Discreteness and Convergence of Complex Hyperbolic Isometry Groups

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Received 10 June 2013; Accepted 2 October 2013

Academic Editor: Pedro M. Lima

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We investigate the discreteness and convergence of complex isometry groups and some discreteness criteria and algebraic convergence theorems for subgroups of $\text{PU}(n,1)$ are obtained. All of the results are generalizations of the corresponding known ones.

1. Introduction

In 1976, Jørgensen obtained a very useful necessary condition for two-generator Kleinian groups of $\text{M}(\mathbb{R}^2)$, which is known as Jørgensen’s inequality. As an application, he obtained the following [1, 2].

**Theorem J.** A nonelementary subgroup $G$ of $\text{M}(\mathbb{R}^2)$ is discrete if and only if each two-generator subgroup in $G$ is discrete.

Furthermore, Gilman [3] and Isachenko [4] showed that the discreteness of all two-generator subgroups of $G$, where each generator is loxodromic, is enough to secure the discreteness of $G$. See [5–8] and so forth for some other discussions along this line.

It is interesting to generalize Theorem J into the higher dimensional case. By adding some conditions, several generalizations of Theorem J into $\text{M}(\mathbb{R}^n)$ ($n \geq 3$) have been obtained; see [9–13] and so forth. In 2005, Wang et al. [14] proved the following.

**Theorem WLC.** Let $G \subset \text{M}(\mathbb{R}^n)$ be nonelementary and $f \in G$ loxodromic. Then $G$ is discrete if and only if $\text{WY}(G)$ is discrete and each nonelementary subgroup $\langle f, gfg^{-1} \rangle$ is discrete, where $g \in G$.

Here

$$\text{WY} (G) = \{h : h|_{\text{M}(G)} = I, h \in G\},$$

and $\text{M}(G)$ is the smallest $G$-invariant hyperbolic subspace whose boundary contains the limit set $L(G)$ of $G$ (cf. [15]).

Since the real hyperbolic plane can be viewed as a complex hyperbolic 1-space $\mathbb{H}^1_c$, it is natural to generalize these results mentioned above to the setting of complex hyperbolic space. Recently, Qin and Jiang [16] proved the following.

**Theorem QJ 1.** Let $G$ be an $n$-dimensional subgroup of $\text{PU}(n,1)$ and $f$ a nonelliptic element in $\text{PU}(n,1)$. If for each loxodromic (resp., regular elliptic) element $g \in G$ the two-generator group $\langle f, g \rangle$ is discrete, then $G$ is discrete.

**Theorem QJ 2.** Let $G$ be an $n$-dimensional subgroup of $\text{PU}(n,1)$ and $f$ a regular elliptic element in $\text{PU}(n,1)$. If for each loxodromic (resp., regular elliptic) element $g \in G$ the two-generator group $\langle f, g \rangle$ is discrete, then $G$ is discrete.

Here $G$ is called $n$-dimensional if it does not leave a point in $\partial \mathbb{H}^n_c$ or a proper totally geodesic submanifold of $\mathbb{H}^n_c$ invariant. Obviously, if $G$ is $n$-dimensional, then $G$ is nonelementary and $\text{M}(G) = \mathbb{H}^n_c$.

Motivated by Theorem WLC, a natural question will be asked: can we use the discreteness of subgroups $\langle f, gfg^{-1} \rangle$ to determine the discreteness of $G$ in Theorems QJ1 and QJ2? In this paper, we will give this question a positive answer (see Section 3).

Let $G$ be the Möbius group $\text{M}(\mathbb{R}^n)$ or the complex hyperbolic isometry group $\text{PU}(n,1)$.
Definition 1. Let \( \{G_{r,i}\} \) be a sequence of subgroups in \( G \) and each \( G_{r,i} \) be generated by \( g_{1,j}, g_{2,j}, \ldots, g_{r,j} \). If, for each \( r \in \{1, 2, \ldots, r\} (r < \infty) \),

\[
g_{r,j} \rightarrow g_r \in G \quad \text{as} \quad i \rightarrow \infty,
\]

then we say that \( \{G_{r,i}\} \) converges algebraically to \( G_r = \langle g_1, g_2, \ldots, g_r \rangle \) and \( G_r \) is called the algebraic limit group of \( \{G_{r,i}\} \). If for each \( i \), \( G_{r,i} \) is a Kleinian group, then the question when \( G_r \) is still a Kleinian group has attracted much attention. Jørgensen and Klein proved that \( G_r \) is still a Kleinian group, when \( n = 2 \). For the higher dimensional case, there are a number of discussions; see [11, 12, 17].

When \( G = PU(n, 1) \), Cao proved [18] the following.

Theorem C.1. Let \( \{G_{r,i}\} \) be a sequence of groups of \( G \). If each \( G_{r,i} \) is discrete, then the algebraic limit group \( G_r \) of \( \{G_{r,i}\} \) is either a complex Kleinian group, or it is elementary, or \( W(G) \) is not finite.

Theorem C.2. Let \( G_r \) be the algebraic limit group of complex Kleinian groups \( \{G_{r,i}\} \) of \( G \). If \( \{G_{r,i}\} \) satisfies IP-condition, then \( G_r \) is a complex Kleinian group.

Here \( \{G_{r,i}\} \) satisfies IP-condition means that \( \{G_{r,i}\} \) satisfies the following conditions: for any sequence \( \{f_i\}, f_i \in G_{r,i} \), if \( \text{card}\{\text{fix}(f_i)\} = \infty \) for each \( i \), and \( f_i \rightarrow f \) as \( i \rightarrow \infty \) with \( f \) being the identity or parabolic, then \( \{f_i\} \) has uniformly bounded torsion (see [18]).

In this paper, we will discuss the discreteness criteria and algebraic convergence theorems for subgroups of \( PU(n, 1) \) further. The rest of this paper is organized as follows: in Section 2, we introduce some preliminary results that we need in the sequel; in Section 3, we show three discreteness criteria for subgroups of \( PU(n, 1) \); finally Section 4 is dedicated to three algebraic convergence theorems for complex Kleinian groups.

2. Preliminaries

Let \( \mathbb{C}^{n+1} \) be the complex vector space of dimension \( n + 1 \) with the Hermitian form

\[
\langle z, w \rangle = z_1 \overline{w}_1 + z_2 \overline{w}_2 + \cdots + z_n \overline{w}_n - z_{n+1} \overline{w}_{n+1},
\]

where \( z, w \) are the column vectors in \( \mathbb{C}^{n+1} \). Consider the following subspaces of \( \mathbb{C}^{n+1} \):

\[
V_0 = \{ z \in \mathbb{C}^{n+1} : \langle z, z \rangle = 0 \},
\]

\[
V = \{ z \in \mathbb{C}^{n+1} : \langle z, z \rangle < 0 \}.
\]

Let \( P : \mathbb{C}^{n+1} - \{0\} \rightarrow \mathbb{CP}^n \) be the complex projective space from \( \mathbb{C}^{n+1} - \{0\} \) onto the complex hyperbolic space \( \mathbb{CP}^n \). The complex hyperbolic space is defined to be \( \mathbb{H}_C^n = PV_+ \) and \( \partial \mathbb{H}_C^n = PV_0 \) is its boundary. The biholomorphic isometry group of \( \mathbb{H}_C^n \) is given by the projective unitary group \( PU(n, 1) \). For a nontrivial element \( g \) of \( PU(n, 1) \), we say that \( g \) is elliptic if it has a fixed point in \( \partial \mathbb{H}_C^n \), \( g \) is parabolic if it has only one fixed point in \( \partial \mathbb{H}_C^n \), and \( g \) is loxodromic if it has exactly two different fixed points in \( \partial \mathbb{H}_C^n \).

For elliptic element \( g \in PU(n, 1) \), let \( \Lambda_0 \) and \( \Lambda_i \) \( (i = 1, 2, \ldots, n) \) be its negative and positive eigenvalues, respectively. Then the fixed point set of \( g \) in \( \mathbb{H}_C^n \) contains only one point if \( \Lambda_0 \neq \Lambda_i \) and is a totally geodesic submanifold, which is equivalent to \( \mathbb{H}_C^n \) if \( \Lambda_0 \) coincides with exact \( m \) of class \( \Lambda_i \) \( (m < n) \). We call \( g \) regular elliptic if \( \Lambda_0 \neq \Lambda_i \), where \( s, t \in \{0, 1, 2, \ldots, n\} \) and \( s \neq t \). Obviously, if \( g \) is regular elliptic, then \( g \) has only one fixed point in \( \mathbb{H}_C^n \). The following proposition follows directly from [19].

Proposition 2. The regular elliptic (resp., loxodromic) elements of \( PU(n, 1) \) form an open set.

Let \( G \) be a subgroup of \( PU(n, 1) \). The limit set \( L(G) \) of \( G \) is defined as

\[
L(G) = \overline{G(p)} \cap \partial \mathbb{H}_C^n, \quad p \in \mathbb{H}_C^n.
\]

\( G \) is called nonelementary if \( L(G) \) contains more than two points; otherwise, it is called elementary. We call a subgroup \( G \) of \( PU(n, 1) \) complex Kleinian group if it is discrete and nonelementary. For a nonelementary subgroup \( G \) of \( PU(n, 1) \), we denote by \( M(G) \) the smallest totally geodesic submanifold of \( G \) whose boundary contains the limit set \( L(G) \). It is easy to see that \( M(G) \) is \( G \)-invariant since \( L(G) \) is \( G \)-invariant. As in [18], the subgroup \( W(G) \) of \( G \) is defined as

\[
W(G) = \{ g : \text{fix}(g) = 1, g \in G \}.
\]

For an element \( f \in PU(n, 1) \), we denote \( N(f) = \| f - I \| \), where \( \| \cdot \| \) is the Hilbert–Schmidt norm. Then we have the following.

Lemma 3 (see [18, 20]). Suppose that two elements \( f, g \in PU(n, 1) \) generate a complex Kleinian group.

(1) If \( f \) is parabolic or loxodromic, then

\[
\max \{ N(f), N([f, g]) \} \geq 2 - \sqrt{3},
\]

where \([f, g] = fgf^{-1}g^{-1}\) is the commutator of \( f \) and \( g \).

(2) If \( f \) is elliptic, then

\[
\max \{ N(f), N([f, g]) : q = 1, 2, 3, \ldots, n + 1 \} \geq 2 - \sqrt{3}.
\]

3. Discreteness Criteria

In this section, we prove the following theorems.

Theorem 4. Let \( G \) be an \( n \)-dimensional subgroup of \( PU(n, 1) \) and \( f \) a nonelliptic element in \( PU(n, 1) \). If for each loxodromic (resp., regular elliptic) element \( g \) in \( G \) the two-generator group \( \langle f, gfg^{-1} \rangle \) is discrete, then \( G \) is discrete.

Theorem 5. Let \( G \) be an \( n \)-dimensional subgroup of \( PU(n, 1) \) and \( f \) a regular elliptic element with finite order \( k \) \( (k \geq 3) \)
in $\text{PU}(n,1)$. If for each loxodromic (resp., regular elliptic) element $g \in G$ the two-generator group $\langle f, gfg^{-1} \rangle$ is discrete, then $G$ is discrete.

When $f$ is elliptic (may not be regular), we have the following.

**Theorem 6.** Let $G$ be an $n$-dimensional subgroup of $\text{PU}(n,1)$ and $f$ an elliptic element with finite order $k$ ($k \geq 3$) in $\text{PU}(n,1)$. If, for each loxodromic (resp., regular elliptic) element $g \in G$ the two-generator group $\langle f, g \rangle$ is discrete, then $G$ is discrete.

In order to prove the above theorems, we need the following lemma which is a classification of elementary subgroups of $\text{PU}(n,1)$.

**Lemma 7.** Let $G$ be a subgroup of $\text{PU}(n,1)$.

1. If $G$ contains a loxodromic element, then $G$ is elementary if and only if it fixes a point in $\partial H^p_C$ or a point-pair $(x, y)$ in $\partial H^n_C$.
2. If $G$ contains a parabolic element but no loxodromic element, then $G$ is elementary if and only if it fixes a point in $\partial H^n_C$.
3. If $G$ is purely elliptic, then $G$ fixes a point in $\mathbb{P}^{n-1}_C$.

**Proof of Theorem 4.** Firstly, we prove the case when each $g$ is loxodromic. Suppose not. Then $G$ is dense in $\text{PU}(n,1)$ according to Corollary 4.5.1 of [15]. By Proposition 2, there exists a sequence $\{g_i\} \subseteq G$ such that each $g_i$ is loxodromic and $g_i \rightarrow I$ as $i \rightarrow \infty$. Then, for large enough $i$, we have

$$N \left( g_i f g_i^{-1} f^{-1} \right) + \sum_{q=1}^{n+1} N \left( \left[ g_i f g_i^{-1} f^{-1}, f^q \right] \right) < 2 - \sqrt{3}. \quad (9)$$

Since $f$ is nonelliptic and $\langle f, g_i f g_i^{-1} f^{-1} \rangle = \langle f, g_i f g_i^{-1} \rangle$, by Lemma 3, we know that, for all large enough $i$, $\langle f, g_i f g_i^{-1} \rangle$ are elementary. This implies that

$$\text{fix}(f) \cap \text{fix}(g_i) \neq \emptyset. \quad (10)$$

Since $G$ is nonelementary, we can find three loxodromic elements $h_s (s = 1, 2, 3)$ in $G$ such that

$$\text{fix}(f) \cap \text{fix}(h_s) = \emptyset, \quad \text{fix}(h_j) \cap \text{fix}(h_k) = \emptyset, \quad (11)$$

where $i, k \in \{1, 2, 3\}$ and $j \neq k$. It follows from a discussion similar to the above that we can obtain that, for large enough $i$,

$$\text{fix}(f) \cap \text{fix}(h_s g_i h_s^{-1}) \neq \emptyset, \quad s = 1, 2, 3. \quad (12)$$

Since $f$ is nonelliptic, that is, fix$(f)$ contains less than three points; it is a contradiction.

Now, we come to prove the case when each $g$ is regular elliptic. Suppose that $G$ is nondiscrete. Similarly, by Proposition 2, we can find a sequence $\{g_i\}$ in $G$ such that each $g_i$ is regular elliptic and $g_i \rightarrow I$ as $i \rightarrow \infty$. This implies that, for sufficiently large $i$, the subgroups $\langle f, g_i f g_i^{-1} \rangle$ are elementary. It follows that

$$\text{fix}(f) = \text{fix}(g_i). \quad (13)$$

It is a contradiction since $f$ is nonelliptic and $g_i$ is regular elliptic.

This completes the proof. \hfill $\Box$

**Proof of Theorem 5.** The proof of Theorem 5 follows from a discussion similar to that in the proof of Theorem 4. \hfill $\Box$

**Proof of Theorem 6.** We only prove the case when $g$ is loxodromic; similar arguments can be applied to the case when $g$ is regular elliptic. Suppose that $G$ is nondiscrete. Then there exists a sequence $\{g_i\} \subseteq G$ such that, for each $i$, $g_i$ is loxodromic and

$$g_i \rightarrow I \quad \text{as} \quad i \rightarrow \infty. \quad (14)$$

Since $G$ is $n$-dimensional, we can find finitely many loxodromic elements $h_1, h_2, \ldots, h_t$ in $G$ such that the set $S = \{A_{h_1(h)}, A_{h_2(h)}, \ldots, A_{h_t(h)}\}$ can span the whole complex hyperbolic space $\mathbb{H}^t_C$, where $A_{h(k)}$ is the attractive fixed point of $h$. For each $k (k = 1, 2, \ldots, t)$, let $U_{A_{h(k)}}$ be a small neighbourhood of $A_{h(k)}$ in $\mathbb{H}^t_C$, then there exists an integer $N$ such that

$$h_k(N) \left( \text{fix}(f) \right) \subset U_{A_{h(k)}}. \quad (15)$$

Since

$$\langle h_k^N f h_k^{-N}, g_i \rangle = h_k(N) \langle f, h_k^N g_i h_k^{-N} \rangle, \quad \max \left\{ N \left( h_k^N g_i h_k^{-N}, f \right), N \left( h_k^N g_i h_k^{-N}, f \right) \right\} < 2 - \sqrt{3}, \quad (16)$$

for large enough $i$, we can see that the subgroups $\langle h_k^N f h_k^{-N}, g_i \rangle$ are elementary. By Lemma 7, we know that, for each $k$ ($k = 1, 2, \ldots, t$),

$$\text{fix}(g_i) \cap U_{A_{h(k)}} \neq \emptyset. \quad (17)$$

Obviously, it is a contradiction. \hfill $\Box$

4. **Algebraic Convergence**

In this section, we discuss the algebraic convergence of complex hyperbolic Kleinian groups. Firstly, we generalize Theorem 1 into the following form.

**Theorem 8.** Let $\{G_r\}$ be a sequence of groups of $\text{PU}(n,1)$ and $G_r$ be its algebraic limit group. Then we have the following.

1. If, for each $i$, $G_{r,i}$ is a complex Kleinian group, then $G_r$ is nonelementary and $G_r$ is discrete if and only if each one-generator subgroup of $W(G_r)$ is discrete.
2. If, for each $i$, $G_{r,i}$ is discrete, then $G_r$ is elementary if and only if for large enough $i$, all $G_{r,i}$ are elementary.
Proof. The proof of (1). The nonelementariness of $G_r$ follows from [21, Theorem 1.4]. Now, we come to prove that if $G_r$ is nondiscrete, then there is an element $f \in W(G_r)$ such that the subgroup $\langle f \rangle$ is nondiscrete. Suppose that $G_r$ is nondiscrete. Since $r < \infty$ (that is, $G_r$ is finitely generated), by Selberg’s Lemma we know that $G_r$ contains a torsion free subgroup $G_{r_i}$ with finite index which is nonelementary and nondiscrete either. Then there exists a sequence $\{f_j\}$ in $G_{r_i}$ such that

$$f_j \to I \quad \text{as } j \to \infty. \quad (18)$$

As $G_{r_i}$ is nonelementary, we can find finitely many loxodromic elements $g_1, g_2, \ldots, g_k$ in $G_{r_i}$ such that the set $\{\text{fix}(g_1), \text{fix}(g_2), \ldots, \text{fix}(g_k)\}$ spans $\partial M(G_{r_i})$, the boundary of $M(G_{r_i})$. Then, for large enough $j$, we have

$$N\left(f_j\right) + \sum_{q=1}^{n+1} N\left(\left[f_j, g_q^1\right]\right) < 2 - \sqrt{3}, \quad s \in \{1, 2, \ldots, k\}. \quad (19)$$

Let $f_{i,j}$ and $g_{i,s}$ be the corresponding elements of $f_j$ and $g_i$ in $G_{r_i,j}$, respectively. Then, for large enough $i$ and $j$,

$$N\left(f_{i,j}\right) + \sum_{q=1}^{n+1} N\left(\left[f_{i,j}, g_q^1\right]\right) < 2 - \sqrt{3}. \quad (20)$$

Lemma 3 implies that, for large enough $i$ and $j$, the subgroups $\langle f_{i,j}, g_{i,s}\rangle$ are elementary. Since the loxodromic elements of $PU(n, 1)$ form an open set, we know that, for sufficiently large $i$, $g_{i,s}$ are loxodromic as well. It follows that

$$\text{fix}(g_{i,s}) \subset \text{fix}(f_{i,j}), \quad (21)$$

which shows that, for $s \in \{1, 2, \ldots, k\}$ and all sufficiently large $j$,

$$\text{fix}(g_i) \subset \text{fix}(f_j). \quad (22)$$

Thus, for all sufficiently large $j$,

$$f_j \in W(G_{r_i}). \quad (23)$$

Since $G_{r_i}$ is torsion free, we know that there exists an element $f \in W(G_{r_i})$ such that $\langle f \rangle$ is nondiscrete. Note that $M(G_{r_i}) = M(G_{r_i})$, so $f \in W(G_{r_i})$. Hence, the conclusion of (1) follows.

The proof of (2). We only need to prove that if, for large enough $i$, all $G_{r_i,j}$ are elementary, then is $G_r$, since the converse is trivial by (1). Suppose that $G_r$ is nonelementary. Then we can find two loxodromic elements $f$ and $g$ in $G_r$ such that

$$\text{fix}(f) \cap \text{fix}(g) = \emptyset. \quad (24)$$

Let $f_i$ and $g_i$ be the corresponding elements of $f$ and $g$ in $G_{r_i,j}$, respectively. Then, for large enough $i$, we have

$$\text{fix}(f_i) \cap \text{fix}(g_i) = \emptyset. \quad (25)$$

It follows a discussion similar to that in the proof of (1) that, for large enough $i$, both $f_i$ and $g_i$ are loxodromic. This shows that, for large enough $i$, all $G_{r_i,j}$ are nonelementary. It is a contradiction.

Definition 9. Let $\{G_i\}$ be a sequence of complex Kleinian groups of $PU(n, 1)$. We say that $\{G_i\}$ satisfies $E$-condition if there is no sequence $\{f_j\}, f_j \in W(G_i)$ such that $f_j \to f$ as $i \to \infty$, where $f$ is an elliptic element with infinite order.

In the following, we give an example which shows that, if the sequence $\{G_i\}$ does not satisfy IP-condition but $E$-condition, then the limit group $G_r$ is still a complex Kleinian groups.

Example 10. Suppose that $H$ is a purely loxodromic nonelementary subgroup of $PU(1, 1)$ and, for each $j$,

$$f_j = \begin{bmatrix} e^{\pi/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (26)$$

Let $H$ be the Poincaré extension of $H$ in $PU(2, 1)$ and $G_j = \langle f_j \rangle$. Then it is easy to see that the algebraic limit group $G$ of $\{G_j\}$ is a complex Kleinian group. Note that $f_j \to I$ as $j \to \infty$; we know that $\{G_j\}$ does not satisfy IP-condition but $E$-condition.

As applications of Theorem 8 and $E$-condition, we have the following.

Theorem 11. Let $G_r$ be the algebraic limit group of complex Kleinian groups $\{G_{r,j}\}$ of $PU(n, 1)$. If $\{G_{r,j}\}$ satisfies $E$-condition, then $G_r$ is a complex Kleinian group.

Proof. By Theorem 8(1), we know that $G_r$ is nonelementary. Suppose that $G_r$ is nondiscrete. Then there exist an elliptic element $f \in W(G_r)$ and an integer sequence $\{n_j\}$ such that $\text{ord}(f) = \infty$ and

$$f^{n_j} \to I \quad \text{as } n_j \to \infty. \quad (27)$$

For each $n_j$, let $f_i^{n_j}$ be the corresponding element of $f^{n_j}$ in $G_{r,i}$. By [21, Lemma 4.2], we know that $f_i^{n_j} \in W(G_{r,i})$. It follows from the hypothesis that $\{G_{r,j}\}$ satisfies $E$-condition; we have $f_i^{n_j} \to I$ for large enough $i$. This implies that $f^{n_j} = I$. It is a contradiction. The proof is completed.

When $r \leq \infty$, Wang [17] proved the following.

Theorem W. Let $r \leq \infty$. If the generator system $\{g_{i,j}\}_{i=1}^r$ of $G_{r,j}$ satisfies that none are elliptic and no two have any fixed point in common, and, if all $G_{r,j}$ are Kleinian groups, then

1. all the generators $g_{i,j} = \lim_{n \to \infty} \delta g_{i,j}$ are neither elliptic nor identity;
2. if $G_r = \langle g_1, g_2, \ldots, g_r \rangle$ is nonelementary and $W(G_r)$ is discrete, then $G_r$ is discrete.

It easily follows a similar argument as in the proof of Theorem 8 and we can obtain the following.

Theorem 12. Let $r \leq \infty$. If the generator system $\{g_{i,j}\}_{i=1}^r$ of $G_{r,j}$ satisfies that none are elliptic and no two have any fixed point in common, and, if all $G_{r,j}$ are discrete, then
(1) \(G_r = \langle g_1, g_2, \ldots, g_r \rangle\) is nonelementary;
(2) \(G_r\) is discrete if and only if \(W(G_r)\) is discrete.

Acknowledgment

The research was partly supported by Tian-Yuan Foundation (no. 11226096).

References


