Research Article

Double Points Local Hardy-Littlewood Maximal Operator

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A double points local Hardy-Littlewood maximal operator $M_{\{a,b\},k,\text{loc}}$ is defined and investigated in Euclidean spaces. It is proved that $M_{\{a,b\},k,\text{loc}}$ is bounded on $L^p(w)$ when $p > 1$ and from $L^1(w)$ to $L^{1,\infty}(w)$ with weight function $w \in A_{\{a,b\},k,\text{loc}}$, the class of double points local A_p weights which is larger than the Muckenhoupt A_p class and the local A_p weights defined by Lin and Stempak.

1. Introduction

The Hardy-Littlewood maximal operator M is defined on the class of the locally integrable functions on $\mathbb{R}^d$ by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(x)| dx, \quad x \in \mathbb{R}^d, \quad (1)$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^d$ containing x , with sides parallel to the coordinate axes. It is one of the most fundamental and important operators in Fourier analysis and often used to majorize other important operators. Many papers are devoted to study the Hardy-Littlewood operator and its generalizations. Li et al. [1] gave the estimates of Hardy-Littlewood operator on the multilinear spaces. Lerner [2] showed that Hardy-Littlewood maximal operator was bounded on variable L^p spaces. Gallardo [3] characterized the pairs of weights (u, w) for which the Hardy-Littlewood maximal operator M satisfies a weak type inequality.

In this paper we consider $\mathbb{R}^d$, $d \geq 1$, equipped with the metric induced by the norm

$$\|x\| = \max\{|x_1|, |x_2|, \dots, |x_d|\}, \quad x = (x_1, x_2, \dots, x_d). \quad (2)$$

By a cube we always mean the usual closed cube with sides parallel to the coordinate axes; in terms of the metric, $Q(x, r) = \{y \in \mathbb{R}^d : \|y - x\| \leq r\}$ denotes the cube centered at $x \in \mathbb{R}^d$ with “radius” $r > 0$. $|E|$ denotes the Lebesgue measure of a measurable set $E \subset \mathbb{R}^d$.

For given $a, b \in \mathbb{R}^d$ and $k > 0$, denote by $\mathcal{O}_{\{a,b\},k}(y)$ the family of all cubes centered at y with radius $r \leq k \min\{\|y - a\|, \|y - b\|\}$, and define

$$\mathcal{O}_{\{a,b\},k} = \bigcup_{y \in \mathbb{R}^d} \mathcal{O}_{\{a,b\},k}(y). \quad (3)$$

Note that for every $Q \in \mathcal{O}_{\{a,b\},k}$, $0 < k < 1$, we have $a, b \notin Q$. In particular, $\mathcal{O}_{\{a,b\},k} = \mathcal{O}_k$ when a is the origin and b tends to ∞ , where $\mathcal{O}_k = \bigcup_{y \in \mathbb{R}^d} \mathcal{O}_k(y)$ and $\mathcal{O}_k(y)$ is the collection of all the cubes centered at y with radius $r \leq k\|y\|$. The cubes in $\mathcal{O}_k$ are away from the origin, while the ones in $\mathcal{O}_{\{a,b\},k}$ are away from two points a and b .

Given $0 < k < 1$, we define the (noncentered) double points local Hardy-Littlewood maximal operator

$$M_{\{a,b\},k,\text{loc}}f(x) = \sup_{x \in Q \in \mathcal{O}_{\{a,b\},k}} \frac{1}{|Q|} \int_Q |f(t)| dt, \quad (4)$$

$$x \in \mathbb{R}^d \setminus \{a, b\},$$

where the supremum is taken over all cubes $Q \in \mathcal{O}_{\{a,b\},k}$ containing x . It generalizes the local maximal operator $M_{k,\text{loc}}$ defined in [4] for locally integrable functions on $\mathbb{R}^d \setminus \{0\}$ by

$$M_{k,\text{loc}}f(x) = \sup_{x \in Q \in \mathcal{O}_k} \frac{1}{|Q|} \int_Q |f(t)| dt, \quad x \in \mathbb{R}^d \setminus \{0\}. \quad (5)$$

The purpose of this paper is to investigate the double points local Hardy-Littlewood maximal operator. The paper was organized as follows.

The definition of the class $A_{\{a,b\},k,\text{loc}}^p$ of double points local A_p weights is given in Section 2, as well as the coincidence of $A_{\{a,b\},k,\text{loc}}^p$ for $0 < k < 1$. That is, $A_{\{a,b\},k,\text{loc}}^p = A_{\{a,b\},m,\text{loc}}^p$ for $0 < k, m < 1$. Section 3 includes some lemmas that need to prove the main result Theorem 6 given in Section 4. We prove that for a weight w satisfying the double points local A_p condition, the operator $M_{\{a,b\},k,\text{loc}}$ is bounded on $L^p(w)$, $1 < p < \infty$ and bounded from $L^1(w)$ to $L^{1,\infty}(w)$.

That w is a weight on $\mathbb{R}^d$ means that w is a nonnegative locally integrable function and finite almost everywhere. $L^p(w)$ and $L^{1,\infty}(w)$ denote the class of weighed L^p space and weighed weak L^1 space, with the norm $\|f\|_{L^p(w)}^p = \int_{\mathbb{R}^d} |f(x)|^p w(x) dx$ and $\|f\|_{L^{1,\infty}(w)} = \sup_{t>0} (t \int_{|f|>t} w(x) dx)$, respectively.

2. Double Points Local A_p Weights

For a weight w on $\mathbb{R}^d$, we say that $w \in A_p$ for $1 < p < \infty$ if there is a constant $C > 0$ such that

$$\sup_{Q \subset \mathbb{R}^d} \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{1-p'} dx \right)^{p-1} \leq C, \quad (6)$$

here and below $1/p + 1/p' = 1$. We say that $w \in A_1$ if there is a constant $C > 0$ such that $Mw(x) \leq Cw(x)$. The smallest constant C is called the A_p constant of w and denoted by $\|w\|_{A_p}$. The definition of A_p weights was first introduced by Muckenhoupt [5] for $d = 1$. Muckenhoupt proved the characterization of the weighed type (p, p) for Hardy-Littlewood maximal operator. That is, for $w \in A_p$, $p \geq 1$, the operator M is bounded on $L^p(w)$ when $p > 1$ and bounded from $L^1(w)$ into $L^{1,\infty}(w)$ when $p = 1$; moreover, we have

$$\begin{aligned} \|Mf\|_{L^p(w)} &\leq C\|w\|_{A_p}^{p'} \|f\|_{L^p(w)}, \quad f \in L^p(w), \quad p > 1, \\ \|Mf\|_{L^{1,\infty}(w)} &\leq C\|w\|_{A_1} \|f\|_{L^1(w)}, \quad f \in L^1(w), \end{aligned} \quad (7)$$

where the constant C is independent of w (cf. [6]).

The class of the local $A_{k,\text{loc}}^p$ weights is defined in [4] by considering the cubes in $\mathcal{O}_k$ and the corresponding constant is denoted as $\|w\|_{A_{k,\text{loc}}^p}$. Lin and Stempak proved that for $w \in A_{k,\text{loc}}^p$, the local Hardy-Littlewood maximal operator is bounded from $L^p(w)$ to $L^p(w)$ when $p > 1$ and from $L^1(w)$ to $L^{1,\infty}(w)$.

In order to prove the coincidence of the classes $A_{\{a,b\},k,\text{loc}}^p$ for $0 < k < 1$, we need the following lemma.

Definition 1. Let $0 < k < 1$, $1 \leq p < \infty$, $a, b \in \mathbb{R}^d$, and $w \in L_{\text{loc}}(\mathbb{R}^d \setminus \{a, b\})$ a nonnegative function. If there exists a constant $C > 0$ such that

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w(x) dx \right)^{1/p} \left(\frac{1}{|Q|} \int_Q w(x)^{-p'/p} dx \right)^{1/p'} \leq C \quad (8)$$

for $1 < p < \infty$, where the supremum takes over all the cubes $Q \in \mathcal{O}_{\{a,b\},k}$, then w is called the double points local A_p weight associated with a and b . And for $p = 1$, the class $A_{\{a,b\},k,\text{loc}}^1$ is defined by

$$\frac{1}{|Q|} \int_Q w(x) dx \leq C \operatorname{essinf}_{x \in Q} w(x), \quad (9)$$

for all $Q \in \mathcal{O}_{\{a,b\},k}$. Denote the collection of all the double points local A_p weights as $A_{\{a,b\},k,\text{loc}}^p = A_{\{a,b\},k,\text{loc}}^p(\mathbb{R}^d)$. The smallest constant C is called the double points local $k - A_p$ constant and denoted as $\|w\|_{A_{\{a,b\},k,\text{loc}}^p}$.

According to the definition above, it is easy to see that $A_p \subset A_{\{a,b\},k,\text{loc}}^p$. Furthermore, if $w \in A_p$, then $\|w\|_{A_{\{a,b\},k,\text{loc}}^p} \leq \|w\|_{A_p}$. But $A_p \neq A_{\{a,b\},k,\text{loc}}^p$. In fact, let $w(x) = \min\{\|x - a\|, \|x - b\|\}$; then for any $t \in \mathbb{R}$, we have $w(x)^t \in A_{\{a,b\},k,\text{loc}}^p$, while $w(x)^t \in A_p$ if and only if $-d < t < d(p-1)$ when $p > 1$ and $-d < t \leq 0$ when $p = 1$.

On the other hand, define $\mathcal{O}_{\{a\},k} = \bigcup_{y \in \mathbb{R}^d} \mathcal{O}_{\{a\},k}(y)$, where $\mathcal{O}_{\{a\},k}(y)$ is the collection of the cubes centered at y with radius no more than $k\|x - a\|$. Then the cubes in $\mathcal{O}_{\{a\},k}$ are just the translations of the cubes in $\mathcal{O}_k$ along a mentioned in [4]. Replacing $\mathcal{O}_{\{a,b\},k}$ with $\mathcal{O}_{\{a\},k}$ in Definition 1 gives the definition of the class $A_{\{a\},k,\text{loc}}^p$. It is also easy to see that $A_{\{a\},k,\text{loc}}^p \subset A_{\{a,b\},k,\text{loc}}^p$. The same process gives $\mathcal{O}_{\{b\},k}$ and consequently the class $A_{\{b\},k,\text{loc}}^p$. We also have the relation $A_{\{b\},k,\text{loc}}^p \subset A_{\{a,b\},k,\text{loc}}^p$.

Lemma 2. Let $0 < k < 1$; for every $0 < r < 1$, there exists a $n = n(k, r)$ satisfying the following property: if R and $\widehat{R}$ are the cubes centered at $x \neq a, b$ with radius $r \min\{\|x - a\|, \|x - b\|\}$ and $(r+k(1-r)) \min\{\|x - a\|, \|x - b\|\}$, respectively, there exist at most n cubes $Q_j \subset \widehat{R}$ and $Q_j \in \mathcal{O}_{\{a,b\},k}$, $1 \leq j \leq n$, covering $\widehat{R} \setminus R$ and $|Q_j \cap R| \geq C|R|$, where Q_j , $1 \leq j \leq n$ are the cubes with centers located on ∂R and radius $k(1-r) \min\{\|x - a\|, \|x - b\|\}$ and $C = \min\{1, (k(1-r)/2r)^d\}$.

Proposition 3. Let $0 < k, m < 1$, and $1 \leq p < \infty$. Then

$$A_{\{a,b\},k,\text{loc}}^p = A_{\{a,b\},m,\text{loc}}^p. \quad (10)$$

The proofs of Lemma 2 and Proposition 3 are analogous to the proofs of Lemma 2.1 and Proposition 2.2 in [4], respectively, so we omit them here.

By Proposition 3, the class $A_{\{a,b\},k,\text{loc}}^p$ is independent of the choice of $k \in (0, 1)$ and we will denote it by $A_{\{a,b\},\text{loc}}^p$.

3. Preparatory Lemmas

For a given $0 < k < 1$, $a \in \mathbb{R}^d$, $a = (a_1, a_2, \dots, a_d)$, considering a grid of $\mathbb{R}^d \setminus \{a\}$ based on the sequence $\{\pm k^{-s}\}_{s \in \mathbb{Z}}$, which results in the collection $\mathcal{G}_a$ of all the rectangles

$$G_a = (I_1 + \{a_1\}) \times (I_2 + \{a_2\}) \times \dots \times (I_d + \{a_d\}), \quad (11)$$

where for a given $s \in \mathbb{Z}$, each I_j is one of the intervals $\pm[k^{-s}, k^{-s-1}]$, $\pm[0, ((1-k)/(1+k))k^{-s}]$, or $\pm[((1-k)/(1+k))k^{-s}, k^{-s}]$ (called intervals of the first, second, or third type, resp.), but at least one of the intervals must be of the first type (if $0 \leq u < v < +\infty$, $-[u, v]$ denotes the interval $[-v, -u]$, while $+ [u, v]$ is just $[u, v]$). If one of the intervals is of the second type, we say that G_a is of the first category, otherwise of the second category.

For $G_a \in \mathcal{G}_a$, define G'_a to be the union of the cubes in $\mathcal{O}_{\{a\},k}$ satisfying $Q \cap G_a \neq \emptyset$. If G_a is of the first category, we define G''_a to be the least cube containing G'_a with the largest distance from the point a along each axis. If G_a is of the second category, we define $G''_{a,j}$ to be the least cube containing $G'_{a,j} = G'_a \cap E_{j,G_a}$ with the largest distance from the point a along each axis, where for every $j = 1, 2, \dots, d$,

$$E_{j,G_a} = \begin{cases} \left\{ y \in \mathbb{R}^d : y_j \geq \frac{1-k}{1+k}k^{-s} + a_j \right\}, \\ \text{when } I_j \subset (0, +\infty), \\ \left\{ y \in \mathbb{R}^d : y_j \leq -\frac{1-k}{1+k}k^{-s} + a_j \right\}, \\ \text{when } I_j \subset (-\infty, 0). \end{cases} \quad (12)$$

According to the definitions above, if G_a is of the first category, we have $G_a \subset G'_a \subset G''_a$, $G'_a \subset \{y \in \mathbb{R}^d : |y_j - a_j| \geq ((1-k)/(1+k))k^{-s}\}$ and $a \notin G''_a$. If G_a is of the second category, then $G_a \subset G'_{a,j}$, $a \notin G''_{a,j}$, $j = 1, 2, \dots, d$ and $G'_a \cap \prod_{j=1}^d E_{j,G_a}^c = \emptyset$ which equals $G'_a = \bigcup_{j=1}^d G'_{a,j}$.

For a point $b \in \mathbb{R}^d$, repeat the process above, we can obtain the corresponding sets G_b , G'_b , G''_b , $G'_{b,j}$, $G''_{b,j}$, and $\mathcal{G}_b$.

For $a, b \in \mathbb{R}^d (a \neq b)$,

$$\begin{aligned} \mathbb{R}^d \setminus \{a, b\} &= \left(\bigcup_{G_a \in \mathcal{G}_a} G_a \right) \cap \left(\bigcup_{G_b \in \mathcal{G}_b} G_b \right) \\ &= \bigcup_{G_a \in \mathcal{G}_a, G_b \in \mathcal{G}_b} (G_a \cap G_b). \end{aligned} \quad (13)$$

Define $\mathcal{G}_{a,b}$ to be the collection of $G_{a,b} = G_a \cap G_b$, where $G_a \in \mathcal{G}_a$ and $G_b \in \mathcal{G}_b$ and $G'_{a,b}$ to be the union of the cubes $Q \in \mathcal{O}_{\{a,b\},k}$ such that $Q \cap G_{a,b} \neq \emptyset$. In the proof of Theorem 6, we will use a result; that is, $G'_{a,b} \subset (G'_a \cap G'_b)$.

In fact, for every $z \in G'_{a,b}$, there exists a cube $Q_0 \in \mathcal{O}_{\{a,b\},k}$, $Q_0 \cap G_{a,b} \neq \emptyset$ such that $z \in Q_0$. The definition of $\mathcal{O}_{\{a,b\},k}$ gives that there exists a $y \in \mathbb{R}^d \setminus \{a, b\}$ such that Q_0 is the cube centered at y with radius $r \leq k\|y - a\|$. Then $Q_0 \in \mathcal{O}_{\{a\},k}$ and $Q_0 \subset G'_a$; consequently $z \in G'_a$. Similarly, $z \in G'_b$. Thus $z \in G'_a \cap G'_b$.

Lemma 4. *Given $0 < k < 1$, there exists $k_0 = k_0(d, k) \in (0, 1)$ such that for each $G_{a,b} = G_a \cap G_b \in \mathcal{G}_{a,b}$, $G''_a \in \mathcal{O}_{\{a\},k_0}$ (or $G''_b \in \mathcal{O}_{\{b\},k_0}$) if G_a (or G_b) is of the first category, and $G''_{a,j} \in \mathcal{O}_{\{a\},k_0}$ (or $G''_{b,j} \in \mathcal{O}_{\{b\},k_0}$), $j = 1, 2, \dots, d$, if G_a (or G_b) is of the second category.*

Proof. The constants k_a and k_b associated with the points a and b , respectively, can be obtained by the same procedure as the one in the proof of Lemma 3.1 in [4]. Then we have $G''_a \in \mathcal{O}_{\{a\},k_a}$ and $G''_b \in \mathcal{O}_{\{b\},k_b}$ when G_a and G_b are of the first category. Let $k_0 = \max\{k_a, k_b\}$, thus $G''_a \in \mathcal{O}_{\{a\},k_0}$ and $G''_b \in \mathcal{O}_{\{b\},k_0}$ by the definition of $\mathcal{O}_{\{a\},k_0}$ and $\mathcal{O}_{\{b\},k_0}$. The case when G_a and G_b are of the second category is similar to the case when G_a and G_b are of the first category. $\square$

Lemma 5. *Given $0 < k < 1$. Let Q and $\tilde{Q}$ be the cubes in $\mathbb{R}^d$ satisfying $Q \subset \tilde{Q}$. If $\tilde{Q} \in \mathcal{O}_{\{a\},k}$ (or $\mathcal{O}_{\{b\},k}$); then $Q \in \mathcal{O}_{\{a\},k}$ (or $\mathcal{O}_{\{b\},k}$).*

Lemma 5 can be obtained from Lemma 3.3 in [4] by translating the cubes in it.

4. The Main Results

Now we come to the main results of the paper.

Theorem 6. *Let $0 < k < 1$ and $1 \leq p < \infty$. If $w \in A^p_{\{a,b\},\text{loc}}$, then the double points local maximal operator $M_{\{a,b\},k,\text{loc}}$ is bounded on $L^p(w)$ when $p > 1$ and bounded from $L^1(w)$ into $L^{1,\infty}(w)$. Moreover, the corresponding constant depends only on the double points local $k_0 - A_p$ constant of w for some k_0 , $0 < k_0 < 1$.*

Proof. Denote $M_{\{a,b\},k,\text{loc}}$ by $M_{k,\text{loc}}$ for short. The definition of $G'_{a,b}$ gives that $M_{k,\text{loc}}f(x) = M_{k,\text{loc}}(f\chi_{G'_{a,b}})(x)$ for $x \in G_{a,b}$. First, we consider the case $p > 1$

$$\begin{aligned} &\int_{\mathbb{R}^d} |M_{k,\text{loc}}f(x)|^p w(x) dx \\ &= \sum_{G_{a,b} \in \mathcal{G}_{a,b}} \int_{G_{a,b}} |M_{k,\text{loc}}f(x)|^p w(x) dx \\ &= \sum_{G_{a,b} \in \mathcal{G}_{a,b}} \int_{G_{a,b}} |M_{k,\text{loc}}(f\chi_{G'_{a,b}})(x)|^p w(x) dx \\ &\leq \sum_{G_{a,b} \in \mathcal{G}_{a,b}} \int_{G_{a,b}} |M(f\chi_{G'_{a,b}})(x)|^p w(x) dx. \end{aligned} \quad (14)$$

Since M is sublinear and $G'_{a,b} \subset (G'_a \cap G'_b)$, we have

$$\begin{aligned} &\int_{\mathbb{R}^d} |M_{k,\text{loc}}f(x)|^p w(x) dx \\ &\leq 2^p \sum_{G_{a,b} \in \mathcal{G}_{a,b}} \left(\int_{G_{a,b}} |M(f\chi_{G'_a})(x)|^p w(x) dx \right. \\ &\quad \left. + \int_{G_{a,b}} |M(f\chi_{G'_b})(x)|^p w(x) dx \right) \end{aligned}$$

$$\begin{aligned}
&= 2^p \left(\int_{\mathbb{R}^d} |M(f\chi_{G'_a})(x)|^p w(x) dx \right. \\
&\quad \left. + \int_{\mathbb{R}^d} |M(f\chi_{G'_b})(x)|^p w(x) dx \right) \\
&= 2^p \left(\sum_{G_a \in \mathcal{G}_a} \int_{G_a} |M(f\chi_{G'_a})(x)|^p w(x) dx \right. \\
&\quad \left. + \sum_{G_b \in \mathcal{G}_b} \int_{G_b} |M(f\chi_{G'_b})(x)|^p w(x) dx \right). \tag{15}
\end{aligned}$$

Translating the cubes along a and b , respectively, in the corresponding part of proof for Theorem 4.1 in [4] gives

$$\begin{aligned}
&\int_{G_a} |M(f\chi_{G'_a})(x)|^p w(x) dx \\
&\leq C \|w\|_{A_{\{a\},k_0,\text{loc}}^{pp'}}^{pp'} \int_{G'_a} |f(x)|^p w(x) dx, \\
&\int_{G_b} |M(f\chi_{G'_b})(x)|^p w(x) dx \\
&\leq C \|w\|_{A_{\{b\},k_0,\text{loc}}^{pp'}}^{pp'} \int_{G'_b} |f(x)|^p w(x) dx. \tag{16}
\end{aligned}$$

Note that the families $\{G'_a\}_{G_a \in \mathcal{G}_a}$ and $\{G'_b\}_{G_b \in \mathcal{G}_b}$ have the finite overlapping property; that is, there exists a constant $m = m(d, k)$ such that every $x \in \mathbb{R}^d \setminus \{a, b\}$ belongs to at most m sets from $\{G'_a\}_{G_a \in \mathcal{G}_a}$ or $\{G'_b\}_{G_b \in \mathcal{G}_b}$. Combining this with (16) gives

$$\begin{aligned}
&\int_{\mathbb{R}^d} |M_{k,\text{loc}} f(x)|^p w(x) dx \\
&\leq 2^p \left(\sum_{G_a \in \mathcal{G}_a} C \|w\|_{A_{\{a\},k_0,\text{loc}}^{pp'}}^{pp'} \int_{G'_a} |f(x)|^p w(x) dx \right. \\
&\quad \left. + \sum_{G_b \in \mathcal{G}_b} C \|w\|_{A_{\{b\},k_0,\text{loc}}^{pp'}}^{pp'} \int_{G'_b} |f(x)|^p w(x) dx \right) \\
&\leq CB_{k_0,p} \left(\int_{\mathbb{R}^d} |f(x)|^p \left(\sum_{G_a \in \mathcal{G}_a} \chi_{G'_a} \right) w(x) dx \right. \\
&\quad \left. + \int_{\mathbb{R}^d} |f(x)|^p \left(\sum_{G_b \in \mathcal{G}_b} \chi_{G'_b} \right) w(x) dx \right) \\
&\leq CmB_{k_0,p} \int_{\mathbb{R}^d} |f(x)|^p w(x) dx, \tag{17}
\end{aligned}$$

where $B_{k_0,p} = \max\{\|w\|_{A_{\{a\},k_0,\text{loc}}^{pp'}}^{pp'}, \|w\|_{A_{\{b\},k_0,\text{loc}}^{pp'}}^{pp'}\}$.

As for the case $p = 1$, we define the level sets

$$\begin{aligned}
E_\lambda &= \{x \in \mathbb{R}^d : M_{k,\text{loc}} f(x) > \lambda\}, \\
E_{\lambda,G'_{a,b}} &= \{x \in \mathbb{R}^d : M_{k,\text{loc}}(f\chi_{G'_{a,b}})(x) > \lambda\}, \\
F_{\lambda,G'_{a,b}} &= \{x \in \mathbb{R}^d : M(f\chi_{G'_{a,b}})(x) > \lambda\}, \tag{18} \\
F_{\lambda,G'_a} &= \{x \in \mathbb{R}^d : M(f\chi_{G'_a})(x) > \lambda\}, \\
F_{\lambda,G'_b} &= \{x \in \mathbb{R}^d : M(f\chi_{G'_b})(x) > \lambda\}.
\end{aligned}$$

Since $F_{\lambda,G'_{a,b}} \subset (F_{\lambda,G'_a} \cup F_{\lambda,G'_b})$, the weak weighed boundedness of the maximal operator M and the similar arguments as the case $p > 1$ give that

$$\begin{aligned}
\int_{E_\lambda} w(x) dx &= \sum_{G_{a,b} \in \mathcal{G}_{a,b}} \int_{G_{a,b}} \chi_{E_{\lambda,G'_{a,b}}}(x) w(x) dx \\
&\leq \sum_{G_{a,b} \in \mathcal{G}_{a,b}} \int_{G_{a,b}} \chi_{F_{\lambda,G'_{a,b}}}(x) w(x) dx \\
&\leq \sum_{G_a \in \mathcal{G}_a} \int_{G_a} \chi_{F_{\lambda,G'_a}}(x) w(x) dx \\
&\quad + \sum_{G_b \in \mathcal{G}_b} \int_{G_b} \chi_{F_{\lambda,G'_b}}(x) w(x) dx \\
&\leq CB_{k_0,1} \lambda^{-1} \left(\sum_{G_a \in \mathcal{G}_a} \int_{G'_a} |f(x)| w(x) dx \right. \\
&\quad \left. + \sum_{G_b \in \mathcal{G}_b} \int_{G'_b} |f(x)| w(x) dx \right) \\
&\leq CmB_{k_0,1} \lambda^{-1} \int_{\mathbb{R}^d} |f(x)| w(x) dx, \tag{19}
\end{aligned}$$

where $B_{k_0,1} = \max\{\|w\|_{A_{\{a\},k_0,\text{loc}}^1}, \|w\|_{A_{\{b\},k_0,\text{loc}}^1}\}$. Thus the proof is finished. $\square$

Theorem 7. Let $0 < k < 1$ and $1 \leq p < \infty$, w be a weight function on $\mathbb{R}^d$. The condition $w \in A_{\{a,b\},\text{loc}}^p$ is necessary for the double points local maximal operator $M_{\{a,b\},k,\text{loc}}$ to be bounded from $L^p(w)$ into $L^{p,\infty}(w)$; that is,

$$\begin{aligned}
&w(\{x \in \mathbb{R}^d : M_{\{a,b\},k,\text{loc}} f(x) > \lambda\}) \\
&\leq \frac{C}{\lambda^p} \int_{\mathbb{R}^d} |f(x)|^p w(x) dx, \quad \lambda > 0. \tag{20}
\end{aligned}$$

The proof of Theorem 7 is analogous to that of Theorem 4.2 in [4], and we omit it here.

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