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Research Article

A New Integro-Differential Equation for Rossby Solitary Waves with Topography Effect in Deep Rotational Fluids

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From rotational potential vorticity-conserved equation with topography effect and dissipation effect, with the help of the multiple-scale method, a new integro-differential equation is constructed to describe the Rossby solitary waves in deep rotational fluids. By analyzing the equation, some conservation laws associated with Rossby solitary waves are derived. Finally, by seeking the numerical solutions of the equation with the pseudospectral method, by virtue of waterfall plots, the effect of detuning parameter and dissipation on Rossby solitary waves generated by topography are discussed, and the equation is compared with KdV equation and BO equation. The results show that the detuning parameter α plays an important role for the evolution features of solitary waves generated by topography, especially in the resonant case; a large amplitude nonstationary disturbance is generated in the forcing region. This condition may explain the blocking phenomenon which exists in the atmosphere and ocean and generated by topographic forcing.

1. Introduction

Among the many wave motions that occur in the ocean and atmosphere, Rossby waves play one of the most important roles. They are largely responsible for determining the ocean's response to atmospheric and other climate changes [1]. In the past decades, the research on nonlinear Rossby solitary waves had been given much attention in the mathematics and physics, and some models had been constructed to describe this phenomenon. Based upon the pioneering work of Long [2] and Benney [3] on barotropic Rossby waves, there had been remarkably exciting developments [4–11] and formed classical solitary waves theory and algebraic solitary waves theory. The so-called classical solitary waves indicate that the evolution of solitary waves is governed by the Korteweg-de Vries (KdV) type model, while the behavior of solitary waves is governed by the Benjamin-Ono (BO) model, it is called algebraic solitary waves. After the KdV model and BO model, a more general evolution model for solitary waves in a finite-depth fluid was given by Kubota, and the model was called intermediate long-wave (ILW) model [12, 13]. Many mathematicians solved the above models by all kinds of method and got a series of results [14–19]. We note that most of the previous researches about solitary waves were carried out in the zonal area and could not be applied directly to the spherical earth, and little attention had been focused on the solitary waves in the rotational fluids [20]. Furthermore, as everyone knows the real oceanic and atmospheric motion is a forced and dissipative system. Topography effect as a forcing factor has been studied by many researchers [21–25]; on the other hand, dissipation effect must be considered in the oceanic and atmospheric motion; otherwise, the motion would grow explosively because of the constant injecting of the external forcing energy. Our aim is to construct a new model to describe the Rossby solitary waves in rotational fluid with topography effect and dissipation effect. It has great difference from the previous researches.

In this paper, from rotational potential vorticity-conserved equation with topography effect and dissipation effect,

with the help of the multiple-scale method, we will first construct a new model to describe Rossby solitary waves in deep rotational fluids. Then we will analyse the conservation relations of the model and derive the conservation laws of Rossby solitary waves. Finally, the model is solved by the pseudospectral method [26]. Based on the waterfall plots, the effect of detuning parameter and dissipation on Rossby solitary waves generated by topography are discussed, the model is compared with KdV model and BO model, and some conclusions are obtained.

2. Mathematics Model

According to [27], taking plane polar coordinates (r,θ), r pointing to lower latitude is positive and the positive rotation is counter-clockwise, and then the rotational potential vorticity-conserved equation including topography effect and turbulent dissipation is, in the nondimensional form, given by

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+\left(\frac{1}{r}\frac{\partial\Psi}{\partial r}\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\frac{\partial}{\partial r}\right)\right]\\&\quad\times\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}+h(r,\theta)\right]+\frac{\beta}{r}\frac{\partial\Psi}{\partial\theta}\\&=-\lambda_0\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}\right]+Q,\end{aligned}\tag{1}$$

where Ψ is the dimensionless stream function; $\beta = (\omega_0/R_0)\cos\phi_0(L^2/U)$, in which R_0 is the Earth's radius, ω_0 is the angular frequency of the Earth's rotation, ϕ_0 is the latitude, L and U are the characteristic horizontal length and velocity scales, $h(r,\theta)$ expresses the topography effect, $\lambda_0[(1/r)(\partial/\partial r)(r(\partial\Psi/\partial r))+(1/r^2)(\partial^2\Psi/\partial\theta^2)]$ denotes the vorticity dissipation which is caused by the Ekman boundary layer and λ_0 is a dissipative coefficient, Q is the external source, and the form of Q will be given in the latter.

In order to consider weakly nonlinear perturbation on a rotational flow, we assume

$$\Psi=\int^r(\Omega(r)-c+\varepsilon\alpha)\,r\,dr+\varepsilon\psi(r,\theta,t),\tag{2}$$

where α is a small disturbance in the basic flow and reflects the proximity of the system to a resonate state; c is a constant, which is regarded as a Rossby waves phase speed; ψ denotes disturbance stream function; $\Omega(r)$ expresses the rotational angular velocity. In order to consider the role of nonlinearity, we assume the following type of rotational angular velocity:

$$\Omega(r)=\begin{cases}\omega(r) & r_1\le r\le r_2,\\ \omega_1 & r>r_2,\end{cases}\tag{3}$$

where ω_1 is constant and $\omega(r)$ is a function of r. For simplicity, $\omega(r)$ is assumed to be smooth across $r=r_2$.

In the domain $[r_1,r_2]$, in order to achieve a balance among topography effect, turbulent dissipation, and nonlinearity and to eliminate the derivative term of dissipation, we assume

$$\begin{aligned}&h(r,\theta)=\varepsilon^2H(r,\theta),\qquad\lambda_0=\varepsilon^{3/2}\lambda,\\&Q=\varepsilon^{3/2}\lambda\frac{1}{r}\frac{\partial}{\partial r}\left[r^2(\omega-c_0+\varepsilon\alpha)\right].\end{aligned}\tag{4}$$

Substituting (2), (3), and (4) into (1) leads to the following equation for the perturbation stream function ψ:

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+(\omega-c+\varepsilon\alpha)\frac{\partial}{\partial\theta}+\varepsilon\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r}\right)\right]\\&\quad\times\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\varepsilon H(r,\theta)\right]\\&\quad+\frac{1}{r}\left[\beta-\frac{d}{dr}\left(\frac{1}{r}\frac{dr^2\omega}{dr}\right)\right]\frac{\partial\psi}{\partial\theta}\\&=-\varepsilon^{3/2}\lambda\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right].\end{aligned}\tag{5}$$

In the domain $[r_2,\infty]$, the parameter β is smaller than that in the domain $[r_1,r_2]$, and we assume $\beta=0$ for $[r_2,\infty]$. Furthermore, the turbulent dissipation and topography effect are absent in the domain and only consider the features of disturbances generated. Substituting (2) and (3) into (1), we have the following governing equations:

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+(\omega_1-c+\varepsilon\alpha)\frac{\partial}{\partial\theta}+\varepsilon\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r}\right)\right]\\&\quad\times\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right]=0.\end{aligned}\tag{6}$$

For (5), we introduce the following stretching transformations:

$$\Theta=\varepsilon^{1/2}\theta,\qquad r=r,\qquad T=\varepsilon^{3/2}t,\tag{7}$$

and the perturbation expansion of $\overline{\psi}$ is in the following form:

$$\overline{\psi}=\psi_1(\Theta,r,T)+\varepsilon\psi_2(\Theta,r,T)+\cdots.\tag{8}$$

Substituting (7) and (8) into (5), comparing the same power of ε term, we can obtain the $\varepsilon^{1/2}$ equation:

$$\mathscr{L}\psi_1=0,\tag{9}$$

where the operator $\mathscr{L}$ is defined as

$$\mathscr{L}=\frac{1}{r}\frac{\partial}{\partial\Theta}\left\{(\omega-c)\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]+\left[\beta-\frac{d}{dr}\left(\frac{1}{r}\frac{dr^2\omega}{dr}\right)\right]\right\}.\tag{10}$$

Assume the perturbation at boundary $r=r_1$ does not exist, that is,

$$\psi_1=\psi_2=\cdots=0,\tag{11}$$

and the perturbation at boundary $r=r_2$ is determined by (6). For the linear solution to be separable, assuming the solution of (9) in the form:

$$\psi_1=A(\Theta,T)\,\phi(r),\tag{12}$$

thus $\phi(r)$ should satisfy the following equation:

$$(\omega-c)\frac{d}{dr}\left(r\frac{d\phi(r)}{dr}\right)+\left[\beta-\frac{d}{dr}\left(\frac{1}{r}\frac{dr^2\omega}{dr}\right)\right]\phi(r)=0.\tag{13}$$

On the other hand, we proceed to the $\varepsilon^{3/2}$ equation:

$$\begin{aligned}&\mathscr{L}\psi_2+\frac{\omega-c}{r^2}\frac{\partial^3\psi_1}{\partial\Theta^3}\\&\quad+\left(\frac{\partial}{\partial T}+\alpha\frac{\partial}{\partial\Theta}+\frac{1}{r}\frac{\partial\psi_1}{\partial r}\frac{\partial}{\partial\Theta}-\frac{1}{r}\frac{\partial\psi_1}{\partial\Theta}\frac{\partial}{\partial r}+\lambda\right)\\&\quad\times\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_1}{\partial r}\right)\right]+(\omega-c)\frac{\partial H(r,\Theta)}{\partial\Theta}=0.\end{aligned}\tag{14}$$

Multiplying the both sides of (14) by $r\phi/(\omega-c)$ and integrating it with respect to r from r_1 to r_2, employing the boundary conditions (11), we get

$$\begin{aligned}&\frac{\partial}{\partial\Theta}r\left(\phi\frac{\partial}{\partial r}\psi_2-\frac{d\phi}{dr}\psi_2\right)\Big|_{r=r_2}\\&\quad+A\frac{\partial A}{\partial\Theta}\int_{r_1}^{r_2}\frac{\phi^3}{\omega-c}\frac{d}{dr}\\&\qquad\times\left[\frac{\beta-(d/dr)\left((1/r)\left(dr^2\omega/dr\right)\right)}{\phi(\omega-c)}\right]dr\\&\quad+\left(\frac{\partial A}{\partial T}+\alpha\frac{\partial A}{\partial\Theta}+\lambda A\right)\int_{r_1}^{r_2}\frac{\phi}{\omega-c}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)dr\\&\quad+\frac{\partial^3 A}{\partial\Theta^3}\int_{r_1}^{r_2}\frac{\phi^2}{r}dr+\int_{r_1}^{r_2}r\phi\frac{\partial H}{\partial\Theta}dr=0.\end{aligned}\tag{15}$$

In (15), if the boundary conditions on ϕ and ψ_2 are known, the equation governing the amplitude A will be determined. Assuming the solution of (5) matches smoothly with the solution of (6) at $r=r_2$, we can solve (6) to seek the solution at $r=r_2$.

For (6), we adopt the transformations in the forms:

$$\rho=\theta,\qquad r=r,\qquad T=\varepsilon^{3/2}t,\tag{16}$$

and the perturbation function is shown $\widetilde{\psi}$; then by substituting (16) into (6), we can get the ε^0 equation:

$$(\omega_1-c)\frac{\partial}{\partial\rho}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\widetilde{\psi}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\widetilde{\psi}}{\partial\rho^2}\right]=0.\tag{17}$$

It is easy to find that (17) can reduce to

$$\begin{aligned}&\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\widetilde{\psi}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\widetilde{\psi}}{\partial\rho^2}=0,\quad r\geq r_2,\\&\qquad\widetilde{\psi}\longrightarrow 0,\quad r\longrightarrow\infty.\end{aligned}\tag{18}$$

Obviously, the solution of (18) is

$$\widetilde{\psi}(\rho,r,T,\varepsilon)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\left(r^2-r_2^2\right)\widetilde{\psi}\left(\rho',r_2,T,\varepsilon\right)}{r_2^2-2r_2r\cos\left(\rho-\rho'\right)+r^2}d\rho'.\tag{19}$$

Taking the derivative with respect to r for both sides of (19) leads to

$$\begin{aligned}&\frac{\partial\widetilde{\psi}}{\partial r}\\&\quad=\frac{r_2}{\pi}\int_0^{2\pi}\widetilde{\psi}\left(\rho',r_2,T,\varepsilon\right)\frac{\left[2r_2r-\left(r^2+r_2^2\right)\cos\left(\rho-\rho'\right)\right]}{\left[r_2^2-2r_2r\cos\left(\rho-\rho'\right)+r^2\right]^2}d\rho'.\end{aligned}\tag{20}$$

Because the solution of (5) matches smoothly with the solution of (6) at $r=r_2$, we obtain

$$\psi_1(\Theta,r_2,T)+\varepsilon\psi_2(\Theta,r_2,T)=\widetilde{\psi}(\rho,r_2,T,\varepsilon)+O\left(\varepsilon^2\right),\tag{21}$$

$$\frac{\partial\psi_1}{\partial r}(\Theta,r_2,T)+\varepsilon\frac{\partial\psi_2}{\partial r}(\Theta,r_2,T)=\frac{\partial\widetilde{\psi}}{\partial r}(\rho,r_2,T,\varepsilon)+O\left(\varepsilon^2\right).\tag{22}$$

From (21), we have

$$A(\Theta,T)\phi(r_2)=\widetilde{\psi}(\rho,r_2,T,\varepsilon),\qquad\psi_2(\Theta,r_2,T)=0.\tag{23}$$

Substituting (23) into (20) leads to

$$\frac{\partial\widetilde{\psi}}{\partial r}(\rho,r_2,T,\varepsilon)=\varepsilon\phi(r_2)\frac{\partial^2\mathscr{J}(A(\Theta,T))}{\partial\Theta^2},\tag{24}$$

where $\mathscr{J}(A(\Theta,T))=(r_2/2\pi)\int_0^{2\pi}A(\Theta',T)\ln|\sin((\Theta-\Theta')/2)|d\Theta'$. Then, based on (22) and (24), we get

$$\phi'(r_2)=0,\qquad\frac{\partial\psi_2}{\partial r}(\Theta,r_2,T)=\phi(r_2)\frac{\partial^2\mathscr{J}(A(\Theta,T))}{\partial\Theta^2}.\tag{25}$$

Substituting the boundary conditions (23) and (25) into (15) yields

$$\begin{aligned}&\frac{\partial A}{\partial T}+\alpha\frac{\partial A}{\partial\Theta}+a_1A\frac{\partial A}{\partial\Theta}+a_2\frac{\partial^3A}{\partial\Theta^3}\\&\quad+a_3\frac{\partial^3}{\partial\Theta^3}\mathscr{J}(A(\Theta,T))+\lambda A=\frac{\partial G}{\partial\Theta}.\end{aligned}\tag{26}$$

Equation (26) can be rewritten as follows:

$$\begin{aligned}&\frac{\partial A}{\partial T}+\alpha\frac{\partial A}{\partial\Theta}+a_1A\frac{\partial A}{\partial\Theta}+a_2\frac{\partial^3A}{\partial\Theta^3}\\&\quad+a_3\frac{\partial^2}{\partial\Theta^2}\mathscr{H}(A(\Theta,T))+\lambda A=\frac{\partial G}{\partial\Theta},\end{aligned}\tag{27}$$

where $\mathscr{H}(A(\Theta,T))=(r_2/4\pi)\int_0^{2\pi}A(\Theta',T)\cot((\Theta-\Theta')/2)d\Theta'$ and $a=\int_{r_1}^{r_2}(\phi/(\omega-c))(d/dr)(r(d\phi/dr))dr$, $a_1=\int_{r_1}^{r_2}(\phi^3/(\omega-c))(d/dr)[\beta-(d/dr)((1/r)(dr^2\omega/dr))/\phi(\omega-c)]dr/a$, $a_2=\int_{r_1}^{r_2}(\phi^2/r)dr/a$, $a_3=r_2\phi^2(r_2)/a$, $G=\int_{r_1}^{r_2}r\phi Hdr/a$. Equation (27) is an integro-differential equation and λA expresses dissipation effect and has the same physical meaning with the term $\partial^2A/\partial\Theta^2$ in Burgers equation. When $a_3=\lambda=H=0$,

the equation degenerates to the KdV equation. When $a_2 = \lambda = H = 0$, the equation degenerates to the so-called rotational BO equation. Here we call (27) forced rotational KdV-BO-Burgers equation. As we know, the forced rotational KdV-BO-Burgers equation as a governing model for Rossby solitary waves is first derived in the paper.

3. Conservation Laws

In this section, the conservation laws are used to explore some features of Rossby solitary waves. In [7], Ono presented four conservation laws of BO equation, and we extend Ono's work to investigate the following questions: Has the rotational KdV-BO-Burgers equation also conservation laws without dissipation effect? Has it four conservation laws or more? How to change of these conservation quantities in the presence of dissipation effect?

In this section, topography effect is ignored; that is, H is taken zero in (27). Based on periodicity condition, we assume that the values of A, A_Θ, $A_{\Theta\Theta}$, $A_{\Theta\Theta\Theta}$ at $\Theta = 0$ equal that at $\Theta = 2\pi$. Then integrating (27) with respect to Θ over $(0, 2\pi)$, we are easy to obtain the following conservation relation:

$$Q_1 = \int_0^{2\pi} A d\Theta = \exp(-\lambda T) \int_0^{2\pi} A(\Theta, 0)\, d\Theta. \tag{28}$$

From (28), it is obvious that Q_1 decreases exponentially with the evolution of time T and the dissipation coefficient λ. By analogy with the KdV equation, Q_1 is regarded as the mass of the solitary waves. This shows that the dissipation effect causes the mass of solitary waves decrease exponentially. When dissipation effect is absent, the mass of the solitary waves is conserved.

In what follows, (27) has another simple conservation law, which becomes clear if we multiply (27) by $A(\Theta, T)$ and carry the integration; by using the property of the operator $\mathscr{H} : \int_0^{2\pi} f(\Theta)\mathscr{H}(f(\Theta))d\Theta = 0$, then we get

$$Q_2 = \int_0^{2\pi} A^2 d\Theta = \exp(-2\lambda T) \int_0^{2\pi} A^2(\Theta, 0)\, d\Theta. \tag{29}$$

Similar to the mass Q_1, Q_2 is regarded as the momentum of the solitary waves and is conserved without dissipation. The momentum of the solitary waves also decreases exponentially with the evolution of time T and the increasing of dissipative coefficient λ in the presence of dissipation effect. Furthermore, the rate of decline of momentum is faster than the rate of mass.

Next, we multiply (27) by $(A^2 - (a_3/a_1)\mathscr{H}(A_\Theta))$ and obtain

$$\begin{aligned}
&\left(\frac{1}{3}A^3\right)_T - \frac{a_3}{a_1}\mathscr{H}(A_\Theta) A_T \\
&\quad + (\alpha + a_1 A) A_\Theta \left(A^2 - \frac{a_3}{a_1}\mathscr{H}(A_\Theta)\right) \\
&\quad + a_2 A_{\Theta\Theta\Theta}\left(A^2 - \frac{a_3}{a_1}\mathscr{H}(A_\Theta)\right) \\
&\quad + a_3\left(A^2 - \frac{a_3}{a_1}\mathscr{H}(A_\Theta)\right)(\mathscr{H}(A))_{\Theta\Theta} \\
&\quad + \lambda\left(A^2 - \frac{a_3}{a_1}\mathscr{H}(A_\Theta)\right) A = 0.
\end{aligned} \tag{30}$$

Then taking the derivative of (27) with respect to Θ and multiplying $(-(2a_2/a_1)A_\Theta + (a_3/a_1)\mathscr{H}(A))$ lead to

$$\begin{aligned}
&\left(-\frac{2a_2}{a_1}A_\Theta + \frac{a_3}{a_1}\mathscr{H}(A)\right) A_{\Theta T} \\
&\quad + [\alpha A_{\Theta\Theta} + a_1(AA_\Theta)_\Theta]\left(-\frac{2a_2}{a_1}A_\Theta + \frac{a_3}{a_1}\mathscr{H}(A)\right) \\
&\quad + a_2 A_{\Theta\Theta\Theta\Theta}\left(-\frac{2a_2}{a_1}A_\Theta + \frac{a_3}{a_1}\mathscr{H}(A)\right) \\
&\quad + a_3\left(-\frac{2a_2}{a_1}A_\Theta + \frac{a_3}{a_1}\mathscr{H}(A)\right)\mathscr{H}(A)_{\Theta\Theta\Theta} \\
&\quad + \lambda A\left(-\frac{2a_2}{a_1}A_\Theta + \frac{a_3}{a_1}\mathscr{H}(A)\right) = 0.
\end{aligned} \tag{31}$$

Adding (30) to (31), by virtue of the property of operator $\mathscr{H}$:

$$\mathscr{H}(A)_{\Theta\Theta} = \mathscr{H}(A_{\Theta\Theta}), \qquad \int_0^{2\pi} u\mathscr{H}v d\Theta = -\int_0^{2\pi} v\mathscr{H}u d\Theta, \tag{32}$$

we have

$$\begin{aligned}
&\left(\frac{1}{3}A^3 - \frac{a_2}{a_1}A_\Theta^2 + \frac{a_3}{a_1}A_\Theta\mathscr{H}(A)\right)_T \\
&\quad + \alpha\left[\frac{1}{3}A^3 - \frac{a_2}{a_1}A_\Theta^2 + \frac{a_3}{a_1}\mathscr{H}(A) A_\Theta\right]_\Theta \\
&\quad + \left(\frac{a_1}{4}A^4\right)_\Theta + \frac{a_3^2}{2a_1}[H(A) H(A)_{\Theta\Theta}]_\Theta \\
&\quad + \frac{a_2 a_3}{a_1}(A_{\Theta\Theta\Theta}H(A) - 2A_\Theta H(A)_{\Theta\Theta})_\Theta \\
&\quad - \frac{2a_2}{a_1}\left(A_\Theta A_{\Theta\Theta\Theta} - \frac{1}{2}A_{\Theta\Theta}^2\right)_\Theta + a_2\left(A^2 A_{\Theta\Theta} - 2AA_\Theta^2\right)_\Theta \\
&\quad + a_3[A(A\mathscr{H}(A))_\Theta]_\Theta + \lambda\left(A^3 - \frac{a_2}{a_1}A_\Theta^2 + \frac{a_3}{a_1}A_\Theta H(A)\right) \\
&= 0.
\end{aligned} \tag{33}$$

Taking $Q_3 = \int_0^{2\pi}((1/3)A^3 - (a_2/a_1)A_\Theta^2 + (a_3/a_1)A_\Theta\mathscr{H}(A))d\Theta$, we are easy to see that when the dissipation effect is absent, that is, $\lambda = 0$, Q_3 is a conserved quantity and regarded as the energy of the solitary waves. So we can conclude that the energy of solitary waves is conserved without dissipation. By analysing (33), we can find the decreasing trend of energy of solitary waves.

Finally, let us consider a quantity related to the phase of solitary waves:

$$\widetilde{Q}_4 = \frac{d}{dT}\int_0^{2\pi} \Theta A d\Theta, \tag{34}$$

and we can get $d\widetilde{Q}_4/dT = 0$ without dissipation. According [7], we present the velocity of the center of gravity for the ensemble of such waves $Q_4 = \widetilde{Q}_4/Q_1$; by employing $dQ_1/dT = 0$ and $d\widetilde{Q}_4/dT = 0$, we have $dQ_4/dT = 0$, which shows that the velocity of the center of gravity is conserved without dissipation.

After the four conservation relations are given, we can proceed to seek the fifth conservation quantity. In fact, after tedious calculation, we can also verify that

$$Q_5 = \int_0^{2\pi}\left(\frac{1}{4}A^4 - \frac{3a_2}{a_1}AA_\Theta^2 + \frac{9a_2}{a_1^2}A_{\Theta\Theta}^2 + \frac{a_3}{4a_1}A^2\mathscr{H}(A)\right)d\Theta \tag{35}$$

is also conservation quantity. According the idea, we can obtain the sixth conservation quantity Q_6 and the seventh conservation quantity $Q_7\ldots$, so we can guess that, similar to the KdV equation, the rotational KdV-BO-Burgers equation without dissipation also owns infinite conservation laws, but it needs to be verified in the future.

4. Numerical Simulation and Discussion

In this section, we will take into account the generation and evolution feature of Rossby solitary waves under the influence of topography and dissipation, so we need to seek the solutions of forced rotational KdV-BO-Burgers equation. But we know that there is no analytic solution for (27), and here we consider the numerical solutions of (27) by employing the pseudospectral method.

The pseudo-spectral method uses a Fourier transform treatment of the space dependence together with a leap-forg scheme in time. For ease of presentation the spatial period is normalized to $[0, 2\pi]$. This interval is divided into $2N$ points, and then $\Delta T = \pi/N$. The function $A(X,T)$ can be transformed to the Fourier space by

$$\widehat{A}(\nu,T) = FA = \frac{1}{\sqrt{2N}}\sum_{j=0}^{2N-1} A(j\Delta X,T)\, e^{-\pi i j\nu/N}, \qquad \nu = 0, \pm1, \ldots, \pm N. \tag{36}$$

The inversion formula is

$$A(j\Delta X,T) = F^{-1}\widehat{A} = \frac{1}{\sqrt{2N}}\sum_{\nu}\widehat{A}(\nu,T)\, e^{\pi i j\nu/N}. \tag{37}$$

These transformations can use Fast Fourier Transform algorithm to efficiently perform. With this scheme, $\partial A/\partial X$ can be evaluated as $F^{-1}\{i\nu FA\}$, $\partial^3 A/\partial X^3$ as $-iF^{-1}\{\nu^3 FA\}$, $\partial H/\partial X$ as $F^{-1}\{i\nu FH\}$, and so on. Combined with a leap-frog time step, (27) would be approximated by

$$\begin{aligned} &A(X,T+\Delta T) - A(X,T-\Delta T) + i\alpha F^{-1}\{\nu FA\}\Delta T \\ &\quad + ia_1 AF^{-1}\{\nu FA\}\Delta T - a_2 iF^{-1}\left\{\nu^3 FA\right\}\Delta T \\ &\quad - a_3 F^{-1}\left\{\nu^2 F\mathscr{H}(A)\right\}\Delta T + \lambda A = iF^{-1}\{\nu FG\}\Delta T. \end{aligned} \tag{38}$$

The computational cost for (38) is six fast Fourier transforms per time step.

Once the zonal flow $\Omega(r)$ and the topography function $H(r,\Theta)$ as well as dissipative coefficient λ are given, it is easy to get the coefficients of (27) by employing (13). In order to simplify the calculation and to focus attention on the time evolution of the solitary waves with topography effect and dissipation effect and to show the difference among the KdV model, BO model, and rotational KdV-BO model, we take $a_1 = 1$, $a_2 = -1$, and $a_3 = -1$. As an initial condition, we take $A(X,0) = 0$. In the present numerical computation, the topography forcing is taken as $G = e^{-[30(\Theta-\pi)]^2/4}$.

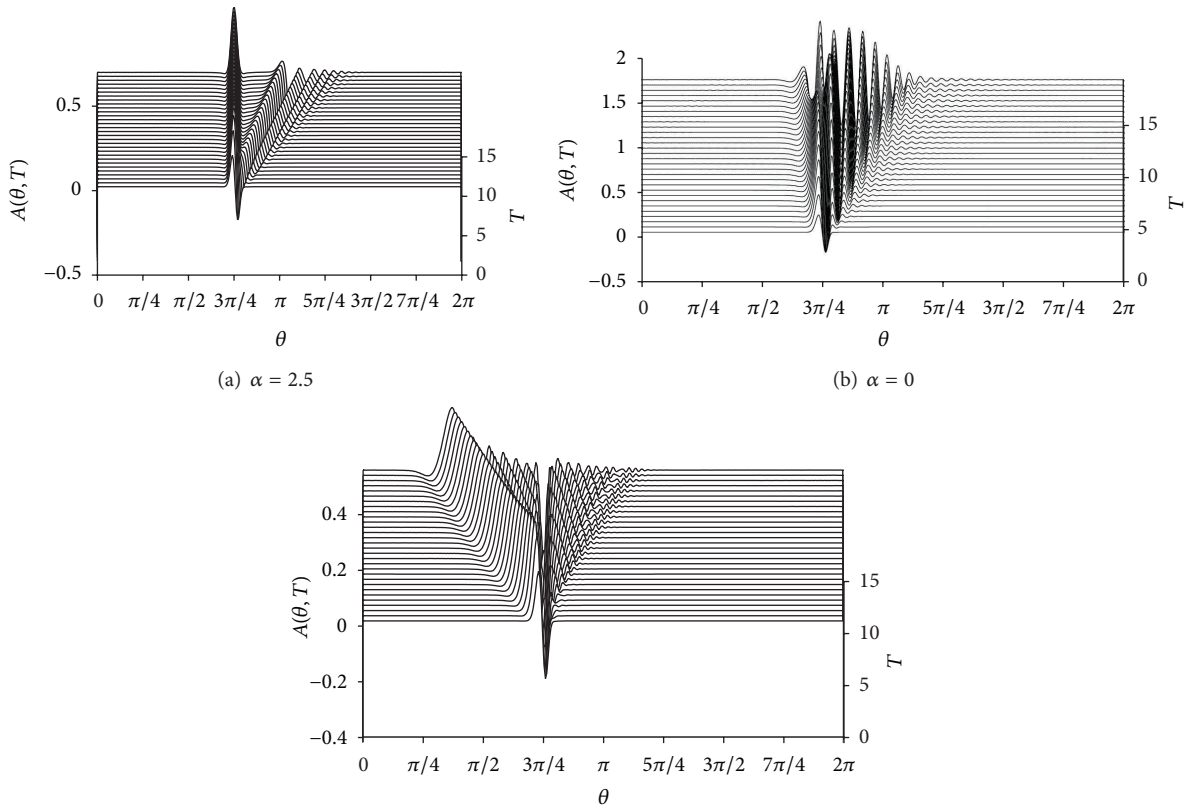
4.1. Effect of Detuning Parameter α and Dissipation. In Figure 1, we consider the effect of detuning parameter α on solitary waves. The evolution features of solitary waves generated by topography are shown in the absence of dissipation with different detuning parameter α. It is easy to find from these waterfall plots that the detuning parameter α plays an important role for the evolution features of solitary waves generated by topography.

When $\alpha > 0$ (Figure 1(a)), a positive stationary solitary wave is generated in the topographic forcing region, and a modulated cnoidal wave-train occupies the downstream region. There is no wave in the upstream region. A flat buffer region exists between the solitary wave in the forcing region and modulated cnoidal wave-train in the downstream. With the detuning parameter α decreasing, the amplitudes of both solitary wave in the forcing region and modulated cnoidal wave-train in the downstream region increase and the modulated cnoidal wave-train closes to the forcing region gradually and the flat buffer region disappears slowly.

Up to $\alpha = 0$ (Figure 1(b)), the resonant case forms. In this case, a large amplitude nonstationary disturbance is generated in the forcing region. To some degree, this condition may explain the blocking phenomenon which exists in the atmosphere and ocean and generated by topographic forcing.

As $\alpha < 0$, from Figure 1(c) we can easy to find that a negative stationary solitary wave is generated in the forcing region, and this is great difference with the former two conditions. Meanwhile, there are both wave-trains in the upstream and downstream region. The amplitude and wavelength of wave-train in the upstream region are larger than those in the downstream regions. Similar to Figure 1(b) and unlike Figure 1(a), the wave-trains in the upstream and downstream regions connect to the forcing region and the flat buffer region disappears.

Figure 2 shows the solitary waves generated by topography in the presence of dissipation with dissipative coefficient $\lambda = 0.3$ and detuning parameter $\alpha = 2.5$. The conditions of $\alpha = 0$ and $\alpha < 0$ are omitted. Compared to Figure 1(a), we will



(a) $\alpha = 2.5$

(b) $\alpha = 0$

(c) $\alpha = -2.5$

Figure 1: Solitary waves generated by topography in the absence of dissipation.

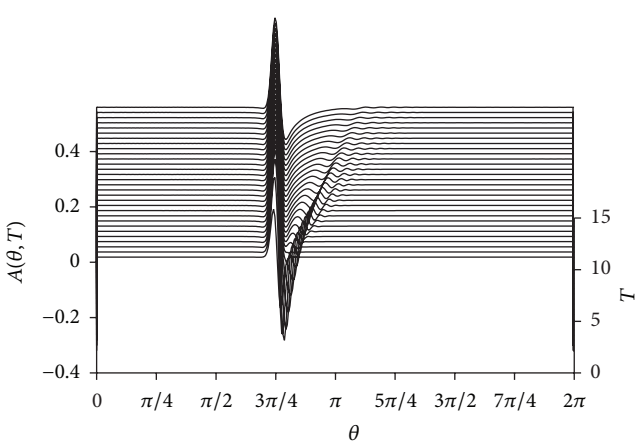
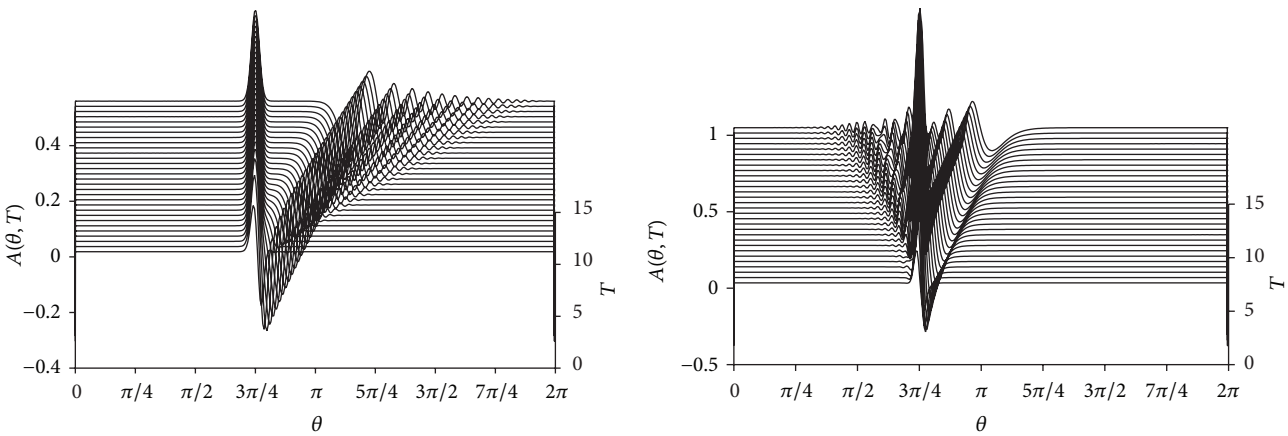


Figure 2: Solitary waves generated by topography in the presence of dissipation ($\lambda = 0.3$, $\alpha = 2.5$).

find that there is also a solitary wave generated in the forcing region, but because of dissipation effect the amplitude of solitary wave in the forcing region decreases as the dissipative coefficient λ increases (Figures omitted) and time evolution. Meanwhile, the modulated cnoidal wave-train in the downstream region is dissipated. When λ is big enough, the modulated cnoidal wave-train in the downstream region disappears.

4.2. Comparison of KdV Model, BO Model, and KdV-BO Model. We know that the rotation KdV-BO equation reduces to the KdV equation as $a_3 = 0$ and to the BO equation as $a_2 = 0$, so, in this subsection by comparing Figure 1(a) with Figure 3, we will look for the difference of solitary waves which is described by KdV-BO model, KdV model, and BO model. The role of detuning parameter α and dissipation effect has been studied in the former subsection, so here we only consider the condition of $\lambda = 0$, $\alpha = 2.5$.

At first, we can find that a positive solitary wave is all generated in the forcing region in Figures 1(a), 3(a) and 3(b), but it is stationary in Figures 1(a) and 3(a), and is nonstationary in Figure 3(b). By surveying carefully we find that the amplitude of stationary wave in the forcing region in Figure 1(a) is larger than that in Figure 3(a). Additionally, a modulated cnoidal wave-train is excited in the downstream region in Figures 1(a) and 3(a), and in both downstream and upstream region in Figure 3(b). The amplitude of modulated cnoidal wave-train in downstream region in Figure 3(b) is the largest and in Figure 1(a) is the smallest among the three models. Furthermore, in Figure 3(a) the wave number of modulated cnoidal wave-train is more than that in Figures 1(a) and 3(b). In a word, by the above analysis and comparison, it is easy to find that Figure 1(a) is similar to Figure 3(a) and has great difference with Figure 3(b). This indicates that the term $a_2(\partial^3 A/\partial\Theta^3)$ plays more important role than the term $a_3(\partial^2/\partial\Theta^2)\mathscr{H}(A)$ in rotational KdV-BO equation.



(a) KdV model ($a_3 = 0, \alpha = 2.5, \lambda = 0$)

(b) BO model ($a_2 = 0, \alpha = 2.5, \lambda = 0$)

Figure 3: Comparison of KdV model, Bo model, and KdV-BO model.

5. Conclusions

In this paper, we presented a new model: rotational KdV-BO-Burgers model to describe the Rossby solitary waves generated by topography with the effect of dissipation in deep rotational fluids. By analysis and computation, five conservation quantities of KdV-BO-Burgers model were derived and corresponding four conservation laws of Rossby solitary waves were obtained; that is, mass, momentum, energy, and velocity of the center of gravity of Rossby solitary waves are conserved without dissipation effect. Further, we presented that the rotational KdV-BO-Burgers equation owns infinite conservation quantities in the absence of dissipation effect. Detailed numerical results obtained using pseudospectral method are presented to demonstrate the effect of detuning parameter α and dissipation. By comparing the KdV model, BO model, and KdV-BO model, we drew the conclusion that the term $a_2(\partial^3 A/\partial\Theta^3)$ plays more important role than the term $a_3(\partial^2/\partial\Theta^2)\mathscr{H}(A)$ in rotational KdV-BO equation. More problems on KdV-BO-Burgers equation such as the analytical solutions, integrability, and infinite conservation quantities are not studied in the paper due to limited space. In fact, there are many methods carried out to solve some equations with special nonhomogenous terms [28] as well as multiwave solutions and other form solution [29, 30] of homogenous equation. These researches have important value for understanding and realizing the physical phenomenon described by the equation and deserve to carry out in the future.

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References

[1] S. G. H. Philander, "Forced oceanic waves," *Reviews of Geophysics and Space Physics*, vol. 16, no. 1, pp. 15–46, 1978.

[2] R. R. Long, "Solitary waves in the westerlies," *Journal of the Atmospheric Sciences*, vol. 21, pp. 197–200, 1964.

[3] D. J. Benney, "Long non-linear waves in fluid flows," *Journal of Mathematical Physics*, vol. 45, pp. 52–63, 1966.

[4] L. G. Redekopp, "On the theory of solitary Rossby waves," *Journal of Fluid Mechanics*, vol. 82, no. 4, pp. 725–745, 1977.

[5] J. I. Yano and Y. N. Tsujimura, "The domain of validity of the KdV-type solitary Rossby waves in the shallow water BETA-plane model," *Dynamics of Atmospheres and Oceans*, vol. 11, no. 2, 1987.

[6] S. Jian and Y. Lian-Gui, "Modified KdV equation for solitary rossby waves with β effect in barotropic fluids," *Chinese Physics B*, vol. 18, no. 7, pp. 2873–2877, 2009.

[7] H. Ono, "Algebraic solitary waves in stratified fluids," vol. 39, no. 4, pp. 1082–1091, 1975.

[8] R. Grimshaw, "Slowly varying solitary waves in deep fluids," *Proceedings of the Royal Society A*, vol. 376, no. 1765, pp. 319–332, 1981.

[9] L. Dehai and J. Liren, "Algebraic rossby solitary wave and blocking in the atmosphere," *Advances in Atmospheric Sciences*, vol. 5, no. 4, pp. 445–454, 1988.

[10] L. Meng and K. L. Lv, "Dissipation and algebraic solitary long-waves excited by localized topography," *Chinese Journal of Computational Physics*, vol. 19, pp. 159–167, 2002.

[11] X. B. Su, G. Wei, and S. Q. Dai, "Two-dimensional algebraic solitary wave and its vertical structure in stratified fluid," *Applied Mathematics and Mechanics*, vol. 26, no. 10, pp. 1255–1265, 2005.

[12] T. Kubota, D. R. S. Ko, and L. D. Dobbs, "Weakly-nonlinear, long internal gravity waves in stratified fluids of finite depth," *Journal of Hydronautics*, vol. 12, no. 4, pp. 157–165, 1978.

[13] Q. P. Zhou, "Second-order solitary waves in stratified fluids of finite depth," *Science China Mathematics*, vol. 10, pp. 924–934, 1984.

[14] W. X. Ma, "Complexiton solutions to the Korteweg-de Vries equation," *Physics Letters A*, vol. 301, no. 1-2, pp. 35–44, 2002.

[15] W.-X. Ma, "Wronskians, generalized Wronskians and solutions to the Korteweg-de Vries equation," *Chaos, Solitons & Fractals*, vol. 19, no. 1, pp. 163–170, 2004.

[16] W.-X. Ma and Y. You, "Solving the Korteweg-de Vries equation by its bilinear form: wronskian solutions," *Transactions of the American Mathematical Society*, vol. 357, no. 5, pp. 1753–1778, 2005.

[17] T. C. Xia, H. Q. Zhang, and Z. Y. Yan, "New explicit and exact travelling wave solutions for a class of nonlinear evolution equations," *Applied Mathematics and Mechanics*, vol. 22, no. 7, pp. 788–793, 2001.

[18] H.-Y. Wei and T.-C. Xia, "Self-consistent sources and conservation laws for nonlinear integrable couplings of the Li soliton hierarchy," *Abstract and Applied Analysis*, vol. 2013, Article ID 598570, 10 pages, 2013.

[19] S. I. A. El-Ganaini, "New exact solutions of some nonlinear systems of partial differential equations using the first integral method," *Abstract and Applied Analysis*, vol. 2013, Article ID 693076, 13 pages, 2013.

[20] D. H. Luo, "Nonlinear schrödinger equation in the rotational atmosphere and atmospheric blocking," *Acta Meteor Sinica*, vol. 48, pp. 265–274, 1990.

[21] O. E. Polukhina and A. A. Kurkin, "Improved theory of nonlinear topographic Rossby waves," *Oceanology*, vol. 45, no. 5, pp. 607–616, 2005.

[22] L. G. Yang, C. J. Da, J. Song, H. Q. Zhang, H. L. Yang, and Y. J. Hou, "Rossby waves with linear topography in barotropic fluids," *Chinese Journal of Oceanology and Limnology*, vol. 26, no. 3, pp. 334–338, 2008.

[23] J. Song and L.-G. Yang, "Force solitary Rossby waves with beta effect and topography effect in stratified flows," *Acta Physica Sinica*, vol. 59, no. 5, pp. 3309–3314, 2010.

[24] H.-W. Yang, B.-S. Yin, D.-Z. Yang, and Z.-H. Xu, "Forced solitary Rossby waves under the influence of slowly varying topography with time," *Chinese Physics B*, vol. 20, no. 12, Article ID 120203, 2011.

[25] H. W. Yang, Y. L. Shi, B. S. Yin, and Q. B. Wang, "Forced ILW-Burgers equation as a model for Rossby solitary waves generated by topography in finite depth fluids," *Journal of Applied Mathematics*, vol. 2012, Article ID 491343, 17 pages, 2012.

[26] B. Fornberg, *A Practical Guide to Pseudospectral Methods*, Cambridge University Press, Cambridge, UK, 1996.

[27] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer, New York, NY, USA, 1979.

[28] W. X. Ma and B. Fuchssteiner, "Explicit and exact solutions to a Kolmogorov-Petrovskii-Piskunov equation," *International Journal of Non-Linear Mechanics*, vol. 31, no. 3, pp. 329–338, 1996.

[29] M. G. Asaad and W.-X. Ma, "Pfaffian solutions to a (3 + 1)-dimensional generalized B-type Kadomtsev-Petviashvili equation and its modified counterpart," *Applied Mathematics and Computation*, vol. 218, no. 9, pp. 5524–5542, 2012.

[30] B. G. Zhang, "Analytical and multishaped solitary wave solutions for extended reduced ostrovsky equation," *Abstract and Applied Analysis*, vol. 2013, Article ID 670847, 8 pages, 2013.