Research Article

Weakly Compact Uniform Attractor for the Nonautonomous Long-Short Wave Equations

Hongyong Cui, Jie Xin, and Anran Li

School of Mathematics and Information, Ludong University, Yantai, 264025, China

Correspondence should be addressed to Jie Xin; fdxinjie@sina.com

Received 8 November 2012; Revised 25 January 2013; Accepted 27 January 2013

Academic Editor: Lucas Jódar

Copyright © 2013 Hongyong Cui et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Solutions and weakly compact uniform attractor for the nonautonomous long-short wave equations with translation compact forces were studied in a bounded domain. We first established the existence and the uniqueness of the solution to the system by using Galerkin method and then obtained the uniform absorbing set and the weakly compact uniform attractor of the problem by applying techniques of constructing skew product flow in the extended phase space.

1. Introduction

The long wave-short wave (LS) resonance equations arise in the study of the interaction of the surface waves with both gravity and capillary modes presence and also in the analysis of internal waves, as well as Rossby wave [1]. In the plasma physics they describe the resonance of the high-frequency electron plasma oscillation and the associated low-frequency ion density perturbation [2]. Benney [3] presents a general theory for the interaction between the short wave and the long wave.

Due to their rich physical and mathematical properties the long wave-short wave resonance equations have drawn much attention of many physicists and mathematicians. For one-dimensional propagation of waves, there are many studies on this interaction. Guo [4, 5] obtains the existence of global solution for long-short wave equations and generalized long-short wave equations, respectively. The existence of global attractor was studied in [6–8]. The orbital stability of solitary waves for this system has been studied in [9]. In [10], Guo investigated the asymptotic behavior of solutions for the long-short wave equations with zero order dissipation in $H^2_{\text{per}} \times H^1_{\text{per}}$. The approximation inertial manifolds for LS type equations have been studied in [11]. The well posedness of the Cauchy problem for the long wave-short wave resonance equations was studied in [8, 12–17].

In this paper, nonautonomous LS equations with translation compact forces were studied. The essential difference between nonautonomous systems and autonomous ones is that the former get much influenced by the time-dependent external forces, which breaks semigroup property of the flow or semiflow created by autonomous systems. Also, attractors of nonautonomous systems are no longer invariant; they change with the changing of the initial time. This makes it impossible for us to consider nonautonomous systems completely in the same way of autonomous ones. Fortunately, Chepyzhov and Vishik [18, 19] developed techniques by which skills in the study of autonomous systems can be used in dealing with nonautonomous problems. Their central idea is that constructing skew product flow in extended phase space is obtained by

$$S(t) (u, \sigma) = (U_{\sigma} (t, 0) u, T(t) \sigma), \quad t \geq 0, \ (u, \sigma) \in E \times \sum_1$$

where $\{U_{\sigma}(t, \tau)\}$ is a family of processes, $\{T(t)\}$ is a translation semigroup, and the flow $\{S(t)\}$ can be proved to be a semigroup under some preconditions, such as the translation identity and $(E \times \sum, E)$-continuity of $\{U_{\sigma}(t, \tau)\}$, and more importantly, the compactness of the symbol space $\sum$. By this means, we can get the uniform attractor by projecting the global attractor of $\{S(t)\}$ to the phase space if the latter...
exists. We consider the following nonautonomous dissipative generalized long-short wave equations:

\[ iu_t + u_{xx} - nu + iau + g(|u|^2)u = a(x,t), \]
\[ n_t + \beta n + |u|^2 + f(|u|^2) = b(x,t), \]
with the initial conditions
\[ u(x,\tau) = u_0(x), \quad n(x,\tau) = n_0(x), \]
and the boundary value conditions
\[ u(x,\tau)|_{\partial \Omega} = 0, \quad n(x,\tau)|_{\partial \Omega} = 0, \]
where \( \Omega = (-D,D) \subset \mathbb{R}, t \geq \tau \in \mathbb{R}_+. \) Nonautonomous terms \( a(x,t) \) and \( b(x,t) \) are time-depended external forces, which are supposed to be translation compact (cf. [18] or generalized long-shortwave equations).

We consider the following nonautonomous dissipative equation:

\[ \begin{align*}
I & \quad \text{Let} \quad H = L^2(\Omega) \quad \text{with usual inner product} \ (\cdot, \cdot), \quad \text{denote by} \quad \| \cdot \| \quad \text{the norm of} \quad L^2(\Omega), \quad \text{for} \quad 1 \leq p \leq \infty (\| \cdot \|_p = \| \cdot \|_L^p), \quad \text{and denote by} \quad \| \cdot \|_{H^k} \quad \text{the norm of a usual Sobolev space} \quad H^k(\Omega) \quad \text{for} \quad 1 \leq k \leq \infty. \quad \text{And we denote different constants by a same constant} \quad c_j, \quad \text{for} \quad j = 1, 2, 3, 4. \\
\text{Uniform means uniform about symbols (}\sigma\text{) in symbol space (} \Sigma \text{) unless there is special explanation. In fact, it is the same if we say uniform about the initial time, since the translation identity and the} \ (E \times \Sigma, E)\text{-continuity of} \quad \{U_\sigma(\tau, \tau)\}\quad \text{hold in our case (cf. [20]).} \\
\text{Throughout this paper, we denote by} \quad \| \cdot \| \quad \text{the norm of} \quad H = L^2(\Omega) \quad \text{with usual inner product} \ (\cdot, \cdot), \quad \text{denote by} \quad \| \cdot \| \quad \text{the norm of} \quad L^2(\Omega), \quad \text{for} \quad 1 \leq p \leq \infty (\| \cdot \|_p = \| \cdot \|_L^p), \quad \text{and denote by} \quad \| \cdot \|_{H^k} \quad \text{the norm of a usual Sobolev space} \quad H^k(\Omega) \quad \text{for} \quad 1 \leq k \leq \infty. \quad \text{And we denote different constants by a same constant} \quad c_j, \quad \text{for} \quad j = 1, 2, 3, 4. \\
\text{We denote all the translation compact functions in} \quad L^2_{\text{loc}}(\mathbb{R}; X), \quad \text{where} \quad X \quad \text{is a Banach space. Apparently,} \quad \varphi \in L^2_{\text{loc}}(\mathbb{R}; X) \quad \text{implies that} \quad \varphi \quad \text{is translation bounded; that is,} \\
\| \varphi \|_{L^2_{\text{loc}}(\mathbb{R}; X)} = \sup_{t \in \mathbb{R}} \int_0^{+\infty} \| \varphi \|^2_X ds < \infty. \]
\end{align*} \]

Let \( E \) be a Banach space, and let a family of two-parameter mappings \( \{U(\tau, \tau)\} = \{U(\tau, \tau) \mid \tau \geq \tau, \tau \in \mathbb{R}\} \) act in \( E \). We also need the following definitions and lemma (cf. [19, 20]).

**Definition 1.** Let \( \Sigma \) be a parameter set. \( \{U_\sigma(\tau, \tau), \tau \geq \tau, \tau \in \mathbb{R}\}, \sigma \in \Sigma \) is said to be a family of processes in Banach space \( E \), if for each \( \sigma \in \Sigma \), \( \{U_\sigma(\tau, \tau)\} \) from \( E \) satisfies

\[ U_\sigma(\tau, \tau) = I, \quad I \text{ is the identity operator, } \tau \in \mathbb{R}. \]

**Definition 2.** \( \{U_\sigma(\tau, \tau)\} \), a family of processes in Banach space \( E \), is called \( (\hat{E} \times \Sigma, \hat{E}) \)-continuous, if for all fixed \( T \) and \( \tau, \tau \geq \tau \), projection \( \{U_\tau, \sigma\} \rightarrow U_\tau(T, \tau) \) is continuous from \( \hat{E} \times \Sigma \) to \( E \).

A set \( B_0 \subset \hat{E} \) is said to be uniformly absorbing set for the family of processes \( \{U_\sigma(\tau, \tau)\}, \) if for any \( \tau \in \mathbb{R} \) and \( B \in \mathcal{B}(E) \) which denotes the set of all bounded subsets of \( E \), there exists \( t_0 = t_0(\tau, B) \geq \tau \), such that \( \{U_\sigma(\tau, \tau)B \supset B_0 \} \) for all \( \tau \geq t_0. \) A set \( Y \subset \hat{E} \) is said to be uniformly attracting for the family of process \( \{U_\tau(\tau, \tau)\}, \sigma \in \Sigma \) if for any fixed \( \tau \in \mathbb{R} \) and every \( B \in \mathcal{B}(E) \),

\[ \lim_{t \to \infty} \sup \{ \text{dist}_E(U_\tau(t, \tau) B, Y) \} = 0. \]

**Definition 3.** A closed set \( \mathcal{A}_E \subset \hat{E} \) is called the uniform attractor of the family of process \( \{U_\sigma(\tau, \tau)\}, \sigma \in \Sigma \) if it is uniformly attracting (attracting property), and it is contained in any closed uniformly attracting set \( \mathcal{A} \) of the family of process \( \{u_\tau(\tau, \tau)\}, \sigma \in \Sigma \) (minimality property).

**Lemma 4.** Let \( \Sigma \) be a compact metric space, and suppose \( \{T(h) \mid h \geq 0\} \) is a family of operators acting on \( \Sigma \), satisfying the following:

(i)

\[ T(h) \Sigma = \Sigma, \quad \forall h \in \mathbb{R}_+. \]

(ii) translation identity:

\[ U_\sigma(t + h, \tau + h) = U_{\tau(h)\sigma}(t, \tau), \quad \forall \sigma \in \Sigma, \quad t \geq \tau, \quad \tau \in \mathbb{R}, \quad h \geq 0, \]

where \( U_\sigma(T, \tau) \) is arbitrarily a process in compact metric space \( E. \) Moreover, if the family of processes \( \{U_\sigma(\tau, \tau)\} \) is \( (\hat{E} \times \Sigma, E) \) continuous, and it has a uniform compact attracting set, then the skew product flow corresponding to it has a global attractor \( \mathcal{A} \) on \( \hat{E} \times \Sigma \), and the projection of \( \mathcal{A} \) on \( \Sigma \), \( \mathcal{A}_E \), is the compact uniform attractor of \( \{U_\sigma(\tau, \tau)\} \).
**Remark 5.** Assumption (13) holds if the system has a unique solution.

For brevity, we rewrite system (2)–(5) in the vector form by introducing $W(x, t) = (u(x, t), n(x, t))$ and $Y(x, t) = (a(x, t), b(x, t))$. We denote by $E_0 = H^2 \cap H^1_0(\Omega)$ the space of vector functions $W(x, t) = (u(x, t), n(x, t))$ with norm

$$
\|W\|_{E_0} = \left\{ \|u\|_{H^2}^2 + \|n\|_{H^1}^2 \right\}^{1/2}.
$$

Similarly, we denote by $\Sigma_0$ the space of $Y(x, t)$ with norm

$$
\|Y\|_{\Sigma_0} = \left\{ \|a\|_{H^2}^2 + \|b\|_{H^1}^2 \right\}^{1/2}.
$$

Then system (2)–(5) can be considered as

$$
\partial_t W = AW + \sigma(t),
$$

$$
W|_{x=s} = (u_s, n_s) = W_s,
$$

$$
W|_{\partial \Omega} = 0,
$$

where $\sigma(t)$ is the symbol of (16).

**Assumption 1.** Assume that the symbol $\sigma(t)$ comes from the symbol space $\Sigma$ defined by

$$
\Sigma = \bigcup \{ Y_0(x, s+r) \mid r \in \mathbb{R}_+ \},
$$

where $Y_0 = (a_0(x, t), b_0(x, t)) \in L^2(\mathbb{R}; E_0)$ and the closure is taken in the sense of local quadratic mean convergence topology in the topological space $L^2_{\text{loc}}(\mathbb{R}; \Sigma_0)$. Moreover, we suppose that $a_0(x, t) \in L^2_{\text{loc}}(\mathbb{R}; H^2)$.

**Remark 6.** By the conception of translation compact/boundedness we remark that

(i) $\forall Y \in \Sigma$, $\|Y\|_{L^2(R; \Sigma_0)} \leq \|Y_0\|_{L^2(R; \Sigma_0)}$;

(ii) $\forall t \in \mathbb{R}$, where $T(t)\sigma(s) = \varphi(s + t)$ is a translation operator.

### 3. Uniform a Priori Estimates in Time

In this section, we derive uniform a priori estimates in time which enable us to show the existence of solutions and the uniform attractor. First we recall the following interpolation inequality (cf. [21]).

**Lemma 7.** Let $j, m, n \in \mathbb{N} \cup \{0\}$, $q, r \in \mathbb{R}^+$, such that $0 \leq j < m$, $1 \leq q, r \leq \infty$. Then one has

$$
\|D^j u\|_p \leq C \|D^m u\|_n \|u\|_q^{1-a},
$$

for $u \in W^{mj}(\Omega) \cap L^q(\Omega)$, where $\Omega \subset \mathbb{R}^1$, $j/m \leq a \leq 1$, and $1/p = j + a(1/r - m) + 1 - a/q$.

**Lemma 8.** If $u(x) \in L^2(\Omega)$ and $Y(x, t) \in \Sigma$, then for the solutions of problem (2)–(5), one has

$$
\|u(t)\| \leq C_1, \quad \forall t \geq t_1,
$$

where $C_1 = C(a, a_0)$, $t_1 = C(a, a_0, \|u_t\|)$.

**Proof.** Taking the inner product of (2) with $u$ in $H$ we get that

$$
\left( iu_t + u_{xx} - nu + i\alpha u + g \left( |u|^2 \right) u \right) = (a(x, t), u).
$$

Taking the imaginary part of (20), we obtain that

$$
\frac{1}{2} \frac{d}{dt} \|u\|^2 + \alpha \|u\|^2 = \text{Im} \left( a(x, t), u \right).
$$

By Young inequality and Remark 6 we have

$$
\frac{d}{dt} \|u\|^2 + \alpha \|u\|^2 \leq \frac{1}{\alpha} \|a(x, t)\|_{L^2(R; H^1)}^2 \leq \frac{1}{\alpha} \|a_0(x, t)\|_{L^2(R; H^1)}^2.
$$

And then by Gronwall lemma we can complete the proof. \(\square\)

In the following, we denote that $\int \cdot \, dx = \int_\Omega \cdot \, dx$, which will not cause confusions.

**Lemma 9.** Under assumptions of (6), (7) and Assumption I, if $W(t) \in H^1 \times H$, solutions of problem (2)–(5) satisfy

$$
\|W(t)\|_{H^1 \times H}^2 \leq C_2, \quad \forall t \geq t_2,
$$

where $C_2 = C(\alpha, \beta, f, g, Y_0, a_0)$ and $t_2 = C(\alpha, \beta, f, g, Y_0, a_0, \|W_0\|_{H^1 \times H})$.

**Proof.** Taking the inner product of (2) with $u_t$ in $H$ and taking the real part, we get that

$$
- \frac{1}{2} \frac{d}{dt} \|u_t\|^2 - \frac{1}{2} \int n \frac{d}{dt} |u|^2 dx + \text{Re} (i\alpha u, u_t)
$$

$$
+ \frac{1}{2} \int g \left( |u|^2 \right) \frac{d}{dt} |u|^2 dx = \text{Re} (a(x, t), u_t).
$$

By (3) we know that

$$
\frac{d}{dt} \int n \frac{d}{dt} |u|^2 dx
$$

$$
= \frac{d}{dt} \int n |u|^2 dx - \int |u|^2 dx
$$

$$
= \frac{d}{dt} \int n |u|^2 dx + \int |u|^2 |u|^2 dx + \beta \int n |u|^2 dx
$$

$$
+ \int f \left( |u|^2 \right) |u|^2 dx - \int b(x, t) |u|^2 dx
$$

$$
= \frac{d}{dt} \int n |u|^2 dx + \beta \int n |u|^2 dx
$$

$$
+ \int f \left( |u|^2 \right) |u|^2 dx - \int b(x, t) |u|^2 dx,
$$

...
which shows that
\[-\frac{1}{2} \frac{d}{dt} \|u_x\|^2 - \frac{1}{2} \left( \frac{d}{dt} \left( n|u|^2 dx + \beta \int |u|^2 dx \right) + \int f (|u|^2) |u|^2 dx - \int b(x, t) |u|^2 dx \right) + \int G (|u|^2) dx + \Re (i\alpha u, u) \]
\[-\frac{1}{2} \frac{d}{dt} \Re (a(x, t), u) + \Re \int a_t(x, t) \overline{u} dx = 0, \tag{26} \]
where \(G(s)\) is introduced by
\[G(s) = \int_0^s g(\xi) \, d\xi. \tag{27} \]
Taking the inner product of (2) with \(u\) in \(H\) and taking the real part, we get that
\[\Re \langle iu_t, u \rangle - \|u_x\|^2 - \int n|u|^2 dx + \int g (|u|^2) |u|^2 dx - \Re (a(x, t), u) = 0. \tag{28} \]
Multiply (28) by \(\alpha\), and add the resulting identity to (26) to get
\[-\frac{1}{2} \frac{d}{dt} \|u_x\|^2 - \frac{1}{2} \frac{d}{dt} \left( n|u|^2 dx - \frac{1}{2} \int f (|u|^2) |u|^2 dx \right)
+ \int b(x, t) |u|^2 dx + \frac{1}{2} \frac{d}{dt} \int G (|u|^2) dx - \alpha \|u_x\|^2 + \alpha \int g (|u|^2) |u|^2 dx - \alpha \Re (a(x, t), u)
- \frac{d}{dt} \Re (a(x, t), u) + \Re \int a_t(x, t) \overline{u} dx = 0. \tag{29} \]
That is,
\[\frac{d}{dt} \left( \|u_x\|^2 + \int n|u|^2 dx - \int G (|u|^2) dx + 2 \Re \int a(x, t) \overline{u} dx \right)
+ \alpha \left( \|u_x\|^2 + \int n|u|^2 dx - \int G (|u|^2) dx \right)
+ 2 \Re \int a_t(x, t) \overline{u} dx + \alpha \|u_x\|^2
= - \int f (|u|^2) |u|^2 dx + \int b(x, t) |u|^2 dx
- \alpha \int G (|u|^2) dx - (\alpha + \beta) \int n|u|^2 dx + 2 \Re \int a_t(x, t) \overline{u} dx. \tag{30} \]
In the following, we denote by \(C\) any constants depending only on the data \((\alpha, \beta, f, g)\), and \(C(\cdot, \cdot)\) means it depends not only on \((\alpha, \beta, f, g)\) but also on parameters in the brackets. \(\forall \rho > 0\), when \(t\) is sufficiently large, by (6) and Lemmas 7 and 8, we have
\[|G(s)| \leq \frac{2}{3} c_3 s^{3/2} + c_2 s, \quad \forall s \geq 0. \tag{32} \]
By (6) we deduce that
\[|G(s)| \leq \frac{2}{3} c_3 s^{3/2} + c_2 s, \quad \forall s \geq 0. \tag{32} \]
And then
\[\left| \alpha \int \Omega G (|u|^2) \, dx \right| \leq C \int \left( \|u\|^3 + |u|^4 \right) dx \leq C \|u\|_3^3 + C \|u\|^2 \tag{33} \]
\[\leq C \|u_x\|^2 \|u\|_3^2 + C \leq \rho \|u\|_3^2 + C_3 (\rho), \tag{34} \]
\[\left| - \alpha \int \Omega G (|u|^2) \, dx \right| \leq C \int \left( \|u\|^3 + |u|^4 \right) dx \leq C \|u\|_3^3 + C \|u\|^2 \tag{33} \]
\[\leq \rho \|u\|^2 + C (\rho) \|u\|_4^4 \tag{34} \]
\[\leq \rho \|u\|^2 + C \rho \|u\|_4^4 + C_4 (\rho), \tag{34} \]
\[\left| 2 \Re \int \Omega a_t(x, t) \overline{u} dx \right| \leq \|a_t(x, t)\|_{L^2(R; \Sigma_0)}^2 + \|u\| \tag{35} \]
\[\leq C \left( \|a_t\|_{L^2(R; \Sigma_0)}^2 \right) \|u\|^2. \tag{35} \]
By (30)–(35) we get that

$$\frac{d}{dt} \left( \|u_x\|^2 + \int n|u|^2 dx - \int G(|u|^2) \, dx + 2 \text{Re} \int a(x, t) \overline{u} \, dx \right)$$

$$+ \alpha \left( \|u_x\|^2 + \int n|u|^2 dx - \int G(|u|^2) \, dx \right)$$

$$+ 2 \text{Re} \int a(x, t) \overline{u} \, dx + \alpha \|u_x\|^2$$

$$\leq \rho \|n\|^2 + 4 \rho \|u_x\|^2 + C(\rho) + C \left( \|u_0\|^2_{L^2([R, Z^0])}, \|u\|^2 \right).$$

(36)

Similarly we can also deduce that

$$\frac{d}{dt} \left( \|u_x\|^2 + \int n|u|^2 dx - \int G(|u|^2) \, dx + 2 \text{Re} \int a(x, t) \overline{u} \, dx \right)$$

$$+ \beta \left( \|u_x\|^2 + \int n|u|^2 dx - \int G(|u|^2) \, dx \right)$$

$$+ 2 \text{Re} \int a(x, t) \overline{u} \, dx + (2 \alpha - \beta) \|u_x\|^2$$

$$\leq \rho \|n\|^2 + 4 \rho \|u_x\|^2 + C(\rho) + C \left( \|u_0\|^2_{L^2([R, Z^0])}, \|u\|^2 \right).$$

(37)

Taking the inner product of (3) with $n$ in $H$, we see that

$$\frac{1}{2} \frac{d}{dt} \|n\|^2 + \int n|u|^2 dx + \beta \|n\|^2$$

$$+ \int f(|u|^2) n \, dx - \int b(x, t) n \, dx = 0.$$  

(38)

By (2) we get that

$$\int n|u|^2 dx = \int n u_x \overline{u} \, dx + \int n u \overline{u} \, dx$$

$$= \int \left( u_x \overline{u} - n u_x \right) dx + 2 \text{Re} \int i \alpha u \overline{u} \, dx$$

$$- 2 \text{Re} \int b(x, t) \overline{u} \, dx,$$

$$\frac{d}{dt} \int (i u \overline{u} - u_x \overline{u}) \, dx$$

$$= \int \left( u_x \overline{u} + u \overline{u}_x - u_x \overline{u} - u_x \overline{u} \right) dx$$

$$= \int \left( u_x \overline{u} - u_x \overline{u} + u \overline{u}_x - u_x \overline{u} \right) dx$$

$$= 2 i \int (u \overline{u}_x - \overline{u}_x u_x) \, dx.$$  

(40)

It comes from (38)–(40) that

$$\frac{d}{dt} \|n\|^2 + \frac{d}{dt} \int i \left( u \overline{u} - u_x \overline{u} \right) dx$$

$$+ 2 \beta \|n\|^2 + i \alpha \int \left( u \overline{u} - u_x \overline{u} \right) dx$$

$$\leq i \alpha \int \left( u \overline{u} - u_x \overline{u} \right) dx - 4 \text{Re} \int i \alpha u \overline{u} \, dx$$

$$+ 4 \text{Re} \int a(x, t) \overline{u} \, dx$$

$$- 2 \int f(|u|^2) n \, dx + 2 \int b(x, t) n \, dx.$$  

(41)

Deal with the right hand side of inequality (41), by Lemmas 7 and 8,

$$\left| \int i \alpha \int \left( u \overline{u} - u_x \overline{u} \right) dx \right| \leq 2 \alpha \|u\| \|u_x\| \leq \rho \|u_x\| + C_1(\rho),$$

(42)

$$\left| - 4 \text{Re} \int i \alpha u \overline{u} \, dx \right| \leq 4 \alpha \|u\| \|u_x\| \leq \rho \|u_x\| + C_2(\rho),$$

(43)

$$\left| - 2 \left( f(|u|^2) n \right) dx \right|$$

$$\leq C \int |u|^p |n| \, dx + C \int |n| \, dx$$

$$\leq \frac{1}{2} \rho \|n\|^2 + C(\rho) \int |u|^p \, dx + \frac{1}{2} \rho \|n\|^2 + C(\rho)$$

$$\leq \rho \|n\|^2 + C(\rho) \left\| u_x \right\|^2 + 4 \rho \|u_x\|^2 + C(\rho)$$

$$\leq \rho \|n\|^2 + 4 \rho \|u_x\|^2 + C_3(\rho),$$

(44)

$$\left| 4 \text{Re} \int a(x, t) \overline{u} \, dx \right|$$

$$\leq 4 \|a(x, t)\|_{L^2([R, Z^0])} \|u_x\| \leq \rho \|u_x\| + C_4(\rho),$$

(45)

$$\left| 2 \int b(x, t) n \, dx \right|$$

$$\leq 2 \|b(x, t)\|_{L^2([R, Z^0])} \|n\| \leq \rho \|n\|^2$$

(46)

So

$$\frac{d}{dt} \left( \|n\|^2 + i \int \left( u \overline{u} - u_x \overline{u} \right) dx \right)$$

$$+ 2 \beta \|n\|^2 + i \alpha \int \left( u \overline{u} - u_x \overline{u} \right) dx$$

$$\leq 2 \rho \|n\|^2 + 4 \rho \|u_x\|^2 + C \left( \rho, \|Y_0(x, t)\|_{L^2([R, Z^0])} \right).$$

(47)
Analogously, we can also deduce that
\[
\frac{d}{dt} \left( \|n\|^2 + i \int (u \bar{\alpha}_x - u_x \bar{u}) \, dx \right)
+ 2\beta \|n\|^2 + i \beta \int (u \bar{\alpha}_x - u_x \bar{u}) \, dx
\leq 2\rho \|n\|^2 + 4\rho \|u_x\|^2 + C \left( \rho s \|Y_0(x, t)\|_{L^2} \right).
\]
Set \( \gamma = \min(\alpha, \beta) \), and
\[
E = \|u_x\|^2 + \|n\|^2 + \int n \|u\|^2 \, dx
- \int G \left( \|u\|^2 \right) \, dx + 2 \int a \bar{u} \, dx
+ i \int (u \bar{\alpha}_x - u_x \bar{u}) \, dx.
\]
Then by (36), (47) and (37), (48) we can, respectively, get
\[
\frac{d}{dt} E + \alpha E + \alpha \|u_x\|^2 + \beta \|n\|^2
\leq 8 \rho \|u_x\|^2 + 3 \rho \|n\|^2 + C \left( \rho s \|Y_0(x, t)\|_{L^2} \right),
\]
\[
\frac{d}{dt} E + \beta E + \alpha \|u_x\|^2 + \beta \|n\|^2
\leq 8 \rho \|u_x\|^2 + 3 \rho \|n\|^2 + C \left( \rho s \|Y_0(x, t)\|_{L^2} \right),
\]
which shows that if we set \( \rho \leq \min(\alpha/8, \beta/3) \), we can deduce that
\[
\frac{d}{dt} E + \gamma E \leq C_0, \quad \forall t \geq t_0,
\]
where \( C_0 = C(\rho, \|Y_0(x, t)\|_{L^2}, \|a_0(x, t)\|_{L^2}) \). By Gronwall lemma we see that
\[
E(t) \leq E(t_0) e^{-\gamma(t-t_0)} + \frac{C_0}{\gamma}, \quad \forall t \geq t_0.
\]
Similar to (33), (34), (45), and (42), for \( t \geq t_0 \) we have
\[
\left| \int n |u|^2 \, dx - \int G \left( |u|^2 \right) \, dx \right|
+ 2 \text{Re} \int a \bar{u} \, dx + i \int (u \bar{\alpha}_x - u_x \bar{u}) \, dx
\leq \rho \|n\|^2 + \rho \|u_x\|^2 + C \left( \rho, \|a_0(x, t)\|_{L^2} \right).
\]
And then
\[
|E(t_0)| \leq \|u_x(t_0)\|^2 + \|n(t_0)\|^2
+ \left| \int n(t_0) |u(t_0)|^2 \, dx - \int G \left( |u(t_0)|^2 \right) \, dx \right|
+ 2 \text{Re} \int a(t_0) \bar{u}(t_0) \, dx
+ i \int (u(t_0) \bar{\alpha}_x (t_0) - u_x (t_0) \bar{u}(t_0)) \, dx
\leq C(R),
\]
where \( C(R) = C(\rho, \|Y_0(x, t)\|_{L^2}, \|a_0(x, t)\|_{L^2}) \) when \( \|W_x\|_{H^2} \leq R \). Then by (52) we infer that
\[
E(t) \leq \frac{C(R) e^{-\gamma(t-t_0)} + C_0}{\gamma}, \quad \forall t \geq t_0,
\]
\[
\leq \frac{2C_0}{\gamma}, \quad \forall t \geq t_*,
\]
where \( t_* = \inf \{ t \mid t \geq t_0 \text{ and } C(R) e^{-\gamma(t-t_0)} \leq C_0/\gamma \} \). By (49), (53), and (55) we infer that
\[
\|u_x(t)\|^2 + \|n(t)\|^2 \leq \rho \|n\|^2 + \rho \|u_x\|^2 + C_0.
\]
Choose \( \rho \leq \min(\alpha/8, \beta/3, 1/2) \); then we have
\[
\|u_x\|^2 + \|n(t)\|^2
\leq C \left( \|Y_0(x, t)\|_{L^2}, \|a_0(x, t)\|_{L^2} \right), \quad \forall t \geq t_*,
\]
which concludes the proof by using Lemma 8. \( \square \)

**Lemma 10.** Under assumptions of Lemma 9, if \( W(t) \in E_0 = H^2 \times H^1 \), solutions of problem (2)–(5) satisfy
\[
\|W(t)\|_{E_0}^2 \leq C_2, \quad \forall t \geq t_3,
\]
where \( C_2 = C(\alpha, \beta, f, g, Y_0, a_0, ||W_x||_{E_0}) \).

**Proof.** Taking the real part of the inner product of (2) with \( u_{xx} \) in \( H \), we have
\[
\frac{d}{dt} \|u_{xx}\|^2 - \text{Re} \int n u \bar{u}_{xx} \, dx + \text{Re} (i a u, u_{xx})
+ \text{Re} \left( g \left( |u|^2 \right) \bar{u}_{xx} \, dx \right)
- \text{Re} \int a(x, t) \bar{u}_{xx} \, dx = 0.
\]
By (2) and (3), we have
\[
- \text{Re} \int n u \bar{u}_{xx} \, dx
= - \frac{d}{dt} \text{Re} (n u \bar{u}_{xx}) \, dx + \text{Re} \int n_x u \bar{u}_{xx} \, dx
+ \text{Re} \int n u_x \bar{u}_{xx} \, dx
= - \frac{d}{dt} \text{Re} (n u \bar{u}_{xx}) \, dx
- \text{Re} \int u \bar{u}_{xx} \left( |u|^2 + \beta n + f \left( |u|^2 \right) - b \right) \, dx
+ \text{Re} \int n u_{xx} (-i n u - a u + i g \left( |u|^2 \right) u - i a) \, dx.
\]
Since
\[ \text{Re}(iau, u_{xx}) = \text{Re} \int iau \overline{u}_{xx} dx = - \text{Re} \int iau \overline{u}_{xx} dx, \quad (61) \]
we see that
\[ \text{Re}(iau, u_{xx}) = a\|u_{xx}\|^2 - \alpha \int n\overline{u}_{xx} dx \]
\[ + \alpha \text{Re} \int g(|u|^2) \overline{u}_{xx} dx \quad (62) \]
\[ - \alpha \text{Re} \int a\overline{u}_{xx} dx. \]

Multiplying (2) by \(\overline{u}\) and taking the real part, we find that
\[ |u|^2 = 2 \text{Re}(iu_{xx}\overline{u}) - 2a|u|^2 - 2 \text{Re}(ia\overline{u}), \quad (63) \]
therefore,
\[ \text{Re} \int g(|u|^2) |u|^2 \overline{u}_{xx} dx \]
\[ = - \int g'(|u|^2) |u|^2 \text{Re}(u\overline{u}_{xx}) dx \]
\[ - \int g(|u|^2) \frac{d}{dt}|u|^2 dx \]
\[ = - \int g'(|u|^2) |u|^2 \text{Re}(u\overline{u}_{xx}) dx \]
\[ - \frac{d}{dt} \int g(|u|^2) |u|^2 dx \]
\[ + \int g'(|u|^2) |u|^2 (\text{Re}(iu_{xx}\overline{u}) - \alpha |u|^2 - \text{Re}(ia\overline{u})) dx. \quad (64) \]

Now we deal with (64) to get (70). Due to equalities
\[ |u|^2 = 2 \text{Re}(u\overline{u}_x), \quad (65) \]
\[ \frac{d}{dt} \text{Re}(u\overline{u}_x) = \text{Re}(u_x\overline{u}) + \text{Re}(u\overline{u}_{xx}), \]
deduce that
\[ \int g'(|u|^2) |u|^2 \text{Re}(u\overline{u}_{xx}) dx \]
\[ = \frac{d}{dt} \int g'(|u|^2) 2 \text{Re}(u\overline{u}_x) \text{Re}(u\overline{u}_x) dx \]
\[ - \int g''(|u|^2) |u|^2 2 \text{Re}(u\overline{u}_x) \text{Re}(u\overline{u}_x) dx \]
\[ - \int g'(|u|^2) 2 \text{Re}(u\overline{u}_x) \text{Re}(u\overline{u}_x) dx \]
\[ - \int g'(|u|^2) 2 \text{Re}(u\overline{u}_x) \text{Re}(u\overline{u}_x) dx. \quad (66) \]

We take care of terms in (66) as follows:
\[ \int g''(|u|^2) |u|^2 2 \text{Re}(u\overline{u}_x) \text{Re}(u\overline{u}_x) dx \]
\[ = 4 \int g''(|u|^2) (\text{Re}(u\overline{u}_x))^2 \]
\[ \times (\text{Re}(iu_{xx}\overline{u}) - \alpha |u|^2 - \text{Re}(ia\overline{u})) dx, \]
\[ \int g'(|u|^2) 2 \text{Re}(u\overline{u}_x) \text{Re}(u\overline{u}_x) dx \]
\[ = \int g'(|u|^2) 2 \text{Re}(u\overline{u}_x) \text{Re}(u\overline{u}_x) dx \]
\[ + \int g'(|u|^2) 2 \text{Re}(u\overline{u}_x) \text{Re}(u\overline{u}_x) dx \]
\[ = 2 \int g'(|u|^2) \text{Re}(u\overline{u}_x) \]
\[ \times \text{Re}(i\overline{u}_x(u_{xx} - nu + iau + g(|u|^2) u - a)) dx, \quad (67) \]

It follows from (66)--(67) that
\[ - \int g'(|u|^2) |u|^2 \frac{d}{dt} \text{Re}(u\overline{u}_{xx}) dx \]
\[ = - \frac{d}{dt} \int g'(|u|^2) 2 \text{Re}(u\overline{u}_x) \text{Re}(u\overline{u}_x) dx \]
\[ + 4 \int g''(|u|^2) (\text{Re}(u\overline{u}_x))^2 \]
\[ \times (\text{Re}(iu_{xx}\overline{u}) - a|u|^2 - \text{Re}(ia\overline{u})) dx, \]
\[ + 4 \int g'(|u|^2) \text{Re}(u\overline{u}_x) \]
\[ \times \text{Re}(i\overline{u}_x(u_{xx} - nu + iau + g(|u|^2) u - a)) dx, \quad (68) \]

And then
\[ - \int g'(|u|^2) |u|^2 \frac{d}{dt} \text{Re}(u\overline{u}_{xx}) dx \]
\[ = - \frac{d}{dt} \int g'(|u|^2) \text{Re}(u\overline{u}_x) \text{Re}(u\overline{u}_x) dx \]
\[ +2 \int g''(|u|^2) \left( \text{Re}(u\overline{u}_x) \right)^2 \times \left( \text{Re}(iu_{xx}\overline{u}) - \alpha |u|^2 - \text{Re}(ia\overline{u}) \right) dx \]

\[ +2 \int g'(|u|^2) \text{Re}(u\overline{u}_x) \times \left( \text{Re}(iu_x(u_{xx} - mu + iau + g(|u|^2)u - a)) \right) dx. \]

(69)

From (64) and (69) we have

\[
\text{Re} \int g(|u|^2)u\overline{u}_{xx}dx
= -\frac{d}{dt} \int g'(|u|^2) \text{Re}(u\overline{u}_x) \text{Re}(u\overline{u}_x) dx
- \frac{1}{2} \frac{d}{dt} \int g(|u|^2) |u_x|^2 dx
+ 2 \int g''(|u|^2) \left( \text{Re}(u\overline{u}_x) \right)^2 \times \left( \text{Re}(iu_{xx}\overline{u}) - \alpha |u|^2 - \text{Re}(ia\overline{u}) \right) dx
+ 2 \int g'(|u|^2) \text{Re}(u\overline{u}_x) \times \left( \text{Re}(iu_{xx}\overline{u}) - \alpha |u|^2 - \text{Re}(ia\overline{u}) \right) dx
+ 2 \int g'(|u|^2) |u_x|^2 \times \left( \text{Re}(iu_{xx}\overline{u}) - \alpha |u|^2 - \text{Re}(ia\overline{u}) \right) dx.
\]

(70)

By (59), (60), (62), and (70) we conclude that

\[
\frac{d}{dt} \left( \|u_x\|^2 \right) - 2 \text{Re} \int nu\overline{u}_{xx} - 2 \int g'(|u|^2) \left( \text{Re}(u\overline{u}_x) \right)^2
- \int g(|u|^2) |u_x|^2 - 2 \text{Re} \int a\overline{u}_{xx}
+ 2\alpha \left( \|u_x\|^2 - 2 \text{Re} \int nu\overline{u}_{xx} \right)
- 2 \int g'(|u|^2) \left( \text{Re}(u\overline{u}_x) \right)^2
- \int g(|u|^2) |u_x|^2 - 2 \text{Re} \int a\overline{u}_{xx}
+ 2\alpha \int nu\overline{u}_{xx} dx + 4\alpha \int g'(|u|^2) \left( \text{Re}(u\overline{u}_x) \right)^2 dx
+ 2\alpha \int g(|u|^2) |u_x|^2 dx + 2\alpha \text{Re} \int a\overline{u}_{xx} dx
- 2 \text{Re} \int u\overline{u}_{xx} (|u_x|^2 + \beta n + f (|u|^2) - b) dx
\]

\[ - 2 \text{Re} \int \overline{u}_{xx} (|u_x|^2 + \beta n + f (|u|^2) - b) dx \]

By later purpose, we let

\[
F(u, n) = -2 \text{Re} \int nu\overline{u}_{xx} dx
- 2 \int g'(|u|^2) \text{Re}(u\overline{u}_x)^2 dx - \int g(|u|^2) |u_x|^2 dx
- 2 \text{Re} \int a\overline{u}_{xx} dx.
\]

(72)

\[
G(u, n)
\]

\[ = 2\alpha \int nu\overline{u}_{xx} dx + 4\alpha \int g'(|u|^2) \left( \text{Re}(u\overline{u}_x) \right)^2 dx
+ 2\alpha \int g(|u|^2) |u_x|^2 dx + 2\alpha \text{Re} \int a\overline{u}_{xx} dx
- 2 \text{Re} \int u\overline{u}_{xx} (|u_x|^2 + \beta n + f (|u|^2) - b) dx
+ 2\alpha \int n\overline{u}_{xx} (-imu - au + ig (|u|^2) - ia) dx
+ 2\alpha \int g(|u|^2) |u_x|^2 dx + 2\alpha \text{Re} \int a\overline{u}_{xx} dx
+ 4 \int g''(|u|^2) \left( \text{Re}(u\overline{u}_x) \right)^2 \times \left( \text{Re}(iu_{xx}\overline{u}) - \alpha |u|^2 - \text{Re}(ia\overline{u}) \right) dx
+ 4 \int g'(|u|^2) \text{Re}(u\overline{u}_x) \times \left( \text{Re}(iu_{xx}\overline{u}) - \alpha |u|^2 - \text{Re}(ia\overline{u}) \right) dx
+ 4 \int g'(|u|^2) \text{Re}(u\overline{u}_x) \times \left( \text{Re}(iu_{xx}\overline{u}) - \alpha |u|^2 - \text{Re}(ia\overline{u}) \right) dx
\]

\[ + g(|u|^2) (|u|^2 - a) dx \]
by (78) we can deduce that

\[
\frac{d}{dt} \|n_x\|^2 + 4 \int \text{Re}(u \bar{u}_{xx} n_x) \, dx \\
+ 4 \int |u_x|^2 n_x \, dx + 2 \beta \|n_x\|^2 \\
+ 2 \int f'(|u|^2) (u_x \bar{u} + u \bar{u}_x) n_x \, dx \\
- 2 \int b_n x \, dx = 0.
\]

From (2) we know that

\[
iut + u_{xxx} - n_x u - n_{xx} + iau_x \\
+ g'(|u|^2) |u|^2 u + g(|u|^2) u_x - a_x(x, t) = 0.
\]

Taking the real part of the inner product to (81) with \(u_{xx}\) in \(H\), we have

\[
\text{Re} \int iu_{txx} \bar{u}_{xx} - \text{Re} \int n_x u \bar{u}_{xx} \, dx \\
- \text{Re} \int nu_{xx} \bar{u}_{xx} \, dx + \text{Re} \int iau_x \bar{u}_{xx} \, dx \\
+ \text{Re} \int g'(|u|^2) |u|^2 u \bar{u}_{xx} \, dx \\
+ \text{Re} \int g(|u|^2) u_x \bar{u}_{xx} \, dx - \text{Re} \int a_x \bar{u}_{xx} \, dx = 0.
\]

Because of

\[
\frac{d}{dt} \text{Re} \int iu_{txx} \bar{u}_{xx} \, dx = 2 \, \text{Re} \int iu_{txx} \bar{u}_{xx} \, dx,
\]

it holds that

\[
\frac{1}{2} \frac{d}{dt} \text{Re} \int iu_x \bar{u}_{xx} \, dx - \text{Re} \int n_x u \bar{u}_{xx} \, dx \\
- \text{Re} \int nu_x \bar{u}_{xx} \, dx + \text{Re} \int iau_x \bar{u}_{xx} \, dx \\
+ \text{Re} \int g'(|u|^2) |u|^2 u \bar{u}_{xx} \, dx \\
+ \text{Re} \int g(|u|^2) u_x \bar{u}_{xx} \, dx \\
- \text{Re} \int a_x \bar{u}_{xx} \, dx = 0.
\]
By (84) and (80), we find that
\[
\begin{align*}
\frac{d}{dt} \|n_x\|^2 &+ 2 \frac{d}{dt} \Re \int i u_x \overline{u}_{xx} dx + 4 \int |u_x|^2 n_x dx + 2 \beta \|n_x\|^2 \\
&+ 2 \int f'(|u|^2) |u|^2 n_x dx \\
&- 2 \int b_x n_x dx - 4 \Re \int n u_x \overline{u}_{xx} dx \\
&+ 4 \alpha \Re \int i u_x \overline{u}_{xx} dx + 4 \Re \int g'(|u|^2) |u|^2 u \overline{u}_{xx} dx \\
&+ 4 \Re \int g(|u|^2) u_x \overline{u}_{xx} dx - 4 \Re \int a u_x \overline{u}_{xx} dx = 0.
\end{align*}
\]
(85)
That is,
\[
\begin{align*}
\frac{d}{dt} \left( \|n_x\|^2 + 2 \Re \int i u_x \overline{u}_{xx} dx \right) \\
&+ 2 \beta \left( \|n_x\|^2 + 2 \Re \int i u_x \overline{u}_{xx} dx \right) \\
&= 4 \beta \Re \int i u_x \overline{u}_{xx} dx - 4 \int |u_x|^2 n_x dx \\
&- 2 \int f'(|u|^2) |u|^2 n_x dx + 2 \int b_x n_x dx \\
&+ 4 \Re \int n u_x \overline{u}_{xx} dx - 4 \alpha \Re \int i u_x \overline{u}_{xx} dx \\
&- 4 \Re \int g'(|u|^2) |u|^2 u \overline{u}_{xx} dx \\
&+ 4 \Re \int g(|u|^2) u_x \overline{u}_{xx} dx + 4 \Re \int a u_x \overline{u}_{xx} dx.
\end{align*}
\]
(86)
Similar to (77), we estimate each term in (86), and then we get
\[
\begin{align*}
\frac{d}{dt} \left( \|n_x\|^2 + 2 \Re \int i u_x \overline{u}_{xx} dx \right) \\
&+ \beta \left( \|n_x\|^2 + 2 \Re \int i u_x \overline{u}_{xx} dx \right) + \beta \|n_x\|^2 \\
&\leq 2 \beta \Re \int i u_x \overline{u}_{xx} dx + C \|n_x\| + C \|n_x\|^2 \\
&+ C \|n_x\| + C \|u_x\| \| \|n_x\| \| \|u_x\| \\
&\leq C \|n_x\| + C \|u_x\|^{3/2} \|u_x\|^{1/2} \|n_x\| \\
&+ C \|n_x\| + C \|n_x\|^2 \|u_x\|^{1/4} \|n\| \|u_x\| \\
&\leq \frac{\alpha}{2} \|u_x\|^{3/2} + \frac{\beta}{2} \|n_x\|^2 + C.
\end{align*}
\]
Let \( \gamma = \min(\alpha, \beta) \), and
\[
E = \|u_x\|^2 + \|n_x\|^2 + F + 2 \Re \int i u_x \overline{u}_{xx} dx.
\]
(88)
By (77) and (87) we deduce that
\[
\frac{d}{dt} E + \gamma E \leq C, \quad \forall t \geq t_2,
\]
(89)
which has the same form with (51) in the proof of Lemma 9. Similar to the study of (51), we can derive that
\[
E(t_2) \leq C(R_2), \quad E(t) \leq \frac{2C}{\gamma}, \quad \forall t \geq t_2,
\]
(90)
where \( t_2 = \inf \{ t \mid t \geq t_2 \} \), \( C(R_2) \leq \gamma (t_2 - t) \leq \gamma (t_2 - t) \leq C_0/\gamma \) and \( C(R_2) = C(\alpha, \beta, f, g, Y_0, \alpha_0, R_2) \) when \( \|W_0\|_{H^2 \times H^2} \leq R_2 \). By (72) we deduce that
\[
\begin{align*}
\|F + 2 \Re \int i u_x \overline{u}_{xx} dx\| \\
&\leq 2 \|u\|_{\infty} \|n\| \|u_x\| + C \|u\|_{\infty} \|u_x\|^2 \\
&+ \|u\|_{\infty} \|u_x\|^2 + \|u\| \|L_1(\Omega, \Sigma)\| \|u_x\| \\
&+ 2 \|u_x\| \|u_x\| + C \\
&\leq C \|u_x\| + C \leq \frac{1}{2} \|u_x\|^2 + C,
\end{align*}
\]
and then by (88), (90), and (91) we deduce that
\[
\|u_x\|^2 + 2 \|n_x\|^2 \leq C, \quad \forall t \geq t_2,
\]
(92)
which concludes the proof by Lemma 9. \( \square \)

4. Solutions for (2)~(5)

**Theorem 11.** Under assumptions of Lemma 10, for each \( W_0 \in E_0 \), system (2)~(5) has a unique global solution \( W(x, t) \in L^\infty(\tau, T; E_0) \), \( \forall T > \tau \).

**Proof.** We prove this theorem briefly by two steps.

**Step 1.** The existence of the solution.

By Galerkin's method, we apply the following approximate solution:
\[
W^l(x, t) = \sum_{j=1}^{l} u_j^l(t) \eta_j(x),
\]
(93)
to approach the solution of the problem (2)~(5), where \{\eta_j\}_{j=1}^{\infty} is an orthogonal basis of \( H(\Omega) \) satisfying \( -\Delta \eta_j = \lambda \eta_j \) (\( j = 1, 2, \ldots \)). And \( W^l(x, t) \) satisfies
\[
\begin{align*}
(i u_j^l + u_j^l - n^l u + i \alpha \eta_j + g(|u|^2) u - a, \eta_j) &= 0, \\
(n_j^l + \beta \eta_j + |u|^2 + f(|u|^2) - b, \eta_j) &= 0, \\
(W^l(x, \tau), \eta_j) &= (W_0, \eta_j), \quad W^l \mid_{\tau=0} = 0,
\end{align*}
\]
(94)
where \( j = 1, 2, \ldots, l \). Then (94) becomes an initial boundary value problem of ordinary differential equations. According
to the standard existence theory for the ordinary differential equations, there exists a unique solution of (94). Similar to [4, 22], by the a priori estimates in Section 3 we know that \( \{W^{(k)}_{\sigma}\}_{k=1}^{\infty} \) converges (weakly star) to a \( W(x, t) \) which solves (2)–(5).

Step 2. The uniqueness of the solution.

Suppose \( W_1, W_2 \) are two solutions of the problem (2)–(5). Let \( W = W_1 - W_2 \), then \( W(x, t) = (u(x, t), n(x, t)) \) satisfies

\[
\begin{align*}
&iu_t + u_{xx} - n_1 u_1 + n_2 u_2 + i\alpha u \\
&+ g \left( \left| u_1 \right|^2 \right) u_1 - g \left( \left| u_2 \right|^2 \right) u_2 = 0, \\
&n_1 + \beta n + \left| u_1 \right|^2 - \left| u_2 \right|^2 = 0, \\
&f \left( \frac{\left| u_1 \right|^2}{2} \right) - f \left( \frac{\left| u_2 \right|^2}{2} \right) = 0, \\
&W\big|_{t=\xi} = 0, \quad W\big|_{t=0} = 0.
\end{align*}
\]

(95)

Similar to [4, 5, 22], we can deduce that \( \|W\| = 0 \). □

5. Uniform Absorbing Set and Unifrom Attractor

From Theorem 11 we know that \( \{U_{\sigma}(t, \tau)\} \), the family of processes corresponding to (2)–(5), is well defined. And assumption (13) is satisfied.

Theorem 12. Under assumptions of Theorem 11, \( \{U_{\sigma}(t, \tau)\} \) possesses a bounded uniformly absorbing set \( B_0 \) in \( E_0 \).

Proof. Let \( B_0 = \{W \in E_0 \mid \|W\|_{E_0} \leq C(\|W_t\|_{E_0}, \|Y_0\|_{L^2(\Sigma)}) \} \).

From Theorem 11 we know that \( B_0 \) is a bounded absorbing set of the process \( U_{\sigma}(t, \tau) \). On the other hand, from Assumption 1 we know that for each \( \Sigma \in E_0, \|Y_t\|_{L^2(\Sigma)}^2 \leq \|Y_0\|_{L^2(\Sigma)}^2 \). Thus, the solution of our system satisfies

\[
\|W\|_{E_0} \leq C(\|W_t\|_{E_0}, \|Y_0\|_{L^2(\Sigma)}).
\]

(96)

So the set \( B_0 = \{W \in E_0 \mid \|W\|_{E_0} \leq \rho_0 \} \leq C(\|W_t\|_{E_0}, \|Y_0\|_{L^2(\Sigma)}) \) is a bounded uniformly absorbing set of \( \{U_{\sigma}(t, \tau)\} \). □

Theorem 13. Under assumptions of Theorem 12, \( \{U_{\sigma}(t, \tau)\} \) admits a weakly compact uniform attractor \( \mathcal{A}_\Sigma \).

Proof. To prove the existence of weakly compact uniform attractor in \( E_0 \), from Lemma 4 and Theorems 11 and 12, the only thing we should do is to verify that \( \{U_{\sigma}(t, \tau)\} \) is \( (E \times \Sigma, E) \)-continuous. Through the following proof, \( \rightharpoonup \) means weak converges, and \( \rightarrow \) means strong converges.

For any fixed \( t_1 \geq t \in \mathbb{R} \), let

\[
(W_{tk}, \sigma_k) \rightharpoonup (W_{\tau}, \sigma) \quad \text{in} \quad E_0 \times \Sigma.
\]

(97)

We will complete the proof if we deduce that

\[
W_{\sigma}(t_1) \rightarrow W_{\sigma}(t_1) \quad \text{in} \quad E_0,
\]

(98)

where \( W_{\sigma}(t_1) = (u(t_1), n(t_1)) = U_{\sigma}(t_1, \tau)W_{\sigma}, \quad W_{\sigma}(t_1) = (u(t_1), n(t_1)) = U_{\sigma}(t_1, \tau)W_{\sigma}. \)

From (97) and Theorem II we know that

\[
\|W_{tk}\|_{E_0} \leq C,
\]

(99)

\[
sup \|W_{tk}(t)\|_{E_0} \leq C.
\]

(100)

By Agmon inequality,

\[
\|v\|_{\infty} \leq C\|v\|_{L^2}.
\]

(101)

We see that

\[
\|W_{\sigma}(t)\|_{E_0} \leq C, \quad \forall 0 \leq t \leq T.
\]

(102)

Note that

\[
iu_{tk} = -u_{xx} + n_k u_k - i\alpha u_k - g \left( \left| u_k \right|^2 \right) u_k + a_k(x, t),
\]

(103)

\[
n_k = -\rho_k - f \left( \left| u_k \right|^2 \right) b_k(x, t),
\]

(104)

and \( a_k = (a_k(x, t), b_k(x, t)) \in \Sigma \). By (100) and (102), we find that \( \partial_t W_{\sigma}(t) \in L^\infty(\tau, T; H) \) and

\[
\|\partial_t W_{\sigma}(t)\|_{L^\infty(\tau, T; H)} \leq C.
\]

(105)

Due to Theorem II and (105), we know that there exist \( \mathcal{W}(t) \equiv (\mathcal{U}(t), \mathcal{N}(t)) \in L^\infty(\tau, T; E_0) \), and subsequences of \( W_{\sigma}(t) \), which are still denoted by \( W_{\sigma}(t) \), such that

\[
W_{\sigma}(t) \rightharpoonup \mathcal{W}(t) \quad \text{in} \quad L^\infty(\tau, T; E_0).
\]

(106)

\[
\partial_t W_{\sigma}(t) \rightharpoonup \partial_t \mathcal{W}(t) \quad \text{in} \quad L^\infty(\tau, T; H).
\]

(107)

Besides, for \( \forall \Sigma \in [\tau, T] \), by (100) we know that there exists \( W_0 \equiv (u^0, n^0) \in E_0 \), such that

\[
W_{\sigma}(t) \rightharpoonup W^0 \quad \text{in} \quad E_0.
\]

(108)

By (106) and a compactness embedding theorem, we claim that

\[
u_k(x, t) \rightharpoonup \bar{u}(x, t) \text{ strongly in } L^2(0, T; H) \quad \text{for all } \Sigma.
\]

(109)

In the following, we shall show that \( \mathcal{W}(t) \) is a solution of the problem (2)–(5). For \( \forall \Sigma \in H, \forall \psi \in C_0^\infty(\tau, T) \), by (103) we find that

\[
\int_\tau^T (iu_{\kappa}, \psi(t) v) dt + \int_\tau^T (u_{kxx}, \psi(t) v) dt
\]

\[
- \int_\tau^T (n_k u_k, \psi(t) v) dt + \int_\tau^T (i\alpha u_k, \psi(t) v) dt
\]

\[
+ \int_\tau^T (g \left( \left| u_k \right|^2 \right) u_k, \psi(t) v) dt
\]

\[
- \int_\tau^T (a_k(x, t), \psi(t) v) dt = 0.
\]

(110)
Since
\[ \int_{\tau}^{T} (u_{k}u_{k,\psi} \psi(t) \nu) dt - \int_{\tau}^{T} (\bar{u}u_{k,\psi} \psi(t) \nu) dt = \int_{\tau}^{T} ((u_{k} - \bar{u}) n_{k,\psi} \psi(t) \nu) dt \]
\[ + \int_{\tau}^{T} (\bar{u}(n_{k} - \bar{n})) \psi(t) \nu) dt, \quad (111) \]
by (102), (109), and (106),
\[ \int_{\tau}^{T} ((u_{k} - \bar{u}) n_{k,\psi} \psi(t) \nu) dt \]
\[ \leq \sup_{0 \leq t \leq T} \| n_{k}(t) \|_{\infty} \| \psi(t) \|_{L^{2}(0,T;H)} \| u_{k} - \bar{u} \|_{L^{2}(0,T;H)} \rightarrow 0, \]
\[ \int_{\tau}^{T} (\bar{u}(n_{k} - \bar{n})) \psi(t) \nu) dt \]
\[ = \int_{\tau}^{T} ((n_{k} - \bar{n})) \psi(t) \bar{u} \nu) dt \rightarrow 0. \quad (112) \]

Then we have
\[ \int_{\tau}^{T} (u_{k}u_{k,\psi} \psi(t) \nu) dt \rightarrow \int_{\tau}^{T} (\bar{u} \bar{u}, \psi(t) \nu) dt. \quad (113) \]

By using the similar methods to the other terms of (110), we have
\[ \int_{\tau}^{T} (\bar{u}_{\nu}, \psi(t) \nu) dt + \int_{\tau}^{T} (\bar{u} \bar{u}, \psi(t) \nu) dt \]
\[ - \int_{\tau}^{T} (\bar{u}u_{\nu}, \psi(t) \nu) dt + \int_{\tau}^{T} (i\bar{u}u_{\nu}, \psi(t) \nu) dt \]
\[ + \int_{\tau}^{T} (g(\bar{u}_{\nu}^{2}) \bar{u}, \psi(t) \nu) dt \]
\[ - \int_{\tau}^{T} (a(x,t), \psi(t) \nu) dt = 0. \quad (114) \]

Therefore, we obtain
\[ \bar{u}_{\nu} + \bar{u} \bar{u}_{\nu} - \bar{u} \bar{u} + i\bar{u} + g(\bar{u}_{\nu}^{2}) \bar{u} = a(x,t), \quad (115) \]
which shows that (\(\bar{u}, \bar{n}, a(t)\)) satisfies (2).

For \(\forall v \in H, \forall \psi \in C_{0}^{\infty}(\tau, T)\) with \(\psi(T) = 0, \psi(\tau) = 1\), by (103) we find that
\[ - \int_{\tau}^{T} (u_{k}, \psi'(t) \nu) dt + \int_{\tau}^{T} (u_{k}a_{\psi} \nu) \psi(t) \nu) dt \]
\[ - \int_{\tau}^{T} (n_{k}u_{k,\psi} \psi(t) \nu) dt + \int_{\tau}^{T} (i\bar{u}u_{k,\psi} \psi(t) \nu) dt \]
\[ + \int_{\tau}^{T} (g(\bar{u}_{\nu}^{2}) u_{k}, \psi(t) \nu) dt - \int_{\tau}^{T} (a_{\psi}(x,t), \psi(t) \nu) dt \]
\[ = i (u_{k}(\tau), v). \quad (116) \]

Assumption (97) implies that
\[ u_{k}(\tau) = u_{\tau} \rightarrow u_{\tau} \text{ in } H. \quad (117) \]

Then by (116) and (117), we have
\[ - \int_{\tau}^{T} (\bar{u}u_{\nu}, \psi(t) \nu) dt + \int_{\tau}^{T} (\bar{u}, \psi'(t) \nu) dt \]
\[ - \int_{\tau}^{T} (\bar{u}u_{\nu}, \psi(t) \nu) dt + \int_{\tau}^{T} (i\bar{u}u_{\nu}, \psi(t) \nu) dt \]
\[ + \int_{\tau}^{T} (g(\bar{u}_{\nu}^{2}) \bar{u}, \psi(t) \nu) dt - \int_{\tau}^{T} (a(x,t), \psi(t) \nu) dt \]
\[ = i (u_{\tau}, v). \quad (118) \]

While from (115) we know that
\[ - \int_{\tau}^{T} (\bar{u}u_{\nu}, \psi(t) \nu) dt + \int_{\tau}^{T} (\bar{u} \bar{u}, \psi(t) \nu) dt \]
\[ - \int_{\tau}^{T} (\bar{u}u_{\nu}, \psi(t) \nu) dt + \int_{\tau}^{T} (i\bar{u}u_{\nu}, \psi(t) \nu) dt \]
\[ + \int_{\tau}^{T} (g(\bar{u}_{\nu}^{2}) \bar{u}, \psi(t) \nu) dt - \int_{\tau}^{T} (a(x,t), \psi(t) \nu) dt \]
\[ = i (\bar{u}(\tau), v). \quad (119) \]

It comes from (118) and (119) that
\[ (u_{\tau}, v) = (\bar{u}(\tau), v), \quad \forall v \in H. \quad (120) \]

And then
\[ \bar{u}(\tau) = u_{\tau}. \quad (121) \]

By (115) and (121), we have
\[ \bar{u}(t) = u(t). \quad (122) \]

For \(\forall v \in H, \forall \psi \in C_{0}^{\infty}(\tau, t_{1})\), with \(\psi(T) = 0, \psi(t_{1}) = 1\), then repeating the procedure of proofs of (116)–(119), by (108) we deduce that
\[ u_{0} = \bar{u}(t_{1}). \quad (123) \]

It comes from (108), (122), and (123) that
\[ u_{k}(t_{1}) \rightarrow u(t_{1}) \text{ in } H^{2}(\Omega). \quad (124) \]

Similarly, we can also deduce that
\[ n_{k}(t_{1}) \rightarrow n(t_{1}) \text{ in } H^{1}(\Omega). \quad (125) \]

By (124) and (125), we derive (98). We complete the proof of Theorem 13. \(\square\)

**Acknowledgments**

The authors were supported by the NSF of China (nos. 1107162, 11271050), the Project of Shandong Province Higher Educational Science and Technology Program (no. J10LA09).
References


Submit your manuscripts at http://www.hindawi.com