Research Article

Exponential Stability and Periodicity of Fuzzy Delayed Reaction-Diffusion Cellular Neural Networks with Impulsive Effect

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Received 3 September 2012; Accepted 4 January 2013

Academic Editor: Tingwen Huang

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This paper considers dynamical behaviors of a class of fuzzy impulsive reaction-diffusion delayed cellular neural networks (FIRDDCNNs) with time-varying periodic self-inhibitions, interconnection weights, and inputs. By using delay differential inequality, M-matrix theory, and analytic methods, some new sufficient conditions ensuring global exponential stability of the periodic FIRDDCNN model with Neumann boundary conditions are established, and the exponential convergence rate index is estimated. The differentiability of the time-varying delays is not needed. An example is presented to demonstrate the efficiency and effectiveness of the obtained results.

1. Introduction

The fuzzy cellular neural networks (FCNNs) model, which combines fuzzy logic with the structure of traditional neural networks (CNNs) [1–3], has been proposed by Yang et al. [4, 5]. Unlike previous CNNs structures, the FCNNs model has fuzzy logic between its template and input and/or output besides the "sum of product" operation. Studies have shown that the FCNNs model is a very useful paradigm for image processing and pattern recognition [6–8]. These applications heavily depend on not only the dynamical analysis of equilibrium points but also on that of the periodic oscillatory solutions. In fact, the human brain is naturally in periodic oscillatory [9], and the dynamical analysis of periodic oscillatory solutions is very important in learning theory [10, 11], because learning usually requires repetition. Moreover, an equilibrium point can be viewed as a special periodic solution of neural networks with arbitrary period. Stability analysis problems for FCNNs with and without delays have recently been probed; see [12–22] and the references therein. Yuan et al. [13] have investigated stability of FCNNs by linear matrix inequality approach, and several criteria have been provided for checking the periodic solutions for FCNNs with time-varying delays. Huang [14] has probed exponential stability of fuzzy cellular neural networks with distributed delay, without considering reaction-diffusion effects.

As we all know, many practical systems in physics, biology, engineering, and information science undergo abrupt changes at certain moments of time because of impulsive inputs [33]. Impulsive differential equations and impulsive neural networks have been received much interest in recent years; see, for example, [34–42] and the references therein. Yang and Xu [36] have investigated existence and exponential stability of periodic solution for impulsive delay differential equations and applications. Li and Lu [38] have discussed global exponential stability and existence of periodic solution of Hopfield-type neural networks with impulses without reaction-diffusion. To the best of our knowledge, few authors have probed the existence and exponential stability of the periodic solutions for the FIRDCCNN model with variable coefficients, and time-varying delays. As a result of the simultaneous presence of fuzziness, pulsing effects, reaction-diffusion phenomena, periodicity, variable coefficients and delays, the dynamical behaviors of this kind of model become much more complex and have not been properly addressed, which still remain important and challenging.

Motivated by the above discussion, we will establish some sufficient conditions for the existence and exponential stability of periodic solutions of this kind of FIRDCCNN model, applying delay differential inequality, \( M \)-matrix theory, and analytic methods. An example is employed to demonstrate the usefulness of the obtained results.

\[ + \sum_{j=1}^{n} b_{ij}(t) u_j(t) + f_j(t), \]

\[ + \sum_{j=1}^{n} a_{ij}(t) f_j(u_j(t, x)) \]

By the above discussion, we will establish some sufficient conditions for the existence and exponential stability of periodic solutions of this kind of FIRDCCNN model, applying delay differential inequality, \( M \)-matrix theory, and analytic methods. An example is employed to demonstrate the usefulness of the obtained results.

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where \( n \geq 2 \) is the number of neurons in the network and \( u_i(t, x) \) corresponds to the state of the \( i \)th neuron at time \( t \) and in space \( x \); \( D = \text{diag}(D_1, D_2, \ldots, D_n) \) is the diffusion-matrix and \( D_j \geq 0; \Delta = \sum_{j=1}^{n} [\partial^2/\partial x_j^2] \) is the Laplace operator; \( f_i(u_i(t, x)) \) denotes the activation function of the \( i \)th unit and \( u_i(t) \) the activation function of the \( i \)th unit; \( f_i(t) \) is an input at time \( t \); \( c_i(t) > 0 \) represents the rate with which the \( i \)th unit will reset its potential to the resting state in isolation when disconnected from the networks and external inputs at time \( t \); \( a_{ij}(t) \) and \( b_{ij}(t) \) are elements of feedback template and feed forward template at time \( t \), respectively. Moreover, in model (1), \( a_{ij}(t) \), \( b_{ij}(t) \), \( T_j(t) \), and \( H_j(t) \) are elements of fuzzy feedback \( MIN \) template, fuzzy feedback \( MAX \) template, fuzzy feed forward \( MIN \) template, and fuzzy feed forward \( MAX \) template at time \( t \), respectively; the symbols \( \wedge \) and \( \vee \) denote the fuzzy AND and fuzzy OR operation, respectively; time-varying delay \( \tau_j(t) \) is the transmission delay along the axon of the \( j \)th unit and satisfies \( 0 \leq \tau_j(t) \leq \tau_j \) (\( \tau_j \) is a constant); the initial condition \( u_i(t_0, x) \) is bounded and continuous on \( [t_0, \infty) \times \Omega \), where \( \tau = \max_{1 \leq j \leq n} \tau_j \). The fixed moments \( t_k \) satisfy \( 0 = t_0 < t_1 < t_2 < \ldots, \lim_{k \to +\infty} t_k = +\infty \), \( k \in \mathbb{N} \), \( u_i(t_k^+, x) \), and \( u_i(t_k^-, x) \) denote the right-hand and left-hand limits at \( t_k \), respectively. We always assume \( u_i(t_k^+, x) = u_i(t_k^-, x) \), for all \( k \in \mathbb{N} \). The initial value functions \( \psi(s, x) \) belong to \( \text{PC}_{2}\{[\tau, 0] \times \Omega; \mathbb{R}^n\} \). \( \text{PC}_{2}\{[\tau, 0] \times \Omega; \mathbb{R}^n\} = \{\psi: I \times \Omega \to L^2(\Omega) \mid \text{for every } t \in I, \psi(t, x) \in L^2(\Omega); \text{for any fixed } x \in \Omega, \psi(t, x) \text{ is continuous for all but at most countable points } s \in I \text{ and at these points, } \psi(s^+, x) \text{ and } \psi(s^-, x) \text{ exist, } \psi(s^+, x) = \psi(s^-, x) \mid, \psi(s^+, x) \text{ and } \psi(s^-, x) \text{ denote the right-hand and left-hand limit of the function } \psi(s, x) \}, \text{respectively.} \) Especially, let \( \text{PC}_{2} = \text{PC}_{2}\{[\tau, 0] \times \Omega; L^2(\Omega)\} \). For any \( \psi(t, x) = (\psi_1(t, x), \ldots, \psi_m(t, x)) \in \text{PC}_{2} \), suppose that \( \psi_i(t, x) = \sup_{-\tau < s < 0} |\psi_i(t + s, x)| \) exists as a finite number.
and introduce the norm \(\|\psi(t)\|_2 = (\sum_{i=1}^{n} \|\psi_i(t)\|_2^2)^{1/2}\), where \(\|\psi_i(t)\|_2 = (\int_{0}^{T} |\psi_i(x)|^2 dx)^{1/2}\).

Throughout the paper, we make the following assumptions.

(H1) There exists a positive diagonal matrix \(F = \text{diag}(F_1, F_2, \ldots, F_n)\), and \(G = \text{diag}(G_1, G_2, \ldots, G_n)\) such that

\[
F_j = \sup_{x \neq y} \frac{|f_j(x) - f_j(y)|}{x - y}, \quad G_j = \sup_{x \neq y} \frac{|g_j(x) - g_j(y)|}{x - y}
\]

for all \(x \neq y, j = 1, 2, \ldots, n\).

(H2) \(c(t) > 0, a_{ij}(t), b_{ij}(t), \alpha_{ij}(t), \beta_{ij}(t), T_i(t), H_i(t), u_i(t), I_i(t), \text{ and } \tau(t) \geq 0\) are periodic functions with a common positive period \(\omega\) for all \(t \geq t_0, i, j = 1, 2, \ldots, n\).

(H3) For \(\omega > 0\), \(i, j = 1, 2, \ldots, n\), there exists \(q \in \mathbb{Z}_+\) such that \(t_k + \omega = t_{k+q}\), \(I_k(u_j) = I_k(u_j)\) and \(I_k(u_i, t_k, x)\) are Lipschitz continuous in \(\mathbb{R}^n\).

**Definition 1.** The model in (1) is said to be globally exponentially periodic if (i) there exists one \(\omega\)-periodic solution and (ii) all other solutions of the model converge exponentially to it as \(t \to +\infty\).

**Definition 2** (see [26]). Let \(C = ([t - \tau, t], \mathbb{R}^n)\), where \(\tau \geq 0\) and \(F(t, x, y) \in C(\mathbb{R}^\tau \times \mathbb{R}^n \times C, \mathbb{R}^n)\). Then the function \(F(t, x, y) = (f_j(t, x, y), f_{j+1}(t, x, y), \ldots, f_n(t, x, y))^T\) is called an \(M\)-function, if (i) for every \(t \in \mathbb{R}^\tau, x \in \mathbb{R}^n, y \in C\), there holds \(F(t, x, y(t)) = F(t, x, y)\), for \(y(t)^{(1)} \leq y(t)^{(2)}\), where \(y(t)^{(1)} = (y_1(t)^{(1)}, \ldots, y_n(t)^{(1)})^T\) and \(y(t)^{(2)} = (y_1(t)^{(2)}, \ldots, y_n(t)^{(2)})^T\); (ii) every \(i\)th element of \(F\) satisfies \(f_i(t, x(t)^{(1)}, y(t)) \leq f_i(t, x(t)^{(2)}, y(t))\) for any \(y(t) \in C, t \in [t_0, t_1] \), where arbitrary \(x(t)^{(1)}\) and \(x(t)^{(2)}\) belong to \(\mathbb{R}^n\) and have the same \(i\)th component \(x_i(t)^{(1)} = x_i(t)^{(2)}\). Here, \(x(t)^{(1)} = (x_1(t)^{(1)}, \ldots, x_n(t)^{(1)})^T, x(t)^{(2)} = (x_1(t)^{(2)}, \ldots, x_n(t)^{(2)})^T\).

**Definition 3** (see [26]). A real matrix \(A = (a_{ij})_{n \times n}\) is said to be a nonsingular \(M\)-matrix if \(a_{ij} \leq 0\) \((i \neq j; i, j = 1, \ldots, n)\) and all successive principal minors of \(A\) are positive.

**Lemma 4** (see [13]). Let \(u\) and \(u^*\) be two states of the model in (1), then we have

\[
\left| \sum_{j=1}^{n} a_{ij}(t) f_j(u_j) - \sum_{j=1}^{n} a_{ij}(t) f_j(u^*_j) \right| \\
= \sum_{j=1}^{n} |a_{ij}(t)| \left| f_j(u_j) - f_j(u^*_j) \right|
\]

**Lemma 5** (see [26]). Assume that \(F(t, x, y)\) is an \(M\)-function, and (i) \(x(t) < y(t), t \in [t_0, t_1]\), (ii) \(D^\alpha y(t) > F(t, y(t), y'(t))\), \(D^\alpha x(t) \leq F(t, x(t), x'(t)), t \geq t_0\), where \(x'(t) = \sup_{t \leq s \leq 0} x(t + s), y'(t) = \sup_{t \leq s \leq 0} y(t + s)\). Then \(x(t) < y(t), t \geq t_0\).

3. Main Results and Proofs

We should first point out that, under assumptions (H1), (H2), and (H3), the FIRDCCN model (1) has at least one \(\omega\)-periodic solution of [26]. The proof of the existence of the \(\omega\)-periodic solution of (1) can be carried out similar to [26, 28] by the nonlinear functional analysis methods such as topological degree and here is omitted. We will mainly discuss the uniqueness of the periodic solution and its exponential stability.

**Theorem 6.** Assume that (H1)–(H3) holds. Furthermore, assume that the following conditions hold

\[
\left| \sum_{j=1}^{n} a_{ij} F_j (u) - \sum_{j=1}^{n} a_{ij} F_j (u^*) \right| \\
\leq \sum_{j=1}^{n} \left| a_{ij} \right| \left| f_j (u_j) - f_j (u^*_j) \right|
\]

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from \( \phi \) and \( u(t, x, \psi) = (u_1(t, x, \psi), \ldots, u_n(t, x, \psi))^T \), a solution of the system (1) starting from \( \psi \). Define
\[
u_i(\phi, x) = u(t + s, x, \phi), \quad s \in [-\tau, 0],
\]
and we can see that \( u_i(\phi, x), u_i(\psi, x) \in PC_{\Omega} \) for all \( t > 0 \). Let \( U_j = u_j(t, x, \phi) - u_j(t, x, \psi) \), then from (1) we get
\[
\frac{\partial U_i}{\partial t} = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left( D_i \frac{\partial U_i}{\partial x_j} \right) - c_i(t) U_i + \sum_{j=1}^{n} a_{ij}(t) \times \left[ f_j(u_j(t, x, \phi)) - f_j(u_j(t, x, \psi)) \right] \\
+ \left[ \sum_{j=1}^{n} \alpha_{ij}(t) f_j(u_j(t - \tau_j(t), x, \phi)) - \sum_{j=1}^{n} \alpha_{ij}(t) f_j(u_j(t - \tau_j(t), x, \psi)) \right] \\
+ \left[ \sum_{j=1}^{n} \beta_{ij}(t) f_j(u_j(t - \tau_j(t), x, \phi)) - \sum_{j=1}^{n} \beta_{ij}(t) f_j(u_j(t - \tau_j(t), x, \psi)) \right],
\]
for all \( t \neq t_k, x \in \Omega, i = 1, \ldots, n \). By boundary condition and Green Formula, we can get
\[
\int_{\Omega} U_j \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left( D_i \frac{\partial U_i}{\partial x_j} \right) dx \leq -D_i \int_{\Omega} (\nabla U_i)^2 dx.
\]
Then, from (8), (9), (H1)-(H2), Lemma 4, and the H"older inequality,
\[
\frac{d}{dt} \| U_i \|^2_2 \\
\leq -\varepsilon \| U_i \|^2_2 + 2 \sum_{j=1}^{n} \| a_{ij} \| F_j \| U_j \|_2 \| U_i \|_2 \\
+ 2 \sum_{j=1}^{n} \| \alpha_{ij} \| G_j \| U_i \|_2 \| U_j \|_2 \\
\times \left[ \| u_j(t - \tau_j(t), x, \phi) - u_j(t - \tau_j(t), x, \psi) \|_2 \\
\right]^2,
\]
for \( t \neq t_k \).

Thus,
\[
D^* \| U_i \|^2_2 \\
\leq -\varepsilon \| U_i \|^2_2 + 2 \sum_{j=1}^{n} \| \alpha_{ij} \| F_j \| U_j \|_2 \| U_i \|_2 \\
+ \sum_{j=1}^{n} \left( \| \alpha_{ij} \| + \| \beta_{ij} \| \right) G_j \| U_i \|_2 \| U_j \|_2,
\]
for \( i = 1, \ldots, n \). Since \( C = (\bar{A}F + (\bar{A} + \bar{B}G) \) is a nonsingular M-matrix, there exists a vector \( \xi = (\xi_1, \ldots, \xi_n)^T > 0 \) such that
\[
-\xi_1 \varepsilon + \sum_{j=1}^{n} \xi_j \left( \| \alpha_{ij} \| F_j + \left( \| \alpha_{ij} \| + \| \beta_{ij} \| \right) G_j \right) < 0.
\]
Considering functions
\[
\Psi_i(y) = \xi_i (y - \xi),
\]
we know from (11) that \( \Psi_i(0) < 0 \) and \( \Psi_i(y) \) is continuous. Since \( d\Psi_i(y)/dy > 0 \, \Psi_i(y) \) is strictly monotonically increasing, there exists a scalar \( \lambda_i > 0 \) such that
\[
\Psi_i(\lambda_i) = \xi_i (\lambda_i - \xi),
\]
for \( i = 1, \ldots, n \).
Choosing $0 < \lambda < \min\{\lambda_1, \ldots, \lambda_n\}$, we have

$$
\xi_i (\lambda_i - \xi)
+ \sum_{j=1}^{n} \xi_j (\|\tilde{a}_{ij}\| F_j + e^{\lambda_j} (\|\tilde{a}_{ij}\| + \|\tilde{B}_{ij}\|) G_j) < 0,
$$

(14)

for $i = 1, \ldots, n$.

That is,

$$
\lambda \xi - (C - AF) \xi + (\pi + \tilde{B}) G \xi e^{\lambda t} < 0.
$$

(15)

Furthermore, choose a positive scalar $p$ large enough such that

$$
pe^{-\lambda t} \xi > (1, 1, \ldots, 1)^T, \quad t \in [-\tau, 0].
$$

(16)

For any $\varepsilon > 0$, let

$$
r(t) = pe^{-\lambda t} (\|\phi - \psi\|_2 + \varepsilon) \xi, \quad t_0 \leq t < t_1.
$$

(17)

From (15)–(17), we obtain

$$
D^+ r(t) > -(C - AF) r(t) + (\pi + \tilde{B}) G r'(t)
= V(t, r(t), r'(t)), \quad t_0 \leq t < t_1,
$$

(18)

where $r'(t) = (r_1'(t), \ldots, r_n'(t))^T$ and $r_1'(t) = \sup_{\tau \leq s \leq 0} pe^{-\lambda (t+s)}. (\|\phi - \psi\|_2 + \varepsilon) \xi_i$. It is easy to verify that $V(t, r(t), r'(t))$ is an $M$-function. It follows also from (16) and (17) that

$$
\|u\|_2 \leq \|\phi - \psi\|_2 < pe^{-\lambda t} \xi_i \|\phi - \psi\|_2
< r_1(t), \quad t \in [-\tau, 0], \quad i = 1, 2, \ldots, n.
$$

(19)

Denote

$$
U^\circ := (\|u_t(t, x, \phi) - u_1(t, x, \psi)\|_2, \ldots, \|u_n(t, x, \phi) - u_1(t, x, \psi)\|_2)^T,
$$

$$
U^{\circ(i)} := (\|u_1(t, x, \phi) - u_1(t, x, \psi)\|_2^{(i)}, \ldots, \|u_n(t, x, \phi) - u_n(t, x, \psi)\|_2^{(i)})^T,
$$

(20)

where $\|u\|_2^{(i)} = \sup_{t \leq s \leq 0} \|u_i(t + s, x, \phi) - u_i(t + s, x, \psi)\|_2$, then

$$
U^\circ < r(t), \quad t \in [-\tau, 0].
$$

(21)

From (10), we can obtain

$$
D^+ U^\circ \leq -(C - AF) U^\circ + (\pi + \tilde{B}) GU^{\circ(i)}
= V(t, U^\circ, U^{\circ(i)}), \quad t \neq t_k.
$$

(22)

Now, it follows from (18)–(22) and Lemma 5 that

$$
U^\circ < r(t) = pe^{-\lambda t} (\|\phi - \psi\|_2 + \varepsilon) \xi, \quad t_0 \leq t < t_1.
$$

(23)

Letting $\varepsilon \to 0$, we have

$$
U^\circ \leq p \xi \|\phi - \psi\| e^{-\lambda t}, \quad t_0 \leq t < t_1.
$$

(24)

And moreover, from (24), we get

$$
\left( \sum_{i=1}^{n} \|u\|_2^2 \right)^{1/2} \leq p \left( \sum_{i=1}^{n} \|u\|_2^2 \right)^{1/2} \|\phi - \psi\| e^{-\lambda t}, \quad t_0 \leq t < t_1.
$$

(25)

Let $M = \rho(\sum_{i=1}^{n} \xi_i^2)^{1/2}$, then $M \geq 1$. Define $W(t) = \|u_t(x, \phi) - u_t(x, \psi)\|_2$; it follows from (25) and the definitions of $u_t(\phi, x)$ and $u_t(\psi, x)$ that

$$
W(t) \leq \eta_0 \cdots \eta_{l-1} M \|\phi - \psi\| e^{-\lambda t}, \quad t_0 \leq t < t_1.
$$

(26)

It is easily observed that

$$
W(t) \leq M \|\phi - \psi\| e^{-\lambda t}, \quad -\tau \leq t \leq t_0 = 0.
$$

(27)

Because (26) holds, we can suppose that for $l \leq k$ inequality

$$
W(t) \leq \eta_0 \cdots \eta_{l-1} M \|\phi - \psi\| e^{-\lambda t},
$$

(28)

holds, where $\eta_0 = 1$. When $l = k + 1$, we note (H5) that

$$
W(t_k) = \|u_t(x, \phi) - u_{t_k}(x, \psi)\|_2
= \|h_k(u_{t_k}(x, \phi) - h_k(u_{t_k}(x, \psi))\|_2
\leq \rho(\Gamma^2_k) \|u_t(x, \phi) - u_{t_k}(x, \psi)\|_2
= \rho(\Gamma^2_k) W(t_k)
\leq \eta_0 \cdots \eta_{l-1} \rho(\Gamma^2_k) M \|\phi - \psi\| e^{-\lambda t_k}
\leq \eta_0 \cdots \eta_{l} \eta_k \rho(\Gamma^2_k) M \|\phi - \psi\| e^{-\lambda t_k},
$$

(29)

where $\rho(\Gamma^2_k)$ is the spectral radius of $\Gamma^2_k$. Let $M = \max\{M, \rho(\Gamma^2_k) M\}$, by (28), (29), and $\eta \geq 1$, we obtain

$$
W(t) \leq \eta_0 \cdots \eta_{l-1} \eta_k M \|\phi - \psi\| e^{-\lambda t},
$$

(30)

$t_k - \tau \leq t \leq t_k$.

Combining (10), (17), (30), and Lemma 5, we get

$$
W(t) \leq \eta_0 \cdots \eta_{l-1} \eta_k M \|\phi - \psi\| e^{-\lambda t},
$$

(31)

$t_k \leq t < t_{k+1}, \quad k \in \mathbb{N}^+$.

Applying mathematical induction, we conclude that

$$
W(t) \leq \eta_0 \cdots \eta_{l-1} M \|\phi - \psi\| e^{-\lambda t},
$$

(32)

$t_{k-1} \leq t < t_k, \quad k \in \mathbb{N}^+$.
From (H6) and (32), we have
\[ W(t) \leq e^{\eta t_1} e^{\eta (t_2 - t_1)} \cdots e^{\eta (t_k - t_{k-1})} \]
\[ \times M \| \phi - \psi \| e^{-(\lambda - \eta)t}, \]
\[ t_{k-1} \leq t < t_k, \quad k \in \mathbb{N}^+. \]
This means that
\[ \| u_t(x, \phi) - u_t(x, \psi) \|_2 \leq M \| \phi - \psi \|_2 e^{-\lambda t}, \]
\[ M = \left\{ \begin{array}{ll}
M & \text{if } t \geq t_0, \\
M 2^n & \text{if } t < t_0,
\end{array} \right. \tag{34} \]
choosing a positive integer $N$ such that
\[ Me^{-(\lambda - \eta)(N\omega - t)} \leq \frac{1}{6}. \tag{35} \]
Define a Poincare mapping $\mathfrak{D} : \Gamma \rightarrow \Gamma$ by
\[ \mathfrak{D}(\phi) = u_{N\omega}(x, \phi), \tag{36} \]
Then
\[ \mathfrak{D}^N(\phi) = u_{N^2\omega}(x, \phi). \tag{37} \]
Setting $t = N\omega$ in (34), from (35) and (37), we have
\[ \| \mathfrak{D}^N(\phi) - \mathfrak{D}^N(\psi) \|_2 \leq \frac{1}{6} \| \phi - \psi \|_2, \tag{38} \]
which implies that $\mathfrak{D}^N$ is a contraction mapping. Thus, there exists a unique fixed point $\phi^* \in \Gamma$ such that
\[ \mathfrak{D}^N(\mathfrak{D}(\phi^*)) = \mathfrak{D}(\mathfrak{D}^N(\phi^*)) = \mathfrak{D}(\phi^*). \tag{39} \]
From (37), we know that $\mathfrak{D}(\phi^*)$ is also a fixed point of $\mathfrak{D}^N$, and then it follows from the uniqueness of the fixed point that
\[ \mathfrak{D}(\phi^*) = \phi^*, \quad \text{that is, } u_{N\omega}(x, \phi^*) = \phi^*. \tag{40} \]
Let $u(t, x, \phi^*)$ be a solution of the model (1), then $u(t + \omega, x, \phi^*)$ is also a solution of the model (1). Obviously,
\[ u_{N\omega}(x, \phi^*) = u_t(x, \phi^*), \tag{41} \]
for all $t \geq t_0$. Hence, $u(t + \omega, x, \phi^*) = u(t, x, \phi^*)$, which shows that $u(t, x, \phi^*)$ is exactly one $\omega$-periodic solution of model (1). It is easy to see that all other solutions of model (1) converge to this periodic solution exponentially as $t \rightarrow +\infty$, and the exponential convergence rate index is $\lambda - \eta$. The proof is completed. \qed

Remark 7. When $\zeta(t) = \zeta, a_i(t) = a_i, b_i(t) = b_i, \alpha_i(t) = \alpha_i, \beta_i(t) = \beta_i, T_{ij}(t) = T_{ij}, H_{ij}(t) = H_{ij}, v_i(t) = v_i, I_i(t) = I_i$, and $\tau_i$ are constants, then the model (1) is changed into
\[ \frac{\partial u_i(t, x)}{\partial t} = \sum_{j=1}^n \frac{\partial}{\partial x_j} \left( D_i \frac{\partial u_i(t, x)}{\partial x_j} \right) \]
\[ - \zeta u_i(t, x) + \sum_{j=1}^n a_{ij} f_j(u_j(t, x)) \]
\[ + \sum_{j=1}^n b_{ij} v_j(t) + I_i(t) \]
\[ + \sum_{j=1}^n \alpha_{ij} g_j(u_j(t - \tau_j(t), x)) \]
\[ + \sum_{j=1}^n \beta_{ij} g_j(u_j(t - \tau_j(t), x)) \tag{42} \]
For any positive constant $\omega \geq 0$, we have $\zeta(t + \omega) = \zeta(t), a_i(t + \omega) = a_i(t), b_i(t + \omega) = b_i(t), \alpha_i(t + \omega) = \alpha_i(t), \beta_i(t + \omega) = \beta_i(t), T_{ij}(t + \omega) = T_{ij}(t), H_{ij}(t + \omega) = H_{ij}(t), v_i(t + \omega) = v_i(t), I_i(t + \omega) = I_i(t), \text{and } \tau_i(t + \omega) = \tau_i(t) \text{ for } t \geq t_0.$ Thus, the sufficient conditions in Theorem 6 are satisfied.

Remark 8. If $I_{k}(\cdot) = 0$, the model (1) is changed into
\[ \frac{\partial u_i(t, x)}{\partial t} = \sum_{j=1}^m \frac{\partial}{\partial x_j} \left( D_i \frac{\partial u_j(t, x)}{\partial x_j} \right) \]
\[ - \zeta_i(t) u_i(t, x) + \sum_{j=1}^n a_{ij}(t) f_j(u_j(t, x)) \]
\[ + \sum_{j=1}^n b_{ij}(t) v_j(t) + I_i(t) \]
\[ + \sum_{j=1}^n \alpha_{ij}(t) g_j(u_j(t - \tau_j(t), x)) \]
\[ + \sum_{j=1}^n \beta_{ij}(t) g_j(u_j(t - \tau_j(t), x)) \]
which has been discussed in [22]. As Song and Wang have pointed out, the model (43) is more general than some well-studied fuzzy neural networks. For example, when \( c_i(t) > 0, a_{ij}(t), b_{ij}(t), \), \( i, j = 1, 2, \ldots, n \), then model (43) reduces the model which has been studied by Huang [19]. Moreover, if \( D_i = 0, \tau_{ij}(t) = 0, \) \( f_j(\theta) = g_j(\theta) = (1/2)[|\theta + 1| - |\theta - 1|], (i = 1, \ldots, n) \), then model (42) covers the model studied by Yang et al. [4, 5] as a special case. If \( D_i = 0 \) and \( \tau_{ij}(t) \) is assumed to be differentiable for \( i, j = 1, 2, \ldots, n \), then model (43) can be specialized to the model investigated in Liu and Tang [12] and Yuan et al. [13]. Obviously, our results are less conservative than that of the above-mentioned literature, because they do not consider impulsive effects.

4. Numerical Examples

Example 9. Consider a two-neuron FIRDCNN model:

\[
\frac{\partial u_i(t, x)}{\partial t} = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left( D_{ij} \frac{\partial u_i(t, x)}{\partial x_j} \right) - c_i(t) u_i(t, x)
\]

where \( i = 1, 2, c_1(t) = 26, c_2(t) = 20.8, a_{11}(t) = -1 - \cos(t), a_{12}(t) = 1 + \cos(t), a_{21}(t) = 1 + \sin(t), a_{22}(t) = -1 - \sin(t), D_1 = 8, D_2 = 4, \partial u_i(t, x)/\partial n = 0 \) (\( t \geq t_0, x = 0, 2\pi \)), \( \gamma_{ie} = 0.4, \gamma_{iec} = 0.2, \), \( j = 1, 2, \), \( g_j(u_j(t-1, x)) = u_j(t-1, x) e^{-u_j(t-1, x)} (j = 1, 2), \) \( \alpha_{11}(t) = -12.8, \alpha_{12}(t) = \alpha_{12}(t) = -1 + \cos(t), \alpha_{22}(t) = -10, \beta_{11}(t) = 12.8, \beta_{12}(t) = -1 + \sin(t) = \beta_{21}(t), \beta_{22}(t) = \)}
and delays have been investigated. By using Halanay’s delay differential inequality, $M$-matrix theory, and analytic methods, some new sufficient conditions have been established to guarantee the existence, uniqueness, and global exponential stability of the periodic solution. Moreover, the exponential convergence rate index can be estimated. An example and its simulation have been given to show the effectiveness of the obtained results. In particular, the differentiability of the time-varying delays has been removed. The dynamic behaviors of fuzzy neural networks with the property of exponential periodicity are of great importance in many areas such as learning systems.

**Acknowledgments**

The authors would like to thank the editors and the anonymous reviewers for their valuable comments and constructive suggestions. This research is supported by the Natural Science Foundation of Guangxi Autonomous Region (no. 2012GXNSFBA053003), the National Natural Science Foundations of China (60973048, 61272077, 60974025, 60673101, and 60939003), National 863 Plan Project (2008 AA04Z401, 2009 AA043404), the Natural Science Foundation of Shandong Province (no. Y2007G30), the Scientific and Technological Project of Shandong Province (no. 2007GG3WZ04016), the Science Foundation of Harbin Institute of Technology (Weihai) (HIT (WH) 200807), the Natural Science Research Innovation Foundation in Harbin Institute of Technology (HIT. NSRIF. 2001120), the China Postdoctoral Science Foundation (2010048 1000), and the Shandong Provincial Key Laboratory of Industrial Control Technique (Qingdao University).

**References**


