Research Article

On General Integral Operator of Analytic Functions

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Let E_β be the integral operator defined by $E_\beta(z) = [\beta \int_0^z t^{\beta-1} (f_1'(t))^{\alpha_1} (f_1(t)/t)^{\gamma_1} P_1^{\zeta_1(t)} \cdots (f_n'(t))^{\alpha_n} (f_n(t)/t)^{\gamma_n} P_n^{\zeta_n(t)} dt]^{1/\beta}$, where each of the functions f_i and P_i is, respectively, analytic functions and functions with positive real part defined in the open unit disk for all $i = 1, \dots, n$. The object of this paper is to obtain several univalence conditions for this integral operator. Our main results contain some interesting corollaries as special cases.

1. Introduction and Definitions

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} : |z| < 1\}$; $S = \{f \in A : f \text{ is univalent in } U\}$. Also, let $\mathcal{P}$ be the class of all functions which are analytic in U and satisfy $P(0) = 1$, $\Re\{P(z)\} > 0$. Frasin and Darus [1] defined the family $B(\delta)$, $0 \leq \delta < 1$, so that it consists of functions $f \in A$ satisfying the condition

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - \delta \quad (z \in U). \quad (2)$$

In this paper, we obtain new sufficient conditions for the univalence of the general integral operator $E_\beta(z)$ defined by

$$E_\beta(z) = \left[\beta \int_0^z t^{\beta-1} \prod_{i=0}^n (f_i'(t))^{\alpha_i} \left(\frac{f_i(t)}{t} \right)^{\gamma_i} P_i^{\zeta_i(t)} dt \right]^{1/\beta}, \quad (3)$$

where $\beta \in \mathbb{C}$, $\alpha_i, \gamma_i, \zeta_i \in \mathbb{C}$, $f_i \in A$, and $P_i \in \mathcal{P}$ for all $i = 1, 2, 3, \dots, n$.

Here and throughout in the sequel, every multivalued functions is taken with the principal branch.

Remark 1. Note that the integral operator E_β generalizes the following operators introduced and studied by several authors as follows.

- (i) For $\alpha_i = 0$, where $i = 1, \dots, n$, we obtain the integral operator

$$N_\beta(z) = \left[\beta \int_0^z t^{\beta-1} \prod_{i=0}^n \left(\frac{f_i(t)}{t} \right)^{\gamma_i} P_i^{\zeta_i(t)} dt \right]^{1/\beta}, \quad (4)$$

introduced and studied by Frasin [2].

- (ii) For $\alpha_i = \gamma_i = 0$, $i = 1, \dots, n$, we obtain the integral operator

$$\mathcal{J}_\beta^\zeta(P_i)(z) = \left[\beta \int_0^z t^{\beta-1} \prod_{i=0}^n P_i^{\zeta_i(t)} dt \right]^{1/\beta}, \quad (5)$$

introduced and studied by Frasin [3].

- (iii) For $\zeta_i = 0$, $i = 1, \dots, n$, we obtain the integral operator

$$\begin{aligned} \mathcal{J}_\beta^{\alpha_i, \gamma_i}(f_1, f_2, \dots, f_n)(z) \\ = \left[\beta \int_0^z t^{\beta-1} \prod_{i=0}^n (f_i'(t))^{\alpha_i} \left(\frac{f_i(t)}{t} \right)^{\gamma_i} dt \right]^{1/\beta}, \end{aligned} \quad (6)$$

introduced and studied by Frasin [4].

- (iv) For $\zeta_i = 0$, $\beta = 1$, and $i = 1, \dots, n$, we obtain the integral operator

$$\mathcal{J}^{\alpha_i, \gamma_i}(f_1, f_2, \dots, f_n)(z) = \int_0^z \prod_{i=0}^n (f_i'(t))^{\alpha_i} \left(\frac{f_i(t)}{t} \right)^{\gamma_i} dt, \quad (7)$$

introduced and studied by Frasin [5].

- (v) For $\zeta_i = \alpha_i = 0$ and $i = 1, \dots, n$, we obtain the integral operator

$$\mathcal{J}_\gamma^\beta(f_1, f_2, \dots, f_n)(z) = \left[\beta \int_0^z t^{\beta-1} \left(\frac{f_1(t)}{t} \right)^{\gamma_1} \dots \left(\frac{f_n(t)}{t} \right)^{\gamma_n} dt \right]^{1/\beta}, \quad (8)$$

introduced and studied by D. Breaz and N. Breaz [6].

- (vi) For $\beta = 1$, $\zeta_i = \alpha_i = 0$, and $i = 1, \dots, n$, we obtain the integral operator

$$F(z) = \int_0^z \left(\frac{f_1(t)}{t} \right)^{\gamma_1} \dots \left(\frac{f_n(t)}{t} \right)^{\gamma_n} dt, \quad (9)$$

introduced and studied by D. Breaz and N. Breaz [6].

- (vii) For $\beta = 1$, $\zeta_i = \gamma_i = 0$, and $i = 1, \dots, n$, we obtain the integral operator

$$F_{\alpha_1, \dots, \alpha_n}(z) = \int_0^z (f_1'(t))^{\alpha_1} \dots (f_n'(t))^{\alpha_n} dt, \quad (10)$$

introduced and studied by Breaz et al. [7].

- (viii) For $\beta = 1$, $n = 1$, $\gamma_1 = \gamma$, and $\alpha_i = \zeta_i = 0$, we obtain the integral operator

$$F_{\alpha, \gamma}(z) = \int_0^z \left(\frac{f(t)}{t} \right)^\gamma (f'(t))^\alpha dt, \quad (11)$$

studied in [8].

- (ix) For $\beta = 1$, $n = 1$, $\gamma_1 = \gamma$, and $\alpha_i = \zeta_i = 0$, we obtain the integral operator

$$F_\gamma(z) = \int_0^z \left(\frac{f(t)}{t} \right)^\gamma dt \quad (12)$$

studied in [9]. In particular, for $\gamma = 1$, we obtain Alexander integral operator which was introduced in [10] as follows

$$F(z) = \int_0^z \frac{f(t)}{t} dt. \quad (13)$$

- (x) For $\beta = 1$, $n = 1$, $\alpha_i = \alpha$, and $\gamma_i = \zeta_i = 0$, we obtain the integral operator

$$G(z) = \int_0^z (f'(t))^\alpha dt, \quad (14)$$

studied in [11].

In order to derive our main results, we have to recall here the following lemmas.

Lemma 2 (see [12]). Let $\eta \in \mathcal{C}$ with $\Re(\eta) > 0$. If $h \in A$ satisfies

$$\frac{1 - |z|^{2\Re(\eta)}}{\Re(\eta)} \left| \frac{zh''(z)}{h'(z)} \right| \leq 1 \quad (15)$$

for all $z \in U$, then the integral operator

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} (f'(t)) dt \right]^{1/\beta} \quad (16)$$

is in the class $\mathcal{S}$.

Lemma 3 (see [13]). Let $\beta \in \mathcal{C}$ with $\Re(\beta) > 0$, $c \in \mathcal{C}$ with $|c| \leq 1$, $c \neq -1$. If $h \in A$ satisfies

$$\left| c|z|^{2\beta} + (1 + |z|^{2\beta}) \frac{zh''(z)}{\beta h'(z)} \right| \leq 1 \quad (17)$$

for all $z \in U$, then the integral operator $F_\beta(z)$ defined by (16) is in the class $\mathcal{S}$.

Lemma 4 (see [14]). If $P(z) \in \mathcal{P}$, then

$$\left| \frac{zP''(z)}{\beta P'(z)} \right| \leq \frac{2|z|}{1 - |z|^2}. \quad (18)$$

Lemma 5 (see [9]). If $f(z) \in B(\delta)$, then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{(1 - \delta)(1 + |z|)}{1 - |z|}. \quad (19)$$

When $\delta = 0$, so $f \in \mathcal{S}$.

Lemma 6 (see [9]). If $f(z) \in B(\delta)$, then

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{(1 - \delta)(2 + |z|)}{1 - |z|}. \quad (20)$$

Also we need the following general Schwarz lemma.

Lemma 7 (see [15]). Let the function f be regular in the disk $U_R = \{z : |z| < R\}$, with $|M| < R$ for fixed M . If $f(z)$ has one zero with multiplicity order bigger than m for $z = 0$, then

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \quad (z \in U_R). \quad (21)$$

The equality holds only if

$$f(z) = e^{i\theta} \left(\frac{M}{R^m} \right) z^m, \quad (22)$$

where θ is constant.

Lemma 8 (see [16]). If $f \in A$, then

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| \leq \frac{5}{4} \left| \frac{zf'(z)}{f(z)} \right|. \quad (23)$$

2. Univalence Conditions for the Operator E_β

We first prove the following theorem.

Theorem 9. Let $f \in B(\delta_i)$, $0 \leq \delta_i < 1$, $P_i(z) \in \mathcal{P}$ for all $i = 1, \dots, n$ and $\eta \in \mathcal{C}$ with $\Re(\eta) = a > 0$. If

$$\sum_{i=1}^n [(3|\alpha_i| + 2|\gamma_i|)(1 - \delta_i) + 2|\zeta_i|] < \min \left\{ a; \frac{1}{2} \right\}, \quad (24)$$

then the integral operator E_β defined by (3) is in the class $\mathcal{S}$.

Proof. Define the regular function $h(z)$ by

$$h(z) = \int_0^z \prod_{i=1}^n \left[(f'_i(t))^{\alpha_i} \left(\frac{f_i(t)}{t} \right)^{\gamma_i} P_i^{\zeta_i(t)} \right] dt. \quad (25)$$

Then it is easy to see that

$$h'(z) = \prod_{i=1}^n \left[(f'_i(t))^{\alpha_i} \left(\frac{f_i(t)}{t} \right)^{\gamma_i} P_i^{\zeta_i(t)} \right], \quad (26)$$

and $h(0) = 1 - h'(0) = 0$. Differentiating both sides of (26) logarithmically, we obtain

$$\begin{aligned} \frac{zh''(z)}{h'(z)} &= \sum_{i=1}^n \alpha_i \left(\frac{zf''_i(z)}{f'_i(z)} \right) + \sum_{i=1}^n \gamma_i \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right) \\ &\quad + \sum_{i=1}^n \zeta_i \left(\frac{zP'_i(z)}{P_i(z)} \right). \end{aligned} \quad (27)$$

Thus, we have

$$\begin{aligned} \left| \frac{zh''(z)}{h'(z)} \right| &\leq \sum_{i=1}^n |\alpha_i| \left| \frac{zf''_i(z)}{f'_i(z)} \right| + \sum_{i=1}^n |\gamma_i| \left| \frac{zf'_i(z)}{f_i(z)} - 1 \right| \\ &\quad + \sum_{i=1}^n |\zeta_i| \left| \frac{zP'_i(z)}{P_i(z)} \right|. \end{aligned} \quad (28)$$

Since $f_i \in B(\delta_i)$, $P_i(z) \in \mathcal{P}$ for all $i = 1, \dots, n$, from (28), (18), (19), and (20), we obtain

$$\begin{aligned} \left| \frac{zh''(z)}{h'(z)} \right| &\leq \sum_{i=1}^n |\alpha_i| (1 - \delta_i) \left(\frac{2 + |z|}{1 - |z|} \right) \\ &\quad + \sum_{i=1}^n |\gamma_i| (1 - \delta_i) \left(\frac{1 + |z|}{1 - |z|} \right) + \sum_{i=1}^n |\zeta_i| \left(\frac{2|z|}{1 - |z|^2} \right) \\ &\leq \sum_{i=1}^n |\alpha_i| (1 - \delta_i) \left(\frac{3}{1 - |z|} \right) \\ &\quad + \sum_{i=1}^n |\gamma_i| (1 - \delta_i) \left(\frac{2}{1 - |z|} \right) + \sum_{i=1}^n |\zeta_i| \left(\frac{2}{1 - |z|} \right). \end{aligned} \quad (29)$$

Multiplying both sides of (29) by $(1 - |z|^{2\Re(\eta)})/\Re(\eta)$, we get

$$\begin{aligned} \frac{1 - |z|^{2\Re(\eta)}}{\Re(\eta)} \left| \frac{zh''(z)}{h'(z)} \right| &\leq \frac{1 - |z|^{2\Re(\eta)}}{(1 - |z|) \Re(\eta)} \\ &\quad \times \sum_{i=1}^n [(3|\alpha_i| + 2|\gamma_i|)(1 - \delta_i) + 2|\zeta_i|], \end{aligned} \quad (30)$$

for all $z \in U$.

Let us denote $|z| = x$, $x \in [0, 1)$, $\Re(\eta) = a > 0$, and $\Phi(x) = (1 - x^{2a})/(1 - x)$. It is easy to prove that

$$\Phi(x) \leq \begin{cases} 1, & 0 < a < 1, \\ 2a, & \frac{1}{2} < a < \infty. \end{cases} \quad (31)$$

From (31), (30), and the hypotheses (24), we have

$$\begin{aligned} \frac{1 - |z|^{2a}}{a} \left| \frac{zh''(z)}{h'(z)} \right| &\leq \begin{cases} \frac{1}{a} \sum_{i=1}^n [(3|\alpha_i| + 2|\gamma_i|)(1 - \delta_i) + 2|\zeta_i|], & 0 < a < 1, \\ 2 \sum_{i=1}^n [(3|\alpha_i| + 2|\gamma_i|)(1 - \delta_i) + 2|\zeta_i|], & \frac{1}{2} < a < \infty \end{cases} \\ &\leq 1, \end{aligned} \quad (32)$$

for all $z \in U$. Applying Lemma 2 for the function $h(z)$, we prove that $E_\beta(z) \in \mathcal{S}$. $\square$

Letting $n = 1$, $\delta_1 = \delta$, $\alpha_1 = \alpha$, $\gamma_1 = \gamma$, and $\zeta_1 = \zeta$ in Theorem 9, we obtain the following corollary.

Corollary 10. Let $f \in B(\delta)$, $0 \leq \delta < 1$, $P(z) \in \mathcal{P}$, and all $\eta, \gamma, \zeta \in \mathcal{C}$ with $\Re(\eta) = a > 0$. If

$$(3|\alpha| + 2|\gamma|)(1 - \delta) + 2|\zeta| < \min \left\{ a; \frac{1}{2} \right\}, \quad (33)$$

and then the integral operator $N_{\beta}^{\alpha, \gamma, \zeta}$ defined by

$$N_{\beta}^{\alpha, \gamma, \zeta}(z) = \left[\beta \int_0^z t^{\beta-1} (f'(t))^{\alpha} \left(\frac{f(t)}{t} \right)^{\gamma} P^{\zeta}(t) dt \right]^{1/\beta} \quad (34)$$

is in the class $\mathcal{S}$.

If we set $\delta = 0$ in Corollary 10, we have the following.

Corollary 11. Let $f \in \mathcal{S}$, $P(z) \in \mathcal{P}$ and all $\eta, \gamma, \zeta \in \mathcal{C}$ with $\Re(\eta) = a > 0$. If

$$3|\alpha| + 2|\gamma| + 2|\zeta| < \min \left\{ a; \frac{1}{2} \right\}, \quad (35)$$

then the integral operator $N_{\beta}^{\alpha, \gamma, \zeta}$ defined by (34) is in the class $\mathcal{S}$.

Next, we prove the following theorem.

Theorem 12. Let $\alpha_i, \gamma_i, \zeta_i \in \mathcal{C}$ for all $i = 1, \dots, n$ and each $f_i \in A$ satisfies condition (15) with $\Re(f_i(z)/z) > 0$ and

$$\sum_{i=1}^n [29|\alpha_i| + 16|\gamma_i + \zeta_i|] \leq \begin{cases} 4\Re(\eta), & \text{if } \Re(\eta) \in (0, 1), \\ 4, & \text{if } \Re(\eta) \in [1, \infty) \end{cases} \quad (36)$$

and then, for any complex number σ with $\Re(\sigma) \geq \Re(\eta) > 0$, the integral operator N_β defined by (4) is in the class $\mathcal{S}$.

Proof. Suppose that $\Re(f_i(z)/z)$ for all $i = 1, \dots, n$. Thus we have

$$\frac{f_i(z)}{z} = P_i(z), \quad (37)$$

where $P_i \in \mathcal{P}$ for all $i = 1, \dots, n$. Differentiating both sides of (37) logarithmically, we obtain

$$\frac{zf'_i(z)}{f_i(z)} - 1 = \frac{zP'_i(z)}{P_i(z)}. \quad (38)$$

Define the regular function $h(z)$ as in (25). Thus from (27) we have

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^n \alpha_i \left(\frac{zf''_i(z)}{f'_i(z)} \right) + \sum_{i=1}^n (\gamma_i + \zeta_i) \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right) \quad (39)$$

and so

$$\begin{aligned} \frac{zh''(z)}{h'(z)} &= \sum_{i=1}^n \alpha_i \left(\frac{zf''_i(z)}{f'_i(z)} + 1 \right) - \sum_{n=i}^0 \alpha_i \\ &\quad + \sum_{i=1}^n (\gamma_i + \zeta_i) \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right). \end{aligned} \quad (40)$$

From Lemma 8, it follows that

$$\begin{aligned} \left| \frac{zh''(z)}{h'(z)} \right| &\leq \frac{5}{4} \sum_{i=1}^n |\alpha_i| \left| \frac{zf'_i(z)}{f_i(z)} \right| \\ &\quad + \sum_{i=1}^n |\gamma_i + \zeta_i| \left| \frac{zf'_i(z)}{f_i(z)} - 1 \right| + \sum_{n=i}^0 |\alpha_i| \\ &\leq \frac{5}{4} \sum_{i=1}^n |\alpha_i| \left[\left| \frac{zf'_i(z)}{f_i(z)} - 1 \right| + 1 \right] \\ &\quad + \sum_{i=1}^n |\gamma_i + \zeta_i| \left| \frac{zf'_i(z)}{f_i(z)} - 1 \right| + \sum_{n=i}^0 |\alpha_i| \\ &\leq \sum_{i=1}^n \left[\left(\frac{5}{4} |\alpha_i| + |\gamma_i + \zeta_i| \right) \left| \frac{zf'_i(z)}{f_i(z)} - 1 \right| \right] \\ &\quad + \frac{9}{4} \sum_{i=1}^n |\alpha_i| \end{aligned}$$

$$\begin{aligned} &\leq \sum_{i=1}^n \left[\left(\frac{5}{4} |\alpha_i| + |\gamma_i + \zeta_i| \right) \left(\left| \frac{zf'_i(z)}{f_i(z)} \right| + 1 \right) \right] \\ &\quad + \frac{9}{4} \sum_{i=1}^n |\alpha_i|. \end{aligned} \quad (41)$$

Multiplying both sides of (41) by $(1 - |z|^{2\Re(\eta)})/\Re(\eta)$, from Lemma 5 with $\delta = 0$, we get

$$\begin{aligned} &\frac{1 - |z|^{2\Re(\eta)}}{\Re(\eta)} \left| \frac{zh''(z)}{h'(z)} \right| \\ &\leq \frac{1 - |z|^{2\Re(\eta)}}{\Re(\eta)} \sum_{i=1}^n \left(\frac{5}{4} |\alpha_i| + |\gamma_i + \zeta_i| \right) \left(\frac{2}{1 - |z|} \right) \\ &\quad + \frac{9(1 - |z|^{2\Re(\eta)}) \sum_{i=1}^n |\alpha_i|}{4\Re(\eta)}. \end{aligned} \quad (42)$$

Suppose that $\Re(\eta) \in (0, 1)$. Define a function $\Phi : (0, 1) \rightarrow R$ by

$$\Phi(x) = 1 - a^{2a} \quad (0 < a < 1). \quad (43)$$

Then Φ is an increasing function and consequently, for $|z| = a; z \in U$, we obtain

$$1 - |z|^{2\Re(\eta)} < 1 - |z|^2. \quad (44)$$

We thus find from (42) and (44) that

$$\begin{aligned} \frac{1 - |z|^{2\Re(\eta)}}{\Re(\eta)} \left| \frac{zh''(z)}{h'(z)} \right| &\leq \frac{\sum_{i=1}^n (5|\alpha_i| + 4|\gamma_i + \zeta_i|)}{\Re(\eta)} \\ &\quad + \frac{9 \sum_{i=1}^n |\alpha_i|}{4\Re(\eta)} \\ &= \frac{\sum_{i=1}^n (29|\alpha_i| + 19|\gamma_i + \zeta_i|)}{4\Re(\eta)}. \end{aligned} \quad (45)$$

Using the hypotheses (36) for $\Re(\eta) \in (0, 1)$, we readily get

$$\frac{1 - |z|^{2\Re(\eta)}}{\Re(\eta)} \left| \frac{zh''(z)}{h'(z)} \right| \leq 1. \quad (46)$$

Now if $\Re(\eta) \in [1, \infty)$, we define a function $\Psi : [1, \infty) \rightarrow R$ by

$$\Psi(x) = \frac{1 - a^{2x}}{x}, \quad (0 < a < 1). \quad (47)$$

We observe that the function Ψ is decreasing and consequently, for $|z| = a; z \in U$, we have

$$\frac{1 - |z|^{2\Re(\eta)}}{\Re(\eta)} < 1 - |z|^2, \quad (48)$$

for all $z \in U$. It follows from (40) and (42) that

$$\frac{1 - |z|^{2\Re(\eta)}}{\Re(\eta)} \left| \frac{zh''(z)}{h'(z)} \right| \leq \sum_{i=1}^n \left(\frac{29}{4} |\alpha_i| + 4|\gamma_i + \zeta_i| \right). \quad (49)$$

Using once again the hypotheses (36) when $\Re(\eta) \in [1, \infty)$, we easily get

$$\frac{1 - |z|^{2\Re(\eta)}}{\Re(\eta)} \left| \frac{zh''(z)}{h'(z)} \right| \leq 1. \quad (50)$$

Finally by applying Lemma 2, we conclude that the integral operator N_β defined by (4) is in the class $\mathcal{S}$. $\square$

Letting $n = 1$, $\delta_1 = \delta$, $\alpha_1 = \alpha$, $\gamma_1 = \gamma$, and $\zeta_1 = \zeta$ in Theorem 12, we obtain the following corollary.

Corollary 13. Let $\alpha, \gamma, \zeta \in \mathcal{C}$, and $f \in A$ satisfies condition (15). If

$$29|\alpha| + 16|\gamma + \zeta| \leq \begin{cases} 4\Re(\eta), & \text{if } \Re(\eta) \in (0, 1), \\ 4, & \text{if } \Re(\eta) \in [1, \infty) \end{cases} \quad (51)$$

then, for any complex number σ with $\Re(\sigma) \geq \Re(\eta) > 0$, the integral operator $N_\beta^{\alpha, \gamma, \zeta}$ defined by (4) is in the class $\mathcal{S}$.

Using Lemma 3, we derive the following theorem.

Theorem 14. Let $\alpha_i, \gamma_i, \zeta_i, \beta \in \mathcal{C}$ for all $i = 1, \dots, n$, $\Re(\beta) > 0$, $c \in \mathcal{C}(|c| \leq 1)$ and each $f_i \in A$ satisfies condition (15). If

$$\sum_{i=1}^n [29|\alpha_i| + 16|\gamma_i + \zeta_i|] \leq \begin{cases} 4\beta(1 - |c|), & \text{if } \Re(\eta) \in (0, 1), \\ 4(1 - |c|), & \text{if } \Re(\eta) \in [1, \infty) \end{cases} \quad (52)$$

then, for any complex number σ with $\Re(\sigma) \geq \Re(\eta) > 0$, the integral operator N_β defined by (4) is in the class $\mathcal{S}$.

Proof. From (40), we have

$$\begin{aligned} & \left| c|z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zh''(z)}{\beta h'(z)} \right| \\ &= \left| c|z|^{2\beta} + \frac{1 - |z|^2\beta}{\beta} \right. \\ & \quad \times \left[\sum_{i=1}^n \alpha_i \left(\frac{zf_i''(z)}{f_i'(z)} + 1 \right) - \sum_{i=1}^n \alpha_i \right. \\ & \quad \left. \left. + \sum_{i=1}^n (\gamma_i + \zeta_i) \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) \right] \right| \\ &\leq |c| + \frac{|1 - |z|^2|}{|\beta|} \\ & \quad \times \left[\sum_{i=1}^n \left(\frac{5}{4} |\alpha_i| + |\gamma_i + \zeta_i| \right) \left(\left| \frac{zf_i'(z)}{f_i(z)} \right| + 1 \right) + \frac{9}{4} \sum_{i=1}^n |\alpha_i| \right] \\ &\leq |c| + \frac{|1 - |z|^2|}{|\beta|} \\ & \quad \times \left[\sum_{i=1}^n \left(\frac{5}{4} |\alpha_i| + |\gamma_i + \zeta_i| \right) \left(\frac{2}{1 - |z|} \right) + \frac{9}{4} \sum_{i=1}^n |\alpha_i| \right]. \end{aligned} \quad (53)$$

Suppose that $\beta \in (0, 1)$. Define a function $\Phi : (0, 1) \rightarrow \mathbb{R}$ by

$$\Phi(x) = 1 - a^{2x} \quad (0 < a < 1). \quad (54)$$

Then Φ is an increasing function and consequently for $|z| = a; z \in U$, we obtain

$$1 - |z|^{2\beta} < 1 - |z|^2. \quad (55)$$

We thus find from (53) that

$$\left| c|z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zh''(z)}{\beta h'(z)} \right| \leq \frac{\sum_{i=1}^n (29|\alpha_i| + 16|\gamma_i + \zeta_i|)}{4|\beta|}. \quad (56)$$

Using the hypotheses (52) for $\beta \in (0, 1)$, we readily get

$$\left| c|z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zh''(z)}{\beta h'(z)} \right| \leq 1. \quad (57)$$

Now if $\beta \in [1, \infty)$, we define a function $\Psi : [1, \infty) \rightarrow \mathbb{R}$ by

$$\Psi(x) = \frac{1 - a^{2x}}{x}, \quad (0 < a < 1). \quad (58)$$

We observe that the function Ψ is decreasing and consequently for $|z| = a; z \in U$, and using once again the hypotheses (36) when $\Re(\eta) \in [1, \infty)$, we easily get

$$\left| c|z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zh''(z)}{\beta h'(z)} \right| \leq 1. \quad (59)$$

Finally, by applying Lemma 3, we conclude that $N_\beta \in \mathcal{S}$. $\square$

Conflict of Interests

The authors declare that they have no conflict interests.

Authors' Contribution

The first author is currently a Ph.D. student under supervision of the second author and jointly worked on deriving the results. All authors read and approved the paper.

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