Research Article

Strong Convergence Iterative Algorithms for Equilibrium Problems and Fixed Point Problems in Banach Spaces

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1. Introduction

Let $E$ be a real reflexive Banach space with norm $\| \cdot \|$ and $E^*$ the dual space of $E$ equipped with the induced norm $\| \cdot \|_*$. Throughout this paper, $f : E \to (-\infty, +\infty]$ is a proper, lower semicontinuous, and convex function and the Fenchel conjugate of $f$ is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(\xi) = \sup \{ \langle \xi, x \rangle - f(x) : x \in E \}.$$  (1)

We denote by $\text{dom } f$ the domain of $f$, that is, the set $\{ x \in E : f(x) < +\infty \}$.

Let $C$ be a nonempty, closed, and convex subset of $E$ and $T : C \to C$ a nonlinear mapping. The fixed points set of $T$ is denoted by

$$F(T) = \{ x \in C : x = Tx \}.$$  (2)

Recall that a mapping $T : C \to C$ is said to be nonexpansive if, for each $x, y \in C$,

$$\|Tx - Ty\| \leq \|x - y\|.$$  (3)

Nakajo-Takahashi [1] introduced the following hybrid method which is the so-called CQ-method for a nonexpansive mapping $T$ in a Hilbert space $H$:

$$x_0 \in C,$$

$$y_n = \alpha_n x_n + (1 - \alpha_n) Tx_n,$$

$$C_n = \{ z \in C : \|y_n - z\| \leq \|x_n - z\| \},$$

$$Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0 \},$$

$$x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,$$  (4)

where $\{\alpha_n\} \subset [0, 1]$ and $P_K$ is the metric projection from $H$ onto a closed and convex subset $K$ of $H$. They proved that the sequence $\{x_n\}$ generated by (4) converges strongly to a fixed point of $T$ under suitable conditions.

Takahashi et al. [2] introduced a new hybrid iterative scheme called the shrinking projection method for a nonexpansive mapping $T$ in a Hilbert space $H$ as follows:

$$x_0 \in H,$$

$$C_1 = C,$$  

$$x_1 = P_{C_1} x_0,$$  

$$x_n = \bar{T}_n \bar{C}_n x_0, \quad \forall n \geq 0,$$  (5)

where $\bar{T}_n = P_{C_n} T_{C_n}$ and $\bar{T}_n$ is the metric projection from $H$ onto a closed and convex subset $K$ of $H$. They proved that the sequence $\{x_n\}$ generated by (5) converges strongly to a fixed point of $T$ under suitable conditions.
\[ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \]
\[ C_{n+1} = \{ z \in C_n : \| y_n - z \| \leq \| x_n - z \| \}, \]
\[ x_{n+1} = P_{C_{n+1}} x, \quad \forall n \geq 1, \] (5)

where \( \{\alpha_n\} \subset [0, 1] \), and they proved that the sequence \( \{x_n\} \) generated by (5) converges strongly to a fixed point of \( T \) under suitable conditions.

In 2010, Reich and Sabach [3] introduced the following two hybrid iterative schemes for Bregman strongly nonexpansive mappings \( T_i : E \rightarrow E \) \( (i = 1, 2, \ldots, N) \) in a reflexive Banach space \( E \):

\[ x_0 \in E, \]
\[ y_n^i = T_i (x_n + e_n^i), \]
\[ C_n^i = \{ z \in E : D_f (z, y_n^i) \leq D_f (z, x_n + e_n^i) \}, \]
\[ C_n = \bigcap_{i=1}^{N} C_n^i, \]
\[ Q_n = \{ z \in E : \langle z - x_n, \nabla f (x_n) - \nabla f (x) \rangle \leq 0 \}, \]
\[ x_{n+1} = P_{C_n \cap Q_n} (x_n), \quad \forall n \geq 0, \]
\[ x_0 \in E, \]
\[ C_0^i = E, \quad i = 1, 2, \ldots, N, \]
\[ y_n^i = T_i (x_n + e_n^i), \]
\[ C_{n+1}^i = \{ z \in C_n^i : D_f (z, y_n^i) \leq D_f (z, x_n + e_n^i) \}, \]
\[ C_{n+1} = \bigcap_{i=1}^{N} C_{n+1}^i, \]
\[ x_{n+1} = P_{C_n \cap Q_n} (x_n), \quad \forall n \geq 0, \] (6) (7)

where \( P_{K}^f \) is the Bregman projection with respect to \( f \) from \( E \) onto a closed and convex subset \( K \) of \( E \). They proved that the sequence \( \{x_n\} \) generated by both (6) and (7) converges strongly to a common fixed point of \( \{T_i\}_{i=1}^{N} \).

The construction of fixed points for Bregman-type mappings via iterative processes has been investigated in, for example, [4–8].

In this paper, we design a new hybrid iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a countable family of Bregman asymptotically quasinonexpansive mappings in reflexive Banach spaces and prove some strong convergence theorems. Our results extend the recent one of Reich and Sabach [3].

\section{2. Preliminaries}

Let \( E \) be a real Banach space. For any \( x \in \text{int} \ dom \ f \) and \( y \in E \), we define the right-hand derivative of \( f \) at \( x \) in the direction \( y \) by

\[ f^0 (x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}, \] (8)

The function \( f \) is said to be Gâteaux differentiable at \( x \) if \( \lim_{t \to 0^+} ((f(x + ty) - f(x))/t) \) exists for any \( y \). In this case, \( f^0 (x, y) \) coincides with \( \nabla f (x) \), the value of the gradient \( \nabla f \) of \( f \) at \( x \). The function \( f \) is said to be Fréchet differentiable at \( x \) if this limit is attained uniformly in \( \| y \| = 1 \). Finally, \( f \) is said to be uniformly Fréchet differentiable on a subset \( C \) of \( E \) if the limit is attained uniformly for \( x \in C \) and \( \| y \| = 1 \).

Let \( E \) be a reflexive Banach space. The Legendre function is defined from a general Banach space \( E \) into \((-\infty, +\infty]\) (see [9]). According to [9], the function \( f \) is Legendre if and only if it satisfies the following conditions

\begin{enumerate}[(L1)]
  \item The interior of the domain of \( f \) (denoted by \( \text{int} \ dom \ f \)) is nonempty; \( f \) is Gâteaux differentiable on \( \text{int} \ dom \ f \) and \( \nabla f = \text{int} \ dom \ f \).
  \item The interior of the domain \( f^* \) (denoted by \( \text{int} \ dom \ f^* \)) is nonempty; \( f^* \) is Gâteaux differentiable on \( \text{int} \ dom \ f^* \) and \( \nabla f^* = \text{int} \ dom \ f^* \).
\end{enumerate}

Since \( E \) is reflexive, we always have \((\partial f)^{-1} = \partial f^* \) (see [10]). This fact, when combined with conditions (L1) and (L2), implies the following equalities:

\[ \nabla f = (\nabla f^*)^{-1}, \]
\[ \text{ran} \nabla f = \text{dom} \nabla f^* = \text{int} \ dom \ f^*, \]
\[ \text{ran} \nabla f^* = \text{dom} \ nabla f = \text{int} \ dom \ f. \] (9)

Also, conditions (L1) and (L2), in conjunction with [9], imply that the functions \( f \) and \( f^* \) are strictly convex on the interior of their respective domains. Several interesting examples of the Legendre functions are presented in [9, 11]. Especially, the functions \((1/s) \| \cdot \|^s\) with \( s \in (1, \infty) \) are Legendre, where the Banach space \( E \) is smooth and strictly convex and, in particular, a Hilbert space.

Let \( f : E \rightarrow (-\infty, +\infty] \) be a convex and Gâteaux differentiable function. The function \( D_f : \text{dom} \ f \times \text{int} \ dom \ f \rightarrow [0, +\infty) \) defined as

\[ D_f (x, y) := f (y) - f (x) - \langle y - x, \nabla f (x) \rangle \] (10)

is called the Bregman distance with respect to \( f \) [12].

By the definition, we know the following property (the three point identity): for any \( x \in \text{dom} \ f \) and \( y, z \in \text{int} \ dom \ f \),

\[ D_f (x, y) + D_f (y, z) - D_f (x, z) = \langle \nabla f (z) - \nabla f (y), x - y \rangle. \] (11)
Recall that the Bregman projection \([13]\) of \(x \in \text{int dom } f\) onto the nonempty, closed, and convex subset \(C\) of \(\text{dom } f\) is the necessarily unique vector \(\text{proj}_C^f(x) \in C\) satisfying
\[
D_f\left(\text{proj}_C^f(x), x\right) = \inf\{D_f(y, x) : y \in C\}. \tag{12}
\]

Let \(f : E \to (-\infty, +\infty]\) be a convex and Gâteaux differentiable function. The function \(f\) is said to be totally convex at \(x \in \text{int dom } f\) if its modulus of total convexity at \(x\), that is, the function \(v_f : \text{int dom } f \times [0, +\infty) \to [0, +\infty)\) defined by
\[
v_f(x, t) := \inf \{v_f(x, t) : x \in C \cap \text{dom } f\}. \tag{13}
\]
is positive whenever \(t > 0\). The function \(f\) is said to be totally convex when it is totally convex at every point \(x \in \text{int dom } f\). In addition, the function \(f\) is said to be totally convex on bounded sets if \(v_f(B, t)\) is positive for any nonempty bounded subset \(B\) of \(E\) and \(t > 0\), where the modulus of total convexity of the function \(f\) on the set \(B\) is the function \(v_f : \text{int dom } f \times [0, +\infty) \to [0, +\infty)\) defined by
\[
v_f(B, t) := \inf \{v_f(x, t) : x \in B \cap \text{dom } f\}. \tag{14}
\]

Some examples of the totally convex functions can be found in [14, 15].

Recall that the function \(f\) is said to be sequentially consistent [15] if, for any two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(E\) such that the first is bounded,
\[
\lim_{n \to \infty} D_f(y_n, x_n) = 0 \implies \lim_{n \to \infty} \|y_n - x_n\| = 0. \tag{15}
\]

Let \(C\) be a nonempty, closed, and convex subset of \(E\) and \(g : C \times C \to \mathbb{R}\) a bifunction that satisfies the following conditions:
\[(C1)\] \(g(x, x) = 0\) for all \(x \in C\);
\[(C2)\] \(g\) is monotone, that is, \(g(x, y) + g(y, x) \leq 0\) for all \(x, y \in C\);
\[(C3)\] \(\limsup_{t \downarrow 0} g(tz + (1 - t)x, y) \leq g(x, y)\) for all \(x, y, z \in C\);
\[(C4)\] for all \(x \in C\), \(g(x, \cdot)\) is convex and lower semicontinuous.

The equilibrium problem with respect to \(g\) is as follows: find \(x \in C\) such that
\[
g(\bar{x}, y) \geq 0, \quad \forall y \in C. \tag{16}
\]
The set of all solutions of \((16)\) is denoted by \(\text{EP}(g)\). The resolvent of a bifunction \(g : C \times C \to \mathbb{R}\) [16] is the operator \(\text{Res}_g^f : E \to 2^C\) denoted by
\[
\text{Res}_g^f(x) = \{z \in C : g(z, y) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \\
\geq 0, \quad \forall y \in C\}. \tag{17}
\]

For any \(x \in E\), there exists \(z \in C\) such that \(z = \text{Res}_g^f(x)\); see [3].

Let \(K\) be a convex subset of \(\text{int dom } f\) and \(T : K \to K\) a mapping. A point \(p\) in the closure of \(K\) is said to be an asymptotic fixed point of \(T\) [17, 18] if \(K\) contains a sequence \(\{x_n\}\) which converges weakly to \(p\) such that the strong \(\lim_{n \to \infty}(x_n - Tx_n) = 0\). The set of asymptotic fixed points of \(T\) will be denoted by \(\text{AEP}(T)\). The mapping \(T\) is called Bregman quasi-nonexpansive if \(\text{AEP}(T) \neq \emptyset\) and
\[
D_f(v, x) \leq D_f(v, x), \quad \forall v \in \text{AEP}(T), \quad x \in K. \tag{18}
\]

\(T\) is said to be Bregman (quasi)-strongly nonexpansive [6] with respect to a nonempty \(\text{AEP}(T)\) if
\[
D_f(p, Tx) \leq D_f(p, x), \tag{19}
\]
for all \(p \in \text{AEP}(T)\) and \(x \in K\), and if whenever \(\{x_n\} \subset K\) is bounded, \(p \in \text{AEP}(T)\), and
\[
\lim_{n \to \infty} (D_f(p, x_n) - D_f(p, Tx_n)) = 0, \tag{20}
\]
it follows that
\[
\lim_{n \to \infty} D_f(Tx_n, x_n) = 0. \tag{21}
\]
The mapping \(T\) is called Bregman firmly nonexpansive if
\[
\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \\
\leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle \tag{22}
\]
for all \(x, y \in K\).

Next, we introduce a new mapping that is called Bregman asymptotically quasinonexpansive mapping which is a natural extension of Bregman quasinonexpansive mapping introduced by Reich and Sabach [3]. The mapping \(T : K \to K\) is said to be Bregman asymptotically quasi-nonexpansive if there exists a sequence \(\{k_n\} \subset [1, \infty)\) satisfying \(\lim_{n \to \infty} k_n = 1\) such that, for every \(n \geq 1\),
\[
D_f(v, T^n x) \leq k_n D_f(v, x), \quad \forall v \in \text{AEP}(T), \quad x \in K. \tag{23}
\]
Obviously, every Bregman quasinonexpansive mapping is a Bregman asymptotically quasi-one with \(k_n = 1\).

Let \(E\) be a Banach space and \(C\) a nonempty subset of \(E\). The mapping \(T : C \to C\) is said to be uniformly asymptotically regular on \(C\) if
\[
\lim_{n \to \infty} \left(\sup_{x \in C} \|T^{n+1}x - T^n x\|\right) = 0. \tag{24}
\]
The mapping \(T\) is said to be closed if, for any sequence \(\{x_n\}\) in \(C\) such that \(\lim_{n \to \infty} x_n = x_0\) and \(\lim_{n \to \infty} Tx_n = y_0\), \(Tx_0 = y_0\).

The following is an important result which will be used in the next section.

**Lemma 1.** Let \(E\) be a reflexive Banach space and \(f : E \to (-\infty, +\infty)\) a Gâteaux differentiable and Legendre function...
which is totally convex on bounded sets. Let $K$ be a nonempty, closed and convex subset of $\intext{dom} f$ and $T : K \to K$ a closed Bergman asymptotically quasi-nonexpansive mapping with the sequence $\{k_n\} \subset [1, +\infty)$ such that $k_n \to 1$ as $n \to \infty$. Then $F(T)$ is closed and convex.

**Proof.** The closedness of $F(T)$ comes directly from the closedness of $T$. Now, for arbitrary $p_1, p_2 \in F(T)$, $t \in (0, 1)$, put $p_3 = tp_1 + (1 - t)p_2$. We prove that $T p_3 = p_3$. Indeed, from the definition of $D_f$, we see that

$$
\begin{align*}
D_f(p_3, T^np_3) &= f(p_3) - f(T^np_3) - \langle \nabla f(T^np_3), p_3 - T^np_3 \rangle \\
&= f(p_3) - f(T^np_3) - \langle \nabla f(T^np_3), tp_1 + (1 - t)p_2 - T^np_3 \rangle \\
&= f(p_3) - f(T^np_3) - t \langle \nabla f(T^np_3), p_1 - T^np_3 \rangle \\
&\quad - (1 - t) \langle \nabla f(T^np_3), p_2 - T^np_3 \rangle \\
&= f(p_3) - f(T^np_3) - t D_f(p_1, T^np_3) - (1 - t) f(p_2)
\end{align*}
$$

$$
\leq f(p_3) + t k_n D_f(p_1, p_3) + (1 - t) f(p_2)
\leq f(p_3) + t k_n D_f(p_2, p_3) + (1 - t) f(p_2)
\leq f(p_3).
$$

This implies that $\lim_{n \to \infty} D_f(p_3, T^np_3) = 0$. It follows from Lemma 3 below that

$$
\lim_{n \to \infty} \| p_3 - T^np_3 \| = 0,
$$

that is, $TT^np_3 - p_3 \to 0$ as $n \to \infty$. In view the closedness of $T$, we can obtain the desired conclusion. This completes the proof.

Finally, we state some lemmas that will be used in the proof of main results in next section.

**Lemma 2** (see [7]). If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of $E$, then $Vf$ is uniformly continuous on bounded subsets of $E$ from the strong topology of $E$ to the strong topology of $E^*$.

**Lemma 3** (see [14]). The function $f$ is totally convex on bounded sets if and only if it is sequentially consistent.

**Lemma 4** (see [15]). Suppose that $f$ is Gâteaux differentiable and totally convex on $\intext{dom} f$. Let $x \in \intext{dom} f$ and $C$ a nonempty, closed, and convex subset of $\intext{dom} f$. If $\hat{x} \in C$, then the following conditions are equivalent.

(i) The vector $\hat{x}$ is the Bregman projection of $x$ onto $C$ with respect to $f$.

(ii) The vector $\hat{x}$ is the unique solution of the variational inequality:

$$
\langle \nabla f(x) - \nabla f(z), z - y \rangle \geq 0, \quad \forall y \in C.
$$

(iii) The vector $\hat{x}$ is the unique solution of the inequality

$$
D_f(y, z) + D_f(z, x) \leq D_f(y, x), \quad \forall y \in C.
$$

**Lemma 5** (see [6]). Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}_{n=1}^{\infty}$ is bounded, then the sequence $\{x_n\}_{n=1}^{\infty}$ is also bounded.

**Lemma 6** (see [3]). Let $f : E \to (-\infty, +\infty)$ be a coercive (i.e., $\lim_{\|x\| \to \infty} f(x)/\|x\| = +\infty$) and Gâteaux differentiable function. Let $C$ be a closed and convex subset of $E$. If the bifunction $g : C \times C \to \mathbb{R}$ satisfies conditions (C1)–(C4), then

(1) $\text{Res}^f_g$ is single-valued;

(2) $\text{Res}^f_g$ is a Bregman firmly nonexpansive mapping;

(3) the set of fixed points of $\text{Res}^f_g$ is the solution set of the equilibrium problem, that is, $F(\text{Res}^f_g) = \text{EP}(g)$;

(4) $\text{EP}(g)$ is a closed and convex subset of $C$;

(5) for all $x \in E$ and $u \in F(\text{Res}^f_g)$, one has

$$
D_f(u, \text{Res}^f_g(x)) + D_f(\text{Res}^f_g(x), x) \leq D_f(u, x).
$$

3. Main Results

Now, we give our main theorems.
Theorem 7. Let $E$ be a reflexive Banach space and $f : E \to \mathbb{R}$ a coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of $E$. Let $K$ be a nonempty, closed, and convex subset of $E$ and $f : E \to \mathbb{R}$ a coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of $E$. Assume that each $T_i (i \geq 1)$ is uniformly asymptotically regular and $\Omega = \bigcap_{i=1}^{\infty} F(T_i) \cap \text{EP}(g)$ is nonempty and bounded. Let $\alpha_n$ be a real sequence in $(0,1)$ with $\sum_{i=1}^{n} \alpha_{ij} = 1$ for every $n \geq 1$ and $\lim_{n \to \infty} \alpha_{ij} > 0$ for every $i \geq 1$. Let $\{x_n\}$ be a sequence generated by the following manner:

$x_1 = x \in K$ chosen arbitrarily,

$$u_{ij} \in K \text{ such that}$$

$$g(u_{ij}, y) + \langle \nabla f (u_{ij}), y - u_{ij} \rangle \geq 0,$$

$$\forall y \in K, \ i = 1, \ldots, n,$$

$$C_n = \left\{ z \in K : \sum_{i=1}^{n} \alpha_{ij} D_f(z, u_{ij}) \leq D_f(z, x_n) + (k_n - 1) M_n \right\},$$

$$D_n = \bigcap_{i=1}^{n} C_i,$$

$$x_{n+1} = \text{proj}_{D_n}^f x, \ n = 1, 2, \ldots, \quad (30)$$

where $M_n = \sup\{D_f(v, x_n) : v \in \Omega\}$ for each $n \geq 1$. Then, $\{x_n\}$ defined by (30) converges strongly to $\text{proj}_{D_n}^f x$ as $n \to \infty$.

Proof. First, we prove that the sequence $\{x_n\}$ is well defined. Note that

$$\sum_{i=1}^{n} \alpha_{ij} D_f(z, u_{ij}) \leq D_f(z, x_n) + (k_n - 1) M_n \quad (31)$$

is

$$\sum_{i=1}^{n} \alpha_{ij} (f(z) - f(u_{ij})) + \langle \nabla f (u_{ij}), z - u_{ij} \rangle$$

$$\leq (f(z) - f(x_n)) + \langle \nabla f (x_n), z - x_n \rangle + (k_n - 1) M_n, \quad (32)$$

that is,

$$f(x_n) - \sum_{i=1}^{n} \alpha_{ij} f(u_{ij}) + \langle \nabla f (x_n), z - x_n \rangle$$

$$\leq \sum_{i=1}^{n} \alpha_{ij} \langle \nabla f (u_{ij}), z - u_{ij} \rangle + (k_n - 1) M_n. \quad (33)$$

This shows that $C_n$ is closed and convex for every $n \geq 1$. From the definition of $D_n$, it is easy to see that $D_n$ is closed and convex for every $n \geq 1$. For every $i \geq 1$ and $n \geq 1$, Lemma 6 shows that $u_{ij} = \text{Res}_{T_i}^f x_n$ and $D_f(v, \text{Res}_{T_i}^f y) \leq D_f(v, y)$ for any $v \in \Omega$ and $y \in E$. Hence,$$D_f(v, u_{ij}) = D_f(v, T_i^* x_n)$$

$$\leq k_{ij} D_f(v, x_n)$$

$$\leq k_{ij} D_f(v, x_n)$$

$$= D_f(v, x_n) + (k_n - 1) M_n,$$

$$\leq D_f(v, x_n) + (k_n - 1) M_n, \quad (34)$$

Since $\sum_{i=1}^{n} \alpha_{ij} = 1$ for every $n \geq 1$, we have

$$\sum_{i=1}^{n} \alpha_{ij} D_f(v, u_{ij})$$

$$\leq \sum_{i=1}^{n} \alpha_{ij} D_f(v, x_n) + (k_n - 1) M_n)$$

$$= D_f(v, x_n) + (k_n - 1) M_n,$$

$$\leq D_f(v, x_n) + (k_n - 1) M_n, \quad (35)$$

This shows that $v \in C_n$ for every $n \geq 1$. Thus $\Omega \subset C_n$ for every $n \geq 1$. Further, we have $\Omega \subset D_n$ for every $n \geq 1$. Thus the sequence $\{x_n\}$ is well defined.

From $\text{proj}_{D_n}^f x = x_{n+1}$, by Lemma 4(iii) we have

$$D_f(x_{n+1}, x) = D_f(\text{proj}_{D_n}^f x, x)$$

$$\leq D_f(v, x) - D_f(v, \text{proj}_{D_n}^f x)$$

$$\leq D_f(v, x) \quad (36)$$

for any $v \in \Omega$. Hence the sequence $D_f(x_n, x)$ is bounded. Therefore by Lemma 5 the sequence $\{x_n\}$ is bounded.

On the other hand, in view of $x_n = \text{proj}_{D_n}^f x$ and $x_{n+2} = \text{proj}_{D_{n+1}}^f x \in D_{n+1} \subset D_n$, from Lemma 4(iii) we have

$$D_f(x_{n+2}, \text{proj}_{D_n}^f x) + D_f(\text{proj}_{D_n}^f x, x) \leq D_f(x_{n+2}, x),$$

$$\text{that is},$$

$$D_f(x_{n+2}, x_{n+1}) \leq D_f(x_{n+2}, x). \quad (37)$$

Therefore the sequence $\{D_f(x_n, x)\}$ is increasing, and since it is also bounded, $\lim_{n \to \infty} D_f(x_n, x)$ exists. By the construction of $D_n$, we have that $D_m \subset D_n$ and $x_m = \text{proj}_{D_{m+1}}^f x \in D_{m+1} \subset D_{m-1}$ for any positive integer $m \geq n$. It follows that

$$D_f(x_m, x_m) = D_f(x_m, \text{proj}_{D_{m+1}}^f x)$$

$$\leq D_f(x_m) - D_f(\text{proj}_{D_{m+1}}^f x, x) \quad (38)$$

$$= D_f(x_m) - D_f(x, x).$$

(39)
Letting $m,n \to \infty$ in (39), we see that $D_f(x_m,x_n) \to 0$. It follows from Lemma 3 that $x_m - x_n \to 0$ as $m,n \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence. Since $E$ is a Banach space and $K$ is closed and convex, we can assume that
\[
\lim_{n \to \infty} x_n = x^* \in K.
\] (40)

By taking $m = n + 1$ in (39), we see that
\[
\lim_{n \to \infty} D_f(x_{n+1},x_n) = 0.
\] (41)

Lemma 3 implies that
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.
\] (42)

Since $x_{n+1} = \text{proj}_{D_n} x \in D_n \subset C_n$, we have
\[
\sum_{i=1}^n \alpha_{i,n} D_f(x_{n+1},u_{i,n}) \leq D_f(x_{n+1},x_n) + (k_n - 1) M_n.
\] (43)

Then (41) implies that
\[
\lim_{n \to \infty} \sum_{i=1}^n \alpha_{i,n} D_f(x_{n+1},u_{i,n}) = 0.
\] (44)

Note that $\alpha_{i,n} D_f(x_{n+1},u_{i,n}) \leq \sum_{i=1}^n \alpha_{i,n} D_f(x_{n+1},u_{i,n})$ and $\lim_{n \to \infty} \alpha_{i,n} > 0$, we have
\[
\lim_{n \to \infty} D_f(x_{n+1},u_{i,n}) = 0
\] (45)

for every $i \geq 1$. It follows from Lemma 3 that
\[
\lim_{n \to \infty} \|x_{n+1} - u_{i,n}\| = 0
\] (46)

for every $i \geq 1$. Note that
\[
\|u_{i,n} - x_n\| \leq \|u_{i,n} - x_{n+1}\| + \|x_{n+1} - x_n\|.
\] (47)

Combining (42) with (46), we see that
\[
\lim_{n \to \infty} \|u_{i,n} - x_n\| = 0
\] (48)

for every $i \geq 1$. This means that the sequence $\{u_{i,n}\}$ is bounded. Since $f$ is uniformly Fréchet differentiable, it follows from Lemma 2 that
\[
\lim_{n \to \infty} \|\nabla f(u_{i,n}) - \nabla f(x_n)\| = 0.
\] (49)

Since $f$ is uniformly Fréchet differentiable, it is also uniformly continuous (see [19, Theorem 1.8, p.13]) and therefore
\[
\lim_{n \to \infty} \|f(u_{i,n}) - f(x_n)\| = 0.
\] (50)

From the definition of the Bregman distance, we obtain that for every $v \in \Omega$. Since every sequence $\{u_{i,n}\}$ is bounded, $\{\nabla f(u_{i,n})\}$ is also bounded for every $i \geq 1$. Now from (48)–(50), we have
\[
\lim_{n \to \infty} D_f(v,x_n) - D_f(v,u_{i,n}) = 0
\] (52)

for any $v \in \Omega$ and for every $i \geq 1$.

In view of $u_{i,n} = \text{Res}_g^T_{\theta} T^n_{\theta} x_n$, by Lemma 6 (5) we have
\[
D_f(u_{i,n},T^n_{\theta} x_n) = D_f(\text{Res}_g^T_{\theta} T^n_{\theta} x_n, T^n_{\theta} x_n)
\]
\[
\leq D_f(v,T^n_{\theta} x_n) - D_f(v,\text{Res}_g^T_{\theta} T^n_{\theta} x_n)
\]
\[
\leq k_n D_f(v,x_n) - D_f(v,\text{Res}_g^T_{\theta} T^n_{\theta} x_n)
\]
\[
\leq D_f(v,x_n) + (k_n - 1) M_n - D_f(v,u_{i,n})
\]

(53)

Note that $M_n$ is bounded and $k_n \to 1$ as $n \to \infty$. It follows from (52) that
\[
\lim_{n \to \infty} D_f(u_{i,n},T^n_{\theta} x_n) = 0
\] (54)

for every $i \geq 1$. Lemma 3 shows that
\[
\lim_{n \to \infty} \|u_{i,n} - T^n_{\theta} x_n\| = 0.
\] (55)

Note that $\|T^n_{\theta} x_n - x_n\| \leq \|T^n_{\theta} x_n - u_{i,n}\| + \|u_{i,n} - x_n\|$. From (48) and (55) we get
\[
\lim_{n \to \infty} \|T^n_{\theta} x_n - x_n\| = 0
\] (56)

for every $i \geq 1$. Note that
\[
\|T^n_{\theta} x_n - x^*\| \leq \|T^n_{\theta} x_n - x_n\| + \|x_n - x^*\|.
\] (57)

It follows from (40) and (56) that
\[
\lim_{n \to \infty} \|T^n_{\theta} x_n - x^*\| = 0
\] (58)

for every $i \geq 1$. On the other hand, we have
\[
\|T^{n+1}_{\theta} x_n - x^*\| \leq \|T^{n+1}_{\theta} x_n - T^n_{\theta} x_n\| + \|T^n_{\theta} x_n - x^*\|.
\] (59)
Since every $T_i$ is uniformly asymptotically regular and (58), we obtain that, for every $i \geq 1,$
\[
\lim_{n \to \infty} \left\| T_i^{n+1} x_n - x^* \right\| = 0, \tag{60}
\]
that is, $T_i^{n+1} x_n \to x^*$ as $n \to \infty.$ From the closedness of $T_i$, we see that $x^* \in F(T_i)$ for every $i \geq 1$. Thus $x^* \in \bigcap_{i=1}^{\infty} F(T_i).
$

Next we prove that $x^* \in EP(g)$ for every $i \geq 1.$ Since $f$ is uniformly Fréchet differentiable, $\nabla f$ is uniformly continuous. Thus, by (55) we have
\[
\lim_{n \to \infty} (\nabla f (u_{i,n}) - \nabla f (T_i^n x_n)) = 0. \tag{61}
\]
Since $u_{i,n} = \text{Res}_{T_i^n} f x_n,$ we have
\[
g (u_{i,n}, y) + \langle \nabla f (u_{i,n}) - \nabla f (T_i^n x_n), y - u_{i,n} \rangle 
\geq 0, \quad \forall y \in K. \tag{62}
\]
We have from (C3) that
\[
\langle \nabla f (u_{i,n}) - \nabla f (T_i^n x_n), y - u_{i,n} \rangle 
\geq -g (u_{i,n}, y) \tag{63}
\geq g (y, u_{i,n}), \quad \forall y \in K.
\]
Letting $n \to \infty,$ we have from (61) and (C4) that
\[
g (y, x^*) \leq 0, \quad \forall y \in K. \tag{64}
\]
For $t$ with $0 < t \leq 1$ and $y \in K,$ let $y_i = ty_i + (1-t) x^*.$ Since $y_i \in K$ and $x^* \in K,$ we have $y_i \in K$ and hence $g (y_i, x^*) \leq 0.$ So, from (C1) we have
\[
0 = g (y_i, y_i) 
\leq t g (y, y) + (1-t) g (y, x^*) \tag{65}
\leq t g (y, y).
\]
Dividing by $t,$ we have
\[
g (y, y) \geq 0, \quad \forall y \in K. \tag{66}
\]
Letting $t \downarrow 0,$ from (C3) we have
\[
g (x^*, y) \geq 0, \quad \forall y \in K. \tag{67}
\]
Therefore, $x^* \in EP(g).$ Thus $x^* \in \bigcap_{i=1}^{\infty} EP(g).
$

Finally, we show that $x^* = \text{proj}_{D_n} x.$ Since $\Omega \subset D_n$ for every $n \geq 1,$ by Lemma 4(ii) we arrive at
\[
\langle x_n - v, \nabla f (x) - \nabla f (x_n) \rangle \geq 0, \quad \forall v \in \Omega. \tag{68}
\]
Taking the limit as $n \to \infty$ in (68), we obtain that
\[
\langle x^* - v, \nabla f (x) - \nabla f (x^*) \rangle \geq 0, \quad \forall v \in \Omega \tag{69}
\]
and hence $x^* = \text{proj}_{\Omega} x$ by Lemma 4(ii). This completes the proof.

**Corollary 8.** Let $E$ be a reflexive Banach space and $f : E \to \mathbb{R}$ a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of $E.$ Let $K$ be a nonempty, closed, and convex subset of $\text{int dom } f$ and $T : K \to K$ a closed Bregman asymptotically quasi-nonexpansive mapping with the sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n \to \infty} k_n = 1.$ Let $g : K \times K \to \mathbb{R}$ be a bifunction satisfying conditions (C1)–(C4). Assume that $T$ is uniformly asymptotically regular and $\Omega = F(T) \cap EP(g)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by the following manner:
\[
x \in K \text{ chosen arbitrarily,}
\]
\[
u_n \in K \text{ such that}
\]
\[
g (u_{i,n}, y) + \langle \nabla f (u_{i,n}) - \nabla f (T_i^n x_n), y - u_{i,n} \rangle 
\geq 0, \quad \forall y \in K, \quad i = 1, \ldots, n,
\]
\[
C_n = \left\{ z \in K : \sum_{i=1}^{n} \alpha_{i,n} D_f (z, u_{i,n}) \leq D_f (z, x_n) \right\}, \tag{70}
\]
where $M_n = \sup \{D_f (v, x_n) : v \in \Omega\}$ for each $n \geq 1.$ Then, $\{x_n\}$ defined by (70) converges strongly to $\text{proj}_{\Omega} x$ as $n \to \infty.$

Since every Bregman quasi-nonexpansive mapping is Bregman quasi-asymptotically nonexpansive, we have the following results.

**Corollary 9.** Let $E$ be a reflexive Banach space and let $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of $E.$ Let $K$ be a nonempty, closed, and convex subset of $\text{int dom } f.$ Let $\{T_i\}_{i=1}^{\infty} : K \to K$ be a countable family of closed Bregman quasi-nonexpansive mappings and $g : K \times K \to \mathbb{R}$ a bifunction satisfying conditions (C1)–(C4). Assume that $\Omega = \bigcap_{i=1}^{\infty} F(T_i) \cap EP(g) \neq \emptyset.$ Let $\{\alpha_{i,n}\}$ be a real sequence in $(0, 1)$ with $\sum_{i=1}^{\infty} \alpha_{i,n} = 1$ and $\lim inf_{n \to \infty} \alpha_{i,n} > 0$ for every $i \geq 1.$ Let $\{x_n\}$ be a sequence generated by the following manner:
\[
x \in K \text{ chosen arbitrarily,}
\]
\[
u_n \in K \text{ such that}
\]
\[
g (u_{i,n}, y) + \langle \nabla f (u_{i,n}) - \nabla f (T_i^n x_n), y - u_{i,n} \rangle 
\geq 0, \quad \forall y \in K, \quad i = 1, \ldots, n,
\]
\[
C_n = \left\{ z \in K : \sum_{i=1}^{n} \alpha_{i,n} D_f (z, u_{i,n}) \leq D_f (z, x_n) \right\}, \tag{70}
\]
where $M_n = \sup \{D_f (v, x_n) : v \in \Omega\}$ for each $n \geq 1.$ Then, $\{x_n\}$ defined by (70) converges strongly to $\text{proj}_{\Omega} x$ as $n \to \infty.$
Abstract and Applied Analysis

\[ D_n = \bigcap_{i=1}^{n} C_i, \]
\[ x_{n+1} = \text{proj}^f_{D_n} x, \quad n = 1, 2, \ldots. \]  
(71)

Then, \( \{x_n\} \) defined by (71) converges strongly to \( \text{proj}^f_{\Omega} x \) as \( n \to \infty. \)

**Corollary 10.** Let \( E \) be a reflexive Banach space and let \( f : E \to \mathbb{R} \) be a coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of \( E. \) Let \( K \) be a nonempty, closed, and convex subset of \( \text{int dom } f. \) Let \( T : K \to K \) be a closed Bregman quasi-nonexpansive mapping and \( g : K \times K \to \mathbb{R} \) a bifunction satisfying conditions (C1)–(C4). Assume that \( \Omega = F(T) \cap \text{EP}(g) \neq \emptyset. \) Let \( \{x_n\} \) be a sequence generated by the following manner:

\[ x \in K \text{ chosen arbitrarily}, \]
\[ u_n \in K \text{ such that} \]
\[ g(u_n, y) + \langle \nabla f(u_n) - \nabla f(T^n x_n), y - u_n \rangle \geq 0, \quad \forall y \in K, \]
\[ C_n = \{ z \in K : D_f(z, u_n) \leq D_f(z, x_n) \}, \]
\[ D_n = \bigcap_{i=1}^{n} C_i, \]
\[ x_{n+1} = \text{proj}^f_{D_n} x, \quad n = 1, 2, \ldots. \]  
(72)

Then, \( \{x_n\} \) defined by (72) converges strongly to \( \text{proj}^f_{\Omega} x \) as \( n \to \infty. \)

**Remark 11.** Set \( \alpha_{nj} = 1/(i+1) + 1/n(n+1) \) for each \( n \geq 1 \) and \( i = 1, 2, \ldots, n \) and \( k_{in} = 1 + 1/in \) for each \( n \geq 1 \) and \( i \geq 1. \) Then \( \Sigma_{j=1}^{n} \alpha_{nj} = 1 \) and \( \lim_{n \to \infty} \alpha_{nj} = 1/(i+1) > 0. \) Also, \( k_{in} = \sup\{k_{ij} : i \geq 1\} = 1 \) for every \( n \geq 1. \) Hence, \( \{\alpha_{nj}\} \) and \( \{k_{in}\} \) satisfy the conditions of Theorem 7.

**Remark 12.** It needs to notice that Corollaries 9 and 10 still hold if we replace the closeness of the mappings with \( F(T) = F(T). \)

Proof. Let \( \{x_n\} \subset E \) converge to \( x^* \) and \( \{\text{Res}_g^j x_n\} \) to \( \tilde{x}. \) To end the conclusion, we need to prove that \( \text{Res}_g^j x^* = \tilde{x}. \) Indeed, for each \( x_n, \) Lemma 6 shows that there exists a unique \( z_n \in C \) such that \( z_n = \text{Res}_g^j x_n, \) that is,

\[ g(z_n, y) + \langle \nabla f(z_n), y - z_n \rangle \geq 0, \quad \forall y \in C. \]

(73)

Since \( f \) is uniformly Fréchet differentiable, \( \nabla f \) is uniformly continuous. So, taking the limit as \( n \to \infty \) in (73), by using (C3') we get

\[ g(\tilde{x}, y) + \langle \nabla f(\tilde{x}), y - \tilde{x} \rangle \geq 0, \quad \forall y \in C, \]  
(74)

which implies that \( \text{Res}_g^j x^* = \tilde{x}. \) This completes the proof. \( \square \)

If the bifunction \( g \) satisfies conditions (C1), (C2), (C3'), and (C4) instead of (C1)–(C4), then we have a simple method to prove that \( x^* \in \text{EP}(g) \) in the proof of Theorem 7. Indeed, from the proof of Theorem 7, we see that

\[ u_{i,n} - T^i_n x_n \to 0, \quad \text{that is, } \text{Res}_g^{i,n} x_n \to 0 \quad \text{as } n \to \infty, \quad \forall i \geq 1. \]

(75)

Note that \( x_n \to x^* \) as \( n \to \infty. \) This shows that \( T^i_n x_n \to x^* \) as \( n \to \infty \) for every \( i \geq 1. \) It follows from the closeness of \( \text{Res}_g^j \) that \( x^* \in \text{EP}(g). \) Lemma 6 shows that \( x^* \in \text{EP}(g). \)

**Remark 14.** Obviously, the proof process of \( x^* \in \text{EP}(g) \) is simple if we replace condition (C3) with (C3') which is such that \( \text{Res}_g^j \) is closed. In fact, although condition (C3') is stronger than (C3), it is not easier to verify condition (C3) than to verify the condition (C3'). Hence, from this viewpoint, the condition (C3') is acceptable.

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**References**


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