Research Article
The Local Strong Solutions and Global Weak Solutions for a Nonlinear Equation

Meng Wu
Department of Mathematics, Southwestern University of Finance and Economics, Chengdu 610074, China

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The existence and uniqueness of local strong solutions for a nonlinear equation are investigated in the Sobolev space $\mathcal{C}([0, T); H^s(\mathbb{R})] \cap \mathcal{C}^1([0, T); H^{s-1}(\mathbb{R}))$ provided that the initial value lies in $H^s(\mathbb{R})$ with $s > 3/2$. Meanwhile, we prove the existence of global weak solutions in $L^\infty([0, \infty); L^2(\mathbb{R}))$ for the equation.

1. Introduction

Coclite and Karlsen [1] investigated the well posedness in classes of discontinuous functions for the generalized Degasperis-Procesi equation:

$$ u_t - u_{xx} + 4h'(u)u_x = h'''(u)u_x^3 + 3h''(u)u_xu_{xx} + h'(u)u_{xxx}, $$

which is subject to the condition

$$ |h'(u)| \leq c|u|, \quad |h(u)| \leq c|u|^2, \quad (2) $$

or

$$ |h'(u)| \leq c, \quad |h(u)| \leq c|u|, \quad (3) $$

where $c$ is a positive constant. The existence and $L^1$ stability of entropy weak solutions belonging to the class $L^1(R) \cap BV(R)$ are established for (1) in paper [1].

In this work, we study the following model:

$$ u_t - u_{xx} + mh'(u)u_x = h'''(u)u_x^3 + 3h''(u)u_xu_{xx} + h'(u)u_{xxx}, \quad (4) $$

where $m$ is a positive constant and $h(u) \in C^3$. If $m = 4$ and $h(u) = u^2/2$, (4) reduces to the classical Degasperis-Procesi model (see [2–13]). Here, we notice that assumptions (2) and (3) do not include the case $h(u) = u^3$. In this paper, we will study the case $h(u) = u^3$, and $m$ is an arbitrary positive constant.

In fact, the Cauchy problem of (4) in the case $h(u) = u^3$ is equivalent to the following system:

$$ u_t - u_{xx} + 3mu^2u_x = 6u_x^3 + 18uu_xu_{xx} + 3u^2u_{xxx}, \quad (5) $$

$$ u(0, x) = u_0(x). $$

Using the operator $(1 - \partial_x^3)^{-1}$ to multiply the first equation of the problem (5), we obtain

$$ u_t + 3u^2u_x + (m - 1)(1 - \partial_x^3)^{-1}\partial_x(u^3) = 0, $$

$$ u(0, x) = u_0(x). \quad (6) $$

It is shown in this work that there exists a unique local strong solution in the Sobolev space $C([0, T); H^s(R)) \cap C^1([0, T); H^{s-1}(R))$ by assuming that the initial value $u_0(x)$ belongs to $H^s(R)$ with $s > 3/2$. In addition, we prove the existence of global weak solutions in $L^\infty([0, \infty); L^2(R))$ for the system (6).

This paper is organized as follows. Section 2 investigates the existence and uniqueness of local strong solutions. The result about global weak solution is given in Section 3.
2. Local Existence

In this section, we will use the Kato theorem in [14] for abstract differential equation to establish the existence of local strong solution for the problem (6). Let us consider the following problem:

\[
\frac{dv}{dt} + H(v)v = g(v), \quad t \geq 0, \quad v(0) = v_0.
\]

Let \(X\) and \(Y\) be Hilbert spaces such that \(Y\) is continuously and densely embedded in \(X\), and let \(Q : Y \to X\) be a topological isomorphism. Let \(L(Y, X)\) be the space of all bounded linear operators from \(Y\) to \(X\). In the case of \(X = Y\), we denote this space by \(L(X)\). We illustrate the following conditions in which \(\sigma_1, \sigma_2, \sigma_3, \) and \(\sigma_4\) are constants depending only on \(m\) such that

\[
\sigma_1 \|u\|_{L^2(\mathbb{R})} \leq \sigma_2 \|u\|_{L^2(\mathbb{R})} \leq \sigma_3 \|u\|_{L^2(\mathbb{R})} \leq \sigma_4 \|u\|_{L^2(\mathbb{R})}.
\]

**Lemma 1.** The operator \(A\) is accretive, uniformly on bounded sets in \(Y\).

**Lemma 2.** Assume that \(H(u) = 3u^2 \partial_x u\) with \(u \in H^2(R, \mathbb{R})\), then, \(H(u) \in L(H^2(R, \mathbb{R}^{1, 1}))\) for all \(u \in H^2(R, \mathbb{R})\). Moreover,

\[
H(u) - H(z) \leq \sigma_1 \|u - z\|_{H^2} + \sigma_2 \|u - z\|_{H^2}.
\]

**Lemma 3.** For \(s > 3/2, u, z \in H^s(R, \mathbb{R})\), it holds that

\[
\|A(u) - A(z)\|_{H^{s-1}} \leq \sigma_3 \|u - z\|_{H^s}.
\]

The above three Lemmas can be found in Ni and Zhou [15].

**Lemma 4.** Let \(u, z \in H^s\) with \(s > 3/2\) and \(g(u) = (m - 1)\Lambda^2 \partial_x u^3\). Then, \(g\) is bounded on bounded sets in \(H^s\) and satisfies

\[
\|g(u) - g(z)\|_{H^s} \leq \sigma_3 \|u - z\|_{H^s}.
\]

**Theorem 5.** Assume that \(u_0 \in H^s(R, \mathbb{R})\), then, there exists a \(T > 0\) such that the system (5) or the problem (6) has a unique solution \(u(t, x)\) satisfying

\[
u(t, x) = C([0, T); H^s(R, \mathbb{R}^{1, 1})) \cap C^1([0, T); H^{s-1}(R)).
\]

3. Weak Solutions

In this section, our aim is to establish the existence of global weak solutions for the system (6). Firstly, we prove that the solution of the problem (5) is bounded in the space \(L^2(\mathbb{R})\) and \(L^{\infty}(\mathbb{R})\).

**Lemma 6.** The solution of the problem (5) with \(m > 0\) satisfies

\[
\int_{\mathbb{R}} K_x K d\xi = \int_{\mathbb{R}} \frac{1 + \xi^2}{m + \xi^2} |\tilde{u}(\xi)|^2 d\xi = \int_{\mathbb{R}} \frac{1 + \xi^2}{m + \xi^2} |\tilde{u}_0(\xi)|^2 d\xi,
\]

where \(K_x = u - \partial_x^2 u\), and \(K = (m - \partial_x^2)^{-1} u\). Moreover, there exist two constants \(c_1 > 0\) and \(c_2 > 0\) depending only on \(m\) such that

\[
c_1 \|u_0\|_{L^2(\mathbb{R})} \leq c_2 \|u\|_{L^2(\mathbb{R})} \leq c_3 \|u_0\|_{L^2(\mathbb{R})}.
\]
Proof. Setting $K_1 = u - \partial_x^2 u$ and $K = (m - \partial_x^2)^{-1} u$ and using the first equation of the problem (5), we obtain $u = my - y_x$ and

$$\frac{d}{dt} \int_R K_1 K dx$$

$$= \int_R \frac{\partial K_1}{\partial t} K dx + \int_R K_1 \frac{\partial K}{\partial t} dx = 2 \int_R \frac{\partial K_1}{\partial t} K dx$$

$$= 2 \int_R [-3mu^2 u_x + 6u_x^3 + 18u_x u_{xx} + 3u^2 u_{xxx}] K dx$$

$$= 2 \int_R [m\partial_x (u^3) + (u^3)_{xxx}] K dx$$

$$= \int_R [m\partial_x (u^3) + u^3 (mK_x - u_x)] K dx$$

$$= \int_R u^3 u_x dx,$n

$$= 0.$$  \quad \text{(21)}$

Using the Parseval identity and (21), we obtain (19) and (20).

From Theorem 5, we know that for any $u_0 \in H^s(R)$ with $s > 3/2$, there exists a maximal $T = T(u_0) > 0$ and a unique strong solution $u$ to the problem (6) such that

$$u \in C([0, T); H^s(R)) \cap C^1([0, T); H^{s-1}(R)).$$  \quad \text{(22)}$

Firstly, we study the following differential equation:

$$p_1 = 3u^2 (t, p), \quad t \in [0, T),$$

$$p_0 (0, x) = x.$$  \quad \text{(23)}$

Lemma 7. Let $u_0 \in H^s, s > 3$, and let $T > 0$ be the maximal existence time of the solution to the problem (6). Then, the problem (23) has a unique solution $p \in C([0, T) \times R, R)$. Moreover, the map $p(t, \cdot)$ is an increasing diffeomorphism of $R$ with $p_x(t, x) > 0$ for $t \in [0, T) \times R$.

Proof. Using Theorem 5, we obtain $u \in C^1([0, T); H^{s-1}(R))$ and $H^s \times H^{s-1} \in C^1(R)$. Therefore, we know that functions $u(t, x)$ and $u_x(t, x)$ are bounded, Lipschitz in space, and $C^1$ in time. Using the existence and uniqueness theorem for ordinary differential equations derives that the problem (23) has a unique solution $p \in C^1([0, T) \times R, R)$.

Differentiating (23) with respect to $x$ gives rise to the following:

$$\frac{d}{dt} p_x = 6uu_x (t, p) p_x, \quad t \in [0, T),$$

$$p_x (0, x) = 1,$$  \quad \text{(24)}$

from which we obtain

$$p_x (t, x) = \exp \left( \int_0^t 6uu_x (\tau, p (\tau, x)) d\tau \right).$$  \quad \text{(25)}$

For every $T' < T$, using the Sobolev imbedding theorem yields that

$$\sup_{(t, x) \in [0, T') \times R} |u_x (t, x)| < \infty.$$  \quad \text{(26)}$

It is inferred that there exists a constant $K_0 > 0$ such that $p_x (t, x) \geq e^{-K_0 t}$ for $(t, x) \in [0, T) \times R$. It completes the proof. \quad \Box

Lemma 8. Assume that $u_0 \in H^s(R), s > 3/2$. Let $T$ be the maximal existence time of the solution $u$ to the problem (6). Then, it has

$$\|u(t, x)\|_{L^\infty} \leq \|u_0\|_{L^\infty} e^{ct} \quad \forall t \in [0, T],$$  \quad \text{(27)}$

where $c > 0$ is a constant independent of $t$.

Proof. Let $\xi(x) = (1/2)e^{-|x|}$, we have $(1 - \partial_x^2)^{-1} g = \xi * f$ for all $g \in L^2(R)$ and $u = \xi * K(t, x)$. Using a simple density argument presented in [7], it suffices to consider $s = 3$ to prove this lemma. Let $T$ be the maximal existence time of the solution $u$ to the problem (6) with the initial value $u_0 \in H^s(R)$ such that $u \in C([0, T), H^s(R)) \cap C^1([0, T); H^{s-1}(R))$. From (6), we have

$$u_t + 3u_x u_x = -(m - 1) \xi * (3u_x u_x).$$  \quad \text{(28)}$

Since

$$\xi * (3u_x u_x) = -\frac{1}{2} \int_{-\infty}^{\infty} e^{-|x| - |y|} 3u_x^2 u_x d\eta$$

$$= -\frac{1}{2} \int_{-\infty}^{\infty} e^{-|x| + 2\eta} u_x^2 u_x d\eta - \frac{1}{2} \int_{-\infty}^{\infty} e^{-|x| - 2\eta} u_x^2 u_x d\eta$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{-|x| - \eta} u_x^2 d\eta - \frac{1}{2} \int_{x}^{\infty} e^{-|x| - \eta} u_x^2 d\eta,$n

$$\frac{du}{dt} (t, p(t, x)) = u_t (t, p(t, x)) + u_x (t, p(t, x)) \frac{dp}{dt} (t, x)$$

$$= (u_t + 3u_x u_x) (t, p(t, x)),$$  \quad \text{(29)}$

from (29), we have

$$\frac{du}{dt} (t, p(t, x)) = \frac{m - 1}{2} \int_{-\infty}^{\infty} e^{-|p(t, x) - \eta|} u_x^2 d\eta$$

$$- \frac{m - 1}{2} \int_{p(t, x)}^{\infty} e^{-|p(t, x) - \eta|} u_x^2 d\eta.$$  \quad \text{(30)}$
Using Lemma 6 and (30) derives that
\[
\left| \frac{du(t, p(t, x))}{dt} \right| \leq \frac{|m-1|}{2} \int_{-\infty}^{\infty} e^{-|p(t, x) - \eta|} \, u^2 \, d\eta \\
\leq \frac{|m-1|}{2} \int_{-\infty}^{\infty} u^2 \, d\eta \\
\leq \frac{|m-1|}{2} \|u\|^2_{L^2(\mathbb{R})} \|u\|_{L^\infty} \\
\leq \epsilon \|u_0\|_{L^\infty(\mathbb{R})} \|u\|_{L^\infty} \\
\leq \epsilon \|u\|_{L^\infty(\mathbb{R})},
\]
where \(c\) is a positive constant independent of \(t\). Using (31) results in the following:
\[
-c \int_0^T \|u\|_{L^\infty(R)} \, dt + u_0 \leq u(t, p(t, x)) \leq c \int_0^T \|u\|_{L^\infty(R)} \, dt + u_0.
\]
(32)

Therefore,
\[
\left| u(t, p(t, x)) \right| \leq \left| u(t, p(t, x)) \right|_{L^\infty} \\
\leq \|u_0\|_{L^\infty} + c \int_0^T \|u\|_{L^\infty(R)} \, dt.
\]
(33)

Using the Sobolev embedding theorem to ensure the uniform boundedness of \(u_\epsilon(s, \eta)\) for \((s, \eta) \in [0, t] \times R\) with \(t \in [0, T']\), from Lemma 7, for every \(t \in [0, T']\), we get a constant \(C(t)\) such that
\[
e^{-C(t)} \leq p_\epsilon(t, x) \leq e^{C(t)}, \quad x \in R.
\]
(34)

We deduce from (34) that the function \(p(t, \cdot)\) is strictly increasing on \(R\) with \(\lim_{t \to -\infty} p(t, x) = \pm \infty\) as long as \(t \in [0, T')\). It follows from (33) that
\[
\|u(t, x)\|_{L^\infty} = \left| u(t, p(t, x)) \right|_{L^\infty} \leq \|u_0\|_{L^\infty} + c \int_0^T \|u\|_{L^\infty(R)} \, dt.
\]
(35)

Using the Gronwall inequality and (35) derives that (27) holds.

For a real number \(s\) with \(s > 0\), suppose that the function \(u_\epsilon(x)\) is in \(H^s(R)\), and let \(u_{\epsilon 0}\) be the convolution \(u_{\epsilon 0} = \phi_* * u_0\) of the function \(\phi(x) = e^{-1/4} \phi(e^{-1/4} x)\) and \(u_0\) such that the Fourier transform \(\hat{\phi}\) of \(\phi\) satisfies \(\hat{\phi} \in C_0^\infty, \hat{\phi}(\xi) \geq 0\), and \(\hat{\phi}(\xi) = 1\) for all \(\xi \in (-1, 1)\). Then, we have \(u_{\epsilon 0}(x) \in C^\infty\). It follows from Theorem 5 that for each \(\epsilon\) satisfying \(0 < \epsilon < 1/2\), the Cauchy problem,
\[
u_{\epsilon} - u_{\epsilon xx} + 3mu_{\epsilon} u_x = 6u_x^3 + 18u u_{x} u_{xx} + 3u^2 u_{xxx},
\]
\[
0, x) = u_{\epsilon 0}(x),
\]
has a unique solution \(u_\epsilon(t, x) \in C^\infty([0, T); H^\infty(R))\). Using Lemmas 6 and 8, for every \(t \in [0, T)\), we obtain
\[
\|u_{\epsilon}(t, x)\|_{L^\infty(\mathbb{R})} \leq c \|u_0(0, x)\|_{L^\infty(\mathbb{R})} \leq c \|u_0\|_{L^\infty(\mathbb{R})},
\]
\[
\|u_{\epsilon}(t, x)\|_{L^\infty} \leq \|u_0(0, x)\|_{L^\infty} e^\epsilon t \leq c \|u_0\|_{L^\infty} e^\epsilon t.
\]
(37)

Sending \(t \to T\), we know that inequalities (37) are still valid. This means that for \(t \in [0, \infty)\), (37) hold.

Now, we state the concepts of weak solutions.

**Definition 9** (weak solution). We call a function \(u : R_+ \times R \to R\) a weak solution of the Cauchy problem (5) provided that
(i) \(u \in L^\infty(R_+; L^2(R))\);
(ii) \(u_t - u_{xxx} + 3mu^2 u_x = 6u_x^3 + 18uu_{x} u_{xx} + 3u^2 u_{xxx}\) in \(D'([0, \infty) \times R)\), that is, for all \(\phi \in C_0^\infty([0, \infty) \times R)\) there holds the following identity:
\[
\int_{R^+} \int_{R} \left( u(\phi_t - \phi_{xxx}) + mu^3 \phi_x - u^3 \phi_{xxx} \right) \, dx \, dt
\]
(38)

**Theorem 10.** Let \(u_\epsilon(x) \in L^2(R)\). Then, there exists a weak solution \(u(t, x) \in L^\infty([0, \infty); L^2(R))\) to the problem (5).

**Proof.** Consider the problem (36). For an arbitrary \(T > 0\), choosing a subsequence \(\epsilon_n \to 0\), from (37), we know that \(u_{\epsilon_n}\) is bounded in \(L^\infty\) and \(\|u_{\epsilon_n}\|^2_{L^2(R)}\) is uniformly bounded in \(L^2(R)\). Therefore, we obtain \(u_{\epsilon_n}^2\) is bounded in \(L^2(R)\). Therefore, there exist subsequences \(\{|u_{\epsilon_n}\}\) and \(\{|u_{\epsilon_k}\}\), still denoted by \(\{|u_{\epsilon_n}\}\) and \(\{|u_{\epsilon_k}\}\), are weakly convergent to \(v\) in \(L^2(R)\). Noticing (38) completes the proof.

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**References**


