(p_1+p_2)\right) \in \mathcal{S}_{(1/2)(h_1+h_2)}([a,b],\mathbb{R}^n). \end{gathered} \tag{62}$$

The proof of this fact is pretty similar to the argument given in Section 3.1 and uses Theorem 12 with the operators $B_k : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $k = 1, 2$,

$$\begin{aligned} (B_k u)(t) := u(t) &- \int_a^t (p_k u)(s)\, dg_1(s) \\ &+ \int_a^t (p_{3-k} u)(s)\, dg_2(s) - h_k(u), \quad t \in [a,b], \end{aligned} \tag{63}$$

instead of those defined by (47).

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