Research Article

Boundedness for Parametrized Littlewood-Paley Operators with Rough Kernels on Weighted Weak Hardy Spaces

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The authors prove that the parametrized area integral $\mu_{\Omega,S}^{\rho}$ and g_{λ}^{*} function $\mu_{\lambda}^{*,\rho}$ are bounded from the weighted weak Hardy space $WH_w^1(\mathbb{R}^n)$ to the weighted weak Lebesgue space $WL_w^1(\mathbb{R}^n)$ as Ω satisfies a class of the integral Dini condition, respectively.

1. Introduction and Main Results

Suppose that $\Omega \in L^1(S^{n-1})$ is homogeneous of degree zero on $\mathbb{R}^n$ and satisfies

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0, \quad (1)$$

where S^{n-1} denotes the unit sphere of $\mathbb{R}^n$ ($n \geq 2$) equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')/|x|$, $x' \neq 0$. The parametrized area integral $\mu_{\Omega,S}^{\rho}$ and g_{λ}^{*} function $\mu_{\lambda}^{*,\rho}$ are defined by

$$\begin{aligned} \mu_{\Omega,S}^{\rho}(f)(x) &= \left(\int \int_{\Gamma(x)} \left| \frac{1}{t^{\rho}} \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}, \\ \mu_{\lambda}^{*,\rho}(f)(x) &= \left(\int \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \right. \\ &\quad \times \left. \left| \frac{1}{t^{\rho}} \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}, \end{aligned} \quad (2)$$

respectively, where

$$\begin{aligned} \Gamma(x) &= \{(y, t) \in \mathbb{R}_+^{n+1} : |x-y| < t\}, \\ \mathbb{R}_+^{n+1} &= \mathbb{R}^n \times (0, \infty), \quad \rho > 0, \quad \lambda > 1. \end{aligned} \quad (3)$$

It is well known that Littlewood-Paley functions are very important tools in harmonic analysis and PDE (see [1–3]). Some well-known results related to the classical Littlewood-Paley operators can be seen in [4–8]. In 1999, inspired by Hörmander's work [9], when Ω satisfies the Lipschitz condition of α , Sakamoto and Yabuta [10] established the L^p ($1 < p < \infty$) boundedness of the parametrized area integral $\mu_{\Omega,S}^{\rho}$ and the parametrized g_{λ}^{*} function $\mu_{\lambda}^{*,\rho}$ and gave the boundedness on BMO spaces and Campanato spaces. For any $0 < \alpha \leq 1$, $1 < q \leq \infty$, it is easy to see that the inclusion relationship

$$\begin{aligned} \text{Lip}_{\alpha}(S^{n-1}) &\subsetneq L^q(S^{n-1}) \subsetneq L \log^+ L(S^{n-1}) \\ &\subsetneq H^1(S^{n-1}) \subsetneq L^1(S^{n-1}) \end{aligned} \quad (4)$$

holds. In 2002, Ding et al. [11] extended the previous L^p -boundedness to the case as Ω belongs to $L \log^+ L(S^{n-1})$. In 2007, Ding et al. [12, 13] gave the boundedness of the parametrized area integral $\mu_{\Omega,S}^{\rho}$ and g_{λ}^{*} function $\mu_{\lambda}^{*,\rho}$ on the Hardy space and weak Hardy space when Ω satisfies a class of the integral Dini conditions. Recently, Wang and Liu [14]

obtained the boundedness on the weighted Hardy space for the parametrized Littlewood-Paley operators with Ω satisfying the logarithmic type Lipschitz conditions. On the other hand, the boundedness properties of the intrinsic square functions on weighted weak Hardy spaces were studied by Wang in [15]. Inspired by the results mentioned previously, in this paper, we will study the boundedness of the parametrized area integral $\mu_{\Omega,S}^p$ and g_λ^* function $\mu_\lambda^{*,p}$ on the weighted weak Hardy spaces.

Before stating our main results, let us recall some definitions. Firstly, let $\Omega(x') \in L^q(S^{n-1})$, $q \geq 1$. Then, the integral modulus $\omega_q(\delta)$ of continuity of order q of Ω is defined by

$$\omega_q(\delta) = \sup_{\|\gamma\| \leq \delta} \left(\int_{S^{n-1}} |\Omega(\gamma x') - \Omega(x')|^q d\sigma(x') \right)^{1/q}, \quad (5)$$

where, γ denotes a rotation on S^{n-1} and $\|\gamma\| = \sup_{x' \in S^{n-1}} |\gamma x' - x'|$. The function Ω is said to satisfy the L^q -Dini condition, if

$$\int_0^1 \frac{\omega_q(\delta)}{\delta} d\delta < \infty. \quad (6)$$

Secondly, given a weight function w on $\mathbb{R}^n$, for $1 \leq p < \infty$, the weighted Lebesgue spaces is defined by

$$L_w^p(\mathbb{R}^n) = \left\{ f : \|f\|_{p,w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty \right\}. \quad (7)$$

And also, the weighted weak Lebesgue spaces is defined by

$$WL_w^p(\mathbb{R}^n) = \left\{ f : \|f\|_{WL_w^p} = \sup_{\lambda > 0} \lambda \cdot w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})^{1/p} < \infty \right\}. \quad (8)$$

Let us now turn to recall the definition of the weighted weak Hardy spaces. The weak Hardy spaces were first introduced in [16]. The atomic decomposition theory of weak H^1 spaces on $\mathbb{R}^n$ was given by Fefferman and Soria in [17]. Later, Liu established the weak H^p spaces on homogeneous groups in [18]. In 2000, Quek and Yang introduced the weighted weak Hardy spaces $WH_w^p(\mathbb{R}^n)$ in [19] and established their atomic decompositions. Moreover, by using the atomic decomposition theory of $WH_w^p(\mathbb{R}^n)$, Quek and Yang also obtained the boundedness of $C-Z$ operators on these weighted spaces in [19]. Let $w \in A_{\infty}$, $0 < p \leq 1$, and $N = [n(q_w/p - 1)]$. Define

$$\mathcal{A}_{N,w} = \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} \sup_{|\alpha| \leq N+1} (1 + |x|)^{N+n+1} |D^\alpha \varphi(x)| \leq 1 \right\}, \quad (9)$$

where, $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $D^\alpha \varphi = \partial^{|\alpha|} \varphi / (\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n})$.

For $f \in \mathcal{S}'(\mathbb{R}^n)$, the grand maximal function of f is defined by

$$G_w f(x) = \sup_{\varphi \in \mathcal{A}_{N,w}} \sup_{|\gamma - x| < t} |(\varphi_t * f)(\gamma)|. \quad (10)$$

Then, weighted weak Hardy space is defined by $WH_w^p(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : G_w f \in WL_w^p(\mathbb{R}^n)\}$. Moreover, we set $\|f\|_{WH_w^p} = \|G_w f\|_{WL_w^p}$.

Our main results are stated as follows.

Theorem 1. Let $\Omega \in L^2(S^{n-1})$ satisfying (1) and the following condition

$$\int_0^1 \frac{\omega_2(\delta)}{\delta^{1+\alpha}} d\delta < \infty, \quad 0 < \alpha \leq 1. \quad (11)$$

Then, for $p > n/2$, $w \in A_1$, there exists a constant $C > 0$ such that

$$\|\mu_{\Omega,S}^p(f)\|_{WL_w^1} \leq C \|f\|_{WH_w^1}. \quad (12)$$

The relationship between condition (11) and $\text{Lip}_\alpha(S^{n-1})$ condition is not clear up to now. We point that the conclusion of Theorem 1 still holds if we replace the condition (11) by the $\text{Lip}_\alpha(S^{n-1})$ ($0 < \alpha \leq 1$) condition. In other words, we have the following result.

Theorem 2. Let $\Omega \in \text{Lip}_\alpha(S^{n-1})$, $0 < \alpha \leq 1$, satisfying (1). Then, for $p > n/2$, $w \in A_1$, there exists a constant $C > 0$ such that

$$\|\mu_{\Omega,S}^p(f)\|_{WL_w^1} \leq C \|f\|_{WH_w^1}. \quad (13)$$

Theorem 3. Let $\Omega \in L^2(S^{n-1})$ satisfying (1) and the following condition

$$\int_0^1 \frac{\omega_2(\delta)}{\delta^{1+\alpha}} d\delta < \infty, \quad 0 < \alpha \leq 1. \quad (14)$$

Then, for $p > n/2$, $w \in A_1$, $\lambda > 2$, there exists a constant $C > 0$ such that

$$\|\mu_\lambda^{*,p}(f)\|_{WL_w^1} \leq C \|f\|_{WH_w^1}. \quad (15)$$

2. Notations and Preliminaries

In this section, we will introduce some notations and preliminary lemmas used in the proofs of our main theorems in the next section.

The classical A_p weighted theory was first introduced by Muckenhoupt in the study of weighted L^p boundedness of Hardy-Littlewood maximal functions in [20]. A weight w is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0, \infty)$ at almost everywhere. Given a ball B and $\lambda > 0$, λB denotes the ball with the same center as B whose radius is λ times that of B . We also denote the weighted measure of E by $w(E)$; that is, $w(E) = \int_E w(x) dx$. We say that $w \in A_p$ with $1 < p < \infty$ if there exists a constant $C > 0$, such that for every ball $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C. \quad (16)$$

We say that $w \in A_1$ if

$$\frac{1}{|B|} \int_B w(x) dx \leq C \cdot \operatorname{ess} \inf_{x \in B} w(x) \quad \text{for every ball } B \subset \mathbb{R}^n. \quad (17)$$

A weight function $w \in A_\infty$ if it satisfies the A_p condition for some $1 < p < \infty$. It is well known that if $w \in A_p$, $1 < p < \infty$, then $w \in A_r$ for all $r > p$, and $w \in A_q$ for some $1 < q < p$. We thus write $q_w \equiv \inf\{q > 1 : w \in A_q\}$ to denote the critical index of w .

Lemma 4 (see [21]). *Let $1 \leq p < \infty$, $w \in A_p$. Then, for any ball B , there exists an absolute constant $C > 0$, such that*

$$w(2B) \leq Cw(B). \quad (18)$$

In general, for any $\lambda > 0$, we have

$$w(\lambda B) \leq C\lambda^{np}w(B), \quad (19)$$

where C does not depend on B nor on λ .

Lemma 5 (see [19]). *Let $0 < p \leq 1$, $w \in A_\infty$. For every $f(x)$ belongs to $WH_w^p(\mathbb{R}^n)$, there exists a sequence of bounded measurable functions $\{f_k(x)\}_{k=-\infty}^\infty$ such that*

$$(i) \quad f(x) = \sum_{k=-\infty}^\infty f_k(x), \text{ in } \mathcal{S}',$$

$$(ii) \quad \text{each } f_k \text{ can be further decomposed into } f_k = \sum_i b_i^k, \text{ where } b_i^k \text{ satisfies the following conditions.}$$

$$(a) \quad \operatorname{supp}(b_i^k) \subset Q_i^k, \text{ where } Q_i^k \text{ denotes the ball with center } x_i^k \text{ and radius } r_i^k. \text{ Moreover,}$$

$$\sum_i w(Q_i^k) \leq C_1 2^{-kp}, \quad \sum_i \chi_{Q_i^k} \leq C_1, \quad (20)$$

where χ_E denotes the characteristic function of the set E and $C_1 \leq \|f\|_{WH_w^p}^p$.

$$(b) \quad \|b_i^k\|_{L^\infty} \leq C2^k, \text{ where } C > 0 \text{ is independent of } i, k.$$

$$(c) \quad \int_{\mathbb{R}^n} b_i^k x^\alpha dx = 0 \text{ for every multi-index } \alpha \text{ with } |\alpha| \leq [n(q_w/p - 1)].$$

Conversely, if $f \in \mathcal{S}'(\mathbb{R}^n)$ have a decomposition satisfying (i) and (ii), then $f \in WH_w^p(\mathbb{R}^n)$. Moreover, we have $\|f\|_{WH_w^p}^p \sim C$.

In the end of this section, we need the following lemmas used in the next section.

Lemma 6 (see [22]). *Suppose that $\Omega \in L^2(S^{n-1})$ satisfies (1) and the following condition*

$$\int_0^1 \frac{\omega_2(\delta)}{\delta} (1 + |\log \delta|)^\sigma d\delta < \infty, \quad \sigma > 1, \quad (21)$$

where $w \in A_p$. Then, for $\rho > n/2$, $\lambda > 2$, and $f \in L^p(\mathbb{R}^n)$ ($1 < p < \infty$), there is a constant C independent of f , such that

$$\begin{aligned} \|\mu_{\Omega, S}^\rho(f)\|_{L_w^p} &\leq C\|f\|_{L_w^p}, \\ \|\mu_{\lambda}^{*, \rho}(f)\|_{L_w^p} &\leq C\|f\|_{L_w^p}. \end{aligned} \quad (22)$$

Lemma 7 (see [23]). *Suppose that $\rho > 0$, Ω is homogeneous of degree zero and satisfies the L^2 -Dini condition. If there exists a constant $0 < \theta < 1/2$ such that $|x| < \theta R$, then we have*

$$\begin{aligned} &\left(\int_{R < |y| < 2R} \left| \frac{\Omega(y-x)}{|y-x|^{n-\rho}} - \frac{\Omega(y)}{|y|^{n-\rho}} \right|^2 dy \right)^{1/2} \\ &\leq CR^{n/2-(n-\rho)} \left\{ \frac{|x|}{R} + \int_{|x|/2R < \delta < |x|/R} \frac{\omega_2(\delta)}{\delta} d\delta \right\}, \end{aligned} \quad (23)$$

where the constant $C > 0$ is independent of R, x .

3. Proof of Main Results

Proof of Theorem 1. In order to prove Theorem 1, it suffices to show that there exists a constant $C > 0$, for any $f \in WH_w^1(\mathbb{R}^n)$ and $\beta > 0$, such that

$$\beta w(\{x \in \mathbb{R}^n : |\mu_{\Omega, S}^\rho(f)(x)| > \beta\}) \leq C\|f\|_{WH_w^1}. \quad (24)$$

Take $k_0 \in \mathbb{Z}$ such that $2^{k_0} \leq \beta < 2^{k_0+1}$; then by Lemma 5 we can write

$$f = \sum_{k=-\infty}^\infty f_k = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+1}^\infty f_k =: F_1 + F_2, \quad (25)$$

where $F_1 = \sum_{k=-\infty}^{k_0} \sum_i b_i^k$, $F_2 = \sum_{k=k_0+1}^\infty \sum_i b_i^k$, and $\{b_i^k\}$ satisfies (a)–(c) in Lemma 5. Then, we have

$$\begin{aligned} &\beta w(\{x \in \mathbb{R}^n : |\mu_{\Omega, S}^\rho(f)(x)| > \beta\}) \\ &\leq \beta w\left(\left\{x \in \mathbb{R}^n : |\mu_{\Omega, S}^\rho(F_1)(x)| > \frac{\beta}{2}\right\}\right) \\ &\quad + \beta w\left(\left\{x \in \mathbb{R}^n : |\mu_{\Omega, S}^\rho(F_2)(x)| > \frac{\beta}{2}\right\}\right) \\ &=: I_1 + I_2. \end{aligned} \quad (26)$$

First, we claim that the following inequality holds:

$$\|F_1\|_{L_w^2} \leq C\beta^{1/2}\|f\|_{WH_w^1}^{1/2}. \quad (27)$$

In fact, since $\operatorname{supp}(b_i^k) \subset Q_i^k = Q(x_i^k, r_i^k)$, $\|b_i^k\|_{L^\infty} \leq C2^k$, then it follows from Minkowski's integral inequality that

$$\begin{aligned} \|F_1\|_{L_w^2} &\leq C \sum_{k=-\infty}^{k_0} \sum_i \|b_i^k\|_{L^\infty} w(Q_i^k)^{1/2} \\ &\leq C \sum_{k=-\infty}^{k_0} 2^k \left(\sum_i w(Q_i^k) \right)^{1/2} \\ &\leq C \sum_{k=-\infty}^{k_0} 2^{k/2} \|f\|_{WH_w^1}^{1/2} \\ &\leq C \sum_{k=-\infty}^{k_0} 2^{(k-k_0)/2} \beta^{1/2} \|f\|_{WH_w^1}^{1/2} \leq C\beta^{1/2} \|f\|_{WH_w^1}^{1/2}. \end{aligned} \quad (28)$$

Let us estimate I_1 . By Chebyshev's inequality, Lemma 6 and (27), we have

$$\begin{aligned} I_1 &\leq \beta \frac{4}{\beta^2} \int_{\mathbb{R}^n} \left| \mu_{\Omega, S}^\rho(F_1)(x) \right|^2 w(x) dx \\ &= C \frac{1}{\beta \left\| \mu_{\Omega, S}^\rho(F_1) \right\|_{L_w^2}^2} \\ &\leq C \frac{1}{\beta \left\| F_1 \right\|_{L_w^2}^2} \\ &\leq C \|f\|_{WH_w^1}. \end{aligned} \quad (29)$$

Now we turn our attention to the estimate of I_2 . If we set

$$A_{k_0} = \bigcup_{k=k_0+1}^{\infty} \bigcup_i \widetilde{Q}_i^k, \quad (30)$$

where $\widetilde{Q}_i^k = Q(x_i^k, 8\tau^{(k-k_0)/n} r_i^k)$ and τ is a fixed positive number such that $1 < \tau < 2$, therefore,

$$I_2 = \beta w \left(\left\{ x \in 2A_{k_0} : \left| \mu_{\Omega, S}^\rho(F_2)(x) \right| > \frac{\beta}{2} \right\} \right)$$

$$\begin{aligned} &+ \beta w \left(\left\{ x \in (2A_{k_0})^c : \left| \mu_{\Omega, S}^\rho(F_2)(x) \right| > \frac{\beta}{2} \right\} \right) \\ &=: I_2' + I_2''. \end{aligned} \quad (31)$$

Since $w \in A_1$, then by Lemma 4 we can get

$$\begin{aligned} I_2' &\leq C\beta \sum_{k=k_0+1}^{\infty} \sum_i w(\widetilde{Q}_i^k) \\ &\leq C\beta \sum_{k=k_0+1}^{\infty} \tau^{k-k_0} \sum_i w(Q_i^k) \\ &\leq C\|f\|_{WH_w^1} \sum_{k=k_0+1}^{\infty} \left(\frac{\tau}{2} \right)^{k-k_0} \\ &\leq C\|f\|_{WH_w^1}. \end{aligned} \quad (32)$$

An application of Chebyshev's inequality and Minkowski integral inequality gives us that

$$\begin{aligned} I_2'' &\leq \beta \frac{2}{\beta} \int_{(2A_{k_0})^c} \mu_{\Omega, S}^\rho(F_2)(x) w(x) dx \\ &= C \int_{(2A_{k_0})^c} \left(\int \int_{|y-x|<t} \left| \sum_{k=k_0+1}^{\infty} \sum_i \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z) dz \right|^2 \frac{dt dy}{t^{n+2\rho+1}} \right)^{1/2} w(x) dx \\ &\leq C \sum_{k=k_0+1}^{\infty} \sum_i \int_{(2A_{k_0})^c} \left(\int \int_{|y-x|<t} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z) dz \right|^2 \frac{dt dy}{t^{n+2\rho+1}} \right)^{1/2} w(x) dx \\ &\leq C \sum_{k=k_0+1}^{\infty} \sum_i \left[\int_{(2A_{k_0})^c} \left(\int \int_{\substack{y \in 4Q_i^k \\ |y-x|<t}} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z) dz \right|^2 \frac{dt dy}{t^{n+2\rho+1}} \right)^{1/2} w(x) dx \right. \\ &\quad \left. + \int_{(2A_{k_0})^c} \left(\int \int_{\substack{y \in (4Q_i^k)^c \\ |y-x|<t}} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z) dz \right|^2 \frac{dt dy}{t^{n+2\rho+1}} \right)^{1/2} w(x) dx \right] \\ &=: C \sum_{k=k_0+1}^{\infty} \sum_i [I_3 + I_4]. \end{aligned} \quad (33)$$

Firstly, let us estimate I_3 . As $y \in 4Q_i^k$, $x \in (2A_{k_0})^c$, $z \in Q_i^k$, it is easy to see that

$$\begin{aligned} |y-x| &\geq |x-x_i^k| - |y-x_i^k| \geq \frac{|x-x_i^k|}{2}, \\ |y-z| &< 8r_i^k, \end{aligned}$$

$$\begin{aligned}
I_3 &\leq C 2^k \int_{(2A_{k_0})^c} \left(\iint_{\substack{y \in 4Q_i^k \\ |x-x_i^k|/2 \leq |y-x| < t}} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} dz \right|^2 \frac{dt dy}{t^{n+2\rho+1}} \right)^{1/2} w(x) dx \\
&\leq C 2^k \int_{(2A_{k_0})^c} \left(\int_{y \in 4Q_i^k} \int_{|x-x_i^k|/2}^\infty \left| \int_{S^{n-1}} \int_0^{8r_i^k} \frac{|\Omega(z')|}{s^{n-\rho}} s^{n-1} ds d\sigma(z') \right|^2 \frac{dt dy}{t^{n+2\rho+1}} \right)^{1/2} w(x) dx \\
&\leq C 2^k \int_{(2A_{k_0})^c} \left(\int_{y \in 4Q_i^k} \int_{|x-x_i^k|/2}^\infty (8r_i^k)^{2\rho} \frac{dt dy}{t^{n+2\rho+1}} \right)^{1/2} w(x) dx \\
&\leq C 2^k (8r_i^k)^\rho \int_{(2A_{k_0})^c} \left(\int_{y \in 4Q_i^k} \frac{dy}{|x-x_i^k|^{n+2\rho}} \right)^{1/2} w(x) dx \\
&\leq C 2^k (8r_i^k)^\rho |4Q_i^k|^{1/2} \int_{(2A_{k_0})^c} \frac{w(x)}{|x-x_i^k|^{n/2+\rho}} dx.
\end{aligned} \tag{34}$$

Since $x \in (2A_{k_0})^c$, $\widetilde{Q}_i^k = Q(x_i^k, 8\tau^{(k-k_0)/n} r_i^k)$, then $|x-x_i^k| > 8\tau^{(k-k_0)/n} r_i^k > 8r_i^k$. By Lemma 4, we obtain that

$$\begin{aligned}
I_3 &\leq C 2^k (r_i^k)^{\rho+n/2} \\
&\quad \times \sum_{j=3}^\infty \int_{2^j \tau^{(k-k_0)/n} r_i^k < |x-x_i^k| < 2^{j+1} \tau^{(k-k_0)/n} r_i^k} \frac{w(x)}{|x-x_i^k|^{n/2+\rho}} dx \\
&\leq C 2^k (r_i^k)^{\rho+n/2} \\
&\quad \times \sum_{j=3}^\infty (2^{j+1} \tau^{(k-k_0)/n} r_i^k)^{-n/2-\rho} \\
&\quad \times \int_{|x-x_i^k| < 2^{j+1} \tau^{(k-k_0)/n} r_i^k} w(x) dx \\
&\leq C 2^k \sum_{j=3}^\infty 2^{-(\rho-n/2)j} \frac{1}{2^{(j+1)n}} (2^{j+1} \tau^{(k-k_0)/n})^n w(Q_i^k)
\end{aligned}$$

$$\begin{aligned}
&\leq C 2^k \sum_{j=3}^\infty 2^{-(\rho-n/2)j} (\tau^{(k-k_0)/n})^{n/2-\rho} w(Q_i^k) \\
&\leq C 2^k (\tau^{(k-k_0)/n})^{n/2-\rho} w(Q_i^k).
\end{aligned} \tag{35}$$

Noticing that $\rho > n/2$, $1 < \tau < 2$, we have

$$\begin{aligned}
\sum_{k=k_0+1}^\infty \sum_i I_3 &\leq C \sum_{k=k_0+1}^\infty 2^k (\tau^{(k-k_0)/n})^{n/2-\rho} \sum_i w(Q_i^k) \\
&\leq C \sum_{k=k_0+1}^\infty (\tau^{(k-k_0)/n})^{n/2-\rho} \|f\|_{WH_w^1} \\
&\leq C \|f\|_{WH_w^1}.
\end{aligned} \tag{36}$$

Now we consider I_4 . Write

$$\begin{aligned}
I_4 &\leq C \int_{(2A_{k_0})^c} \left(\iint_{\substack{y \in (4Q_i^k)^c \\ |y-x|<t \\ t \leq |y-x_i^k|+2r_i^k}} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z) dz \right|^2 \frac{dt dy}{t^{n+2\rho+1}} \right)^{1/2} w(x) dx \\
&\quad + C \int_{(2A_{k_0})^c} \left(\iint_{\substack{y \in (4Q_i^k)^c \\ |y-x|<t \\ |y-x_i^k|+2r_i^k < t}} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z) dz \right|^2 \frac{dt dy}{t^{n+2\rho+1}} \right)^{1/2} w(x) dx \\
&=: C(I_{41} + I_{42}).
\end{aligned} \tag{37}$$

Take $0 < \varepsilon < \min\{1/2, \alpha, \rho - n/2\}$. First we deal with I_{41} . As $|y - z| < t$, we have

$$\begin{aligned} t &> |y - z| \geq |y - x_i^k| - |x_i^k - z| \\ &\geq |y - x_i^k| - 2r_i^k, \end{aligned}$$

$$|x - x_i^k| \leq |x - y| + |y - x_i^k| \leq t + |y - x_i^k|$$

Using the Minkowski inequality, we get that

$$\leq 2|y - x_i^k| + 2r_i^k \leq 3|y - x_i^k|,$$

$$\begin{aligned} &\left| \frac{1}{(|y - x_i^k| - 2r_i^k)^{n+2\rho}} - \frac{1}{(|y - x_i^k| + 2r_i^k)^{n+2\rho}} \right| \\ &\leq C \frac{r_i^k}{|y - x_i^k|^{n+2\rho+1}}. \end{aligned}$$

(38)

$$\begin{aligned} I_{41} &\leq \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} |b_i^k(z)| \left(\int \int_{\substack{y \in (4Q_i^k)^c \\ |y-x| < t \\ t \leq |y-x_i^k| + 2r_i^k \\ |y-z| < t}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dt dy}{t^{n+2\rho+1}} \right)^{1/2} dz w(x) dx \\ &\leq \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} |b_i^k(z)| \left(\int_{\substack{y \in (4Q_i^k)^c \\ |x-x_i^k| \leq 3|y-x_i^k|}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \left(\int_{|y-x_i^k| - 2r_i^k}^{|y-x_i^k| + 2r_i^k} \frac{dt}{t^{n+2\rho+1}} \right) dy \right)^{1/2} dz w(x) dx \\ &\leq C \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} |b_i^k(z)| \\ &\quad \times \left(\int_{\substack{y \in (4Q_i^k)^c \\ |x-x_i^k| \leq 3|y-x_i^k|}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \left| \frac{1}{(|y-x_i^k| - 2r_i^k)^{n+2\rho}} - \frac{1}{(|y-x_i^k| + 2r_i^k)^{n+2\rho}} \right| dy \right)^{1/2} dz w(x) dx \\ &\leq C \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} |b_i^k(z)| \left(\int_{\substack{y \in (4Q_i^k)^c \\ |x-x_i^k| \leq 3|y-x_i^k|}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{r_i^k}{|y-x_i^k|^{n+2\rho+1}} dy \right)^{1/2} dz w(x) dx \\ &\leq C \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} |b_i^k(z)| \left(\int_{y \in (4Q_i^k)^c} \frac{|\Omega(y-z)|^2}{|y-z|^{n-2\varepsilon+1}} \frac{r_i^k}{|x-x_i^k|^{2n+2\varepsilon}} dy \right)^{1/2} dz w(x) dx \\ &\leq C(r_i^k)^{1/2} \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} \frac{|b_i^k(z)|}{|x-x_i^k|^{n+\varepsilon}} \left(\int_{y \in (4Q_i^k)^c} \frac{|\Omega(y-z)|^2}{|y-z|^{n-2\varepsilon+1}} dy \right)^{1/2} dz w(x) dx \\ &\leq C(r_i^k)^{1/2} \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} \frac{|b_i^k(z)|}{|x-x_i^k|^{n+\varepsilon}} \left(\int_{S^{n-1}} \int_{2r_i^k}^{\infty} \frac{|\Omega(z')|^2}{s^{2-2\varepsilon}} ds d\sigma(z') \right)^{1/2} dz w(x) dx \\ &\leq C(r_i^k)^\varepsilon \int_{(2A_{k_0})^c} \frac{w(x)}{|x-x_i^k|^{n+\varepsilon}} dx \int_{Q_i^k} |b_i^k(z)| dz \\ &\leq C 2^k (r_i^k)^{\varepsilon+n} \sum_{j=3}^{\infty} \int_{2^j \tau^{(k-k_0)/n} r_i^k < |x-x_i^k| < 2^{j+1} \tau^{(k-k_0)/n} r_i^k} \frac{w(x)}{|x-x_i^k|^{n+\varepsilon}} dx \\ &\leq C 2^k (r_i^k)^{\varepsilon+n} \sum_{j=3}^{\infty} (2^{j+1} r_i^k)^{-n-\varepsilon} \tau^{(k-k_0)(-n-\varepsilon)/n} \int_{|x-x_i^k| < 2^{j+1} \tau^{(k-k_0)/n} r_i^k} w(x) dx \end{aligned}$$

$$\begin{aligned}
&\leq C 2^k \sum_{j=3}^{\infty} (2^{j+1})^{-n-\varepsilon} \tau^{(k-k_0)(-n-\varepsilon)/n} 2^{(j+1)n} \tau^{k-k_0} w(Q_i^k) \\
&\leq C 2^k \left[\tau^{(k-k_0)/n} \right]^{-\varepsilon} \sum_{j=3}^{\infty} 2^{-\varepsilon(j+1)} w(Q_i^k).
\end{aligned} \tag{39}$$

By Lemma 5, we have

$$\begin{aligned}
&\sum_{k=k_0+1}^{\infty} \sum_i I_{41} \\
&\leq C \sum_{k=k_0+1}^{\infty} 2^k \left(\tau^{(k-k_0)/n} \right)^{-\varepsilon} \sum_i w(Q_i^k)
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{k=k_0+1}^{\infty} \left(\tau^{(k-k_0)/n} \right)^{-\varepsilon} \|f\|_{WH_w^1} \\
&\leq C \|f\|_{WH_w^1}.
\end{aligned} \tag{40}$$

Now let us consider I_{42} . It is easy to check that $Q_i^k \subset \{z : |y-z| < t\}$ as $y \in (4Q_i^k)^c$, $t > |y-x_i^k| + 2r_i^k$. Thus, we can obtain by the condition (c) of b_i^k in Lemma 5

$$\begin{aligned}
I_{42} &= \int_{(2A_{k_0})^c} \left(\iint_{\substack{y \in (4Q_i^k)^c \\ |y-x| < t \\ |y-x_i^k| + 2r_i^k < t}} \left| \int_{|y-z| < t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_i^k)}{|y-x_i^k|^{n-\rho}} b_i^k(z) dz \right|^2 \frac{dt dy}{t^{n+2\rho+1}} \right)^{1/2} w(x) dx \\
&\leq \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} |b_i^k(z)| \left(\iint_{\substack{y \in (4Q_i^k)^c \\ \max\{|y-x|, |y-x_i^k| + 2r_i^k, |y-z|\} < t \\ |x-x_i^k| \leq 2|y-x_i^k|}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_i^k)}{|y-x_i^k|^{n-\rho}} \right|^2 \frac{dt dy}{t^{n+2\rho+1}} \right)^{1/2} dz w(x) dx \\
&\quad + \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} |b_i^k(z)| \left(\iint_{\substack{y \in (4Q_i^k)^c \\ \max\{|y-x|, |y-x_i^k| + 2r_i^k, |y-z|\} < t \\ |x-x_i^k| > 2|y-x_i^k|}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_i^k)}{|y-x_i^k|^{n-\rho}} \right|^2 \frac{dt dy}{t^{n+2\rho+1}} \right)^{1/2} dz w(x) dx \\
&= I'_{42} + I''_{42}. \\
I'_{42} &\leq \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} |b_i^k(z)| \left(\int_{\substack{y \in (4Q_i^k)^c \\ |x-x_i^k| \leq 2|y-x_i^k|}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_i^k)}{|y-x_i^k|^{n-\rho}} \right|^2 \left(\int_{|y-x_i^k| + 2r_i^k}^{\infty} \frac{dt}{t^{n+2\rho+1}} \right) dy \right)^{1/2} dz w(x) dx \\
&\leq C \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} |b_i^k(z)| \left(\int_{\substack{y \in (4Q_i^k)^c \\ |x-x_i^k| \leq 2|y-x_i^k|}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_i^k)}{|y-x_i^k|^{n-\rho}} \right|^2 \frac{1}{(|y-x_i^k| + 2r_i^k)^{n+2\rho}} dy \right)^{1/2} dz w(x) dx \\
&\leq C \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} \frac{|b_i^k(z)|}{(|x-x_i^k| + 2r_i^k)^{n+\varepsilon}} \left(\int_{y \in (4Q_i^k)^c} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_i^k)}{|y-x_i^k|^{n-\rho}} \right|^2 \frac{1}{(|y-x_i^k| + 2r_i^k)^{2\rho-n-2\varepsilon}} dy \right)^{1/2} dz w(x) dx.
\end{aligned} \tag{41}$$

By using Lemma 7, we can get

$$\begin{aligned}
I'_{42} &\leq C \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} \frac{|b_i^k(z)|}{(|x - x_i^k| + 2r_i^k)^{n+\varepsilon}} \\
&\quad \times \sum_{j=1}^{\infty} \left(\int_{2^j r_i^k < |y - x_i^k| < 2^{j+1} r_i^k} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_i^k)}{|y-x_i^k|^{n-\rho}} \right|^2 \frac{1}{(|y-x_i^k| + 2r_i^k)^{2\rho-n-2\varepsilon}} dy \right)^{1/2} dz w(x) dx \\
&\leq C \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} \frac{|b_i^k(z)|}{(|x - x_i^k| + 2r_i^k)^{n+\varepsilon}} \sum_{j=1}^{\infty} \frac{1}{(2^j r_i^k + 2r_i^k)^{\rho-n/2-\varepsilon}} \\
&\quad \times \left(\int_{2^j r_i^k < |y - x_i^k| < 2^{j+1} r_i^k} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_i^k)}{|y-x_i^k|^{n-\rho}} \right|^2 dy \right)^{1/2} dz w(x) dx \\
&\leq C \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} \frac{|b_i^k(z)|}{(|x - x_i^k| + 2r_i^k)^{n+\varepsilon}} \sum_{j=1}^{\infty} \frac{1}{(2^j r_i^k + 2r_i^k)^{\rho-n/2-\varepsilon}} \left(2^j r_i^k \right)^{n/2-(n-\rho)} \left\{ \frac{|z - x_i^k|}{2^j r_i^k} + \int_{|z-x_i^k|/2^{j+1}r_i^k}^{|z-x_i^k|/2^j r_i^k} \frac{w_2(\delta)}{\delta} d\delta \right\} dz w(x) dx \\
&\leq C \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} \frac{|b_i^k(z)|}{(|x - x_i^k| + 2r_i^k)^{n+\varepsilon}} \sum_{j=1}^{\infty} (2^j r_i^k)^{\varepsilon} \left\{ \frac{1}{2^j} + \int_{|z-x_i^k|/2^{j+1}r_i^k}^{|z-x_i^k|/2^j r_i^k} \frac{w_2(\delta)}{\delta} d\delta \right\} dz w(x) dx \\
&\leq C \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} \frac{|b_i^k(z)|}{(|x - x_i^k| + 2r_i^k)^{n+\varepsilon}} \sum_{j=1}^{\infty} (2^j r_i^k)^{\varepsilon} \frac{1}{2^j} dz w(x) dx \\
&\quad + C \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} \frac{|b_i^k(z)|}{(|x - x_i^k| + 2r_i^k)^{n+\varepsilon}} \sum_{j=1}^{\infty} (2^j r_i^k)^{\varepsilon} \int_{|z-x_i^k|/2^{j+1}r_i^k}^{|z-x_i^k|/2^j r_i^k} \frac{w_2(\delta)}{\delta} d\delta dz w(x) dx \\
&=: C(U_1 + U_2).
\end{aligned} \tag{42}$$

It is easy to see that

$$\begin{aligned}
U_1 &\leq C 2^k (r_i^k)^{\varepsilon+n} \\
&\quad \times \int_{(A_{k_0})^c} \frac{w(x)}{(|x - x_i^k| + 2r_i^k)^{n+\varepsilon}} dx \sum_{j=1}^{\infty} 2^{j(\varepsilon-1)} \\
&\leq C 2^k (r_i^k)^{\varepsilon+n} \int_{(A_{k_0})^c} \frac{w(x)}{(|x - x_i^k| + 2r_i^k)^{n+\varepsilon}} dx.
\end{aligned} \tag{43}$$

Using the same method as what used to deal with the inequality (39), we can obtain that

$$U_1 \leq C 2^k [\tau^{(k-k_0)/n}]^{-\varepsilon} w(Q_i^k). \tag{44}$$

For U_2 , we have

$$\begin{aligned}
U_2 &\leq C \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} \frac{|b_i^k(z)|}{(|x - x_i^k| + 2r_i^k)^{n+\varepsilon}} \\
&\quad \times \sum_{j=1}^{\infty} \frac{(2^j r_i^k)^{\varepsilon}}{2^{j\alpha}} \int_{|z-x_i^k|/2^{j+1}r_i^k}^{|z-x_i^k|/2^j r_i^k} \frac{w_2(\delta)}{\delta^{1+\alpha}} d\delta dz w(x) dx \\
&\leq C \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} \frac{(r_i^k)^{\varepsilon} |b_i^k(z)|}{(|x - x_i^k| + 2r_i^k)^{n+\varepsilon}} dz w(x) dx \\
&\quad \times \sum_{j=1}^{\infty} 2^{j(\varepsilon-\alpha)} \int_0^1 \frac{w_2(\delta)}{\delta^{1+\alpha}} d\delta \\
&\leq C 2^k (r_i^k)^{\varepsilon+n} \int_{(2A_{k_0})^c} \frac{w(x)}{(|x - x_i^k| + 2r_i^k)^{n+\varepsilon}} dx \\
&\leq C 2^k [\tau^{(k-k_0)/n}]^{-\varepsilon} w(Q_i^k).
\end{aligned} \tag{45}$$

Hence, by the inequalities (44) and (45), we have

$$I'_{42} \leq C(U_1 + U_2) \leq C2^k [\tau^{(k-k_0)/n}]^{-\varepsilon} w(Q_i^k). \quad (46)$$

Now we give the estimate for I''_{42} . Since $|x - x_i^k| > 2|y - x_i^k|$, $t > |y - x_i^k| + 2r_i^k$, then

$$t > |x - y| \geq |x - x_i^k| - |y - x_i^k| \geq \frac{|x - x_i^k|}{2}. \quad (47)$$

Thus,

$$\begin{aligned} I''_{42} &\leq \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} |b_i^k(z)| \\ &\quad \times \left(\int \int_{\substack{y \in (4Q_i^k)^c \\ \max\{|y-x|, |y-x_i^k|+2r_i^k, |y-z|\} < t}} \frac{1}{(|y-x_i^k| + 2r_i^k)^{2\rho-n-2\varepsilon}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_i^k)}{|y-x_i^k|^{n-\rho}} \right|^2 \frac{dt dy}{t^{2n+2\varepsilon+1}} \right)^{1/2} dz w(x) dx \\ &\leq \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} |b_i^k(z)| \left(\int_{y \in (4Q_i^k)^c} \frac{1}{(|y-x_i^k| + 2r_i^k)^{2\rho-n-2\varepsilon}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_i^k)}{|y-x_i^k|^{n-\rho}} \right|^2 \left(\int_{|x-x_i^k|/2}^{\infty} \frac{dt}{t^{2n+2\varepsilon+1}} \right) dy \right)^{1/2} dz w(x) dx \\ &\leq \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} |b_i^k(z)| \left(\int_{y \in (4Q_i^k)^c} \frac{1}{(|y-x_i^k| + 2r_i^k)^{2\rho-n-2\varepsilon}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_i^k)}{|y-x_i^k|^{n-\rho}} \right|^2 \frac{1}{|x-x_i^k|^{2n+2\varepsilon}} dy \right)^{1/2} dz w(x) dx \\ &\leq \int_{(2A_{k_0})^c} \int_{\mathbb{R}^n} \frac{|b_i^k(z)|}{|x-x_i^k|^{n+\varepsilon}} \left(\int_{y \in (4Q_i^k)^c} \frac{1}{(|y-x_i^k| + 2r_i^k)^{2\rho-n-2\varepsilon}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_i^k)}{|y-x_i^k|^{n-\rho}} \right|^2 dy \right)^{1/2} dz w(x) dx. \end{aligned} \quad (48)$$

Repeating this process which is similar to the one of estimating I'_{42} (from (42) to (46)), we may have

$$I''_{42} \leq C2^k [\tau^{(k-k_0)/n}]^{-\varepsilon} w(Q_i^k). \quad (49)$$

Thus by (36) and (40), we get

$$\begin{aligned} I_2 &= I'_2 + I''_2 \\ &\leq I'_2 + C \sum_{k=k_0+1}^{\infty} \sum_i (I_3 + I_4) \\ &\leq I'_2 + C \sum_{k=k_0+1}^{\infty} \sum_i (I_3 + I_{41} + I_{42}) \\ &\leq I'_2 + C \sum_{k=k_0+1}^{\infty} \sum_i (I_3 + I_{41} + I'_{42} + I''_{42}) \\ &\leq C \|f\|_{WH^1_w}. \end{aligned} \quad (50)$$

This completes the proof of Theorem 1. $\square$

Proof of Theorem 2. Combining the idea of proving Theorem 1 with the similar steps as in [12] and the following inequalities

$$\begin{aligned} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_i^k)}{|y-x_i^k|^{n-\rho}} \right| &\leq C \left| \frac{1}{|y-z|^{n-\rho}} - \frac{1}{|y-x_i^k|^{n-\rho}} \right| \\ &\leq C \frac{r_i^k}{|y-x_i^k|^{n-\rho+1}}, \\ \left| \Omega(y-z) - \Omega(y-x_i^k) \right| &\leq C \left| \frac{y-z}{|y-z|} - \frac{y-x_i^k}{|y-x_i^k|} \right|^\alpha \\ &\leq C \frac{(r_i^k)^\alpha}{|y-x_i^k|^\alpha}, \end{aligned} \quad (51)$$

it is not difficult to get the proof of Theorem 2. We omit the details here. $\square$

Proof of Theorem 3. We follow the strategy of the proof of Theorem 1. It suffices to show that there exists a constant $C > 0$, such that, for any $f \in WH^1_w(\mathbb{R}^n)$, $\beta > 0$,

$$\beta w(\{x \in \mathbb{R}^n : |\mu_\lambda^{*,\rho}(f)(x)| > \beta\}) \leq C \|f\|_{WH^1_w}. \quad (52)$$

Take $k_0 \in \mathbb{Z}$ such that $2^{k_0} \leq \beta < 2^{k_0+1}$, we have

$$\begin{aligned} & \beta w(\{x \in \mathbb{R}^n : |\mu_\lambda^{*,\rho}(f)(x)| > \beta\}) \\ & \leq \beta w\left(\left\{x \in \mathbb{R}^n : |\mu_\lambda^{*,\rho}(F_1)(x)| > \frac{\beta}{2}\right\}\right) \\ & \quad + \beta w\left(\left\{x \in \mathbb{R}^n : |\mu_\lambda^{*,\rho}(F_2)(x)| > \frac{\beta}{2}\right\}\right) \\ & =: J_1 + J_2, \end{aligned} \quad (53)$$

where the notations F_1, F_2 are the same as in the proof of Theorem 1. Using the same method of the proof of Theorem 1, we can get

$$\begin{aligned} J_1 & \leq 4\beta^{-1} \int_{\mathbb{R}^n} |\mu_\lambda^{*,\rho}(F_1)(x)| w(x) dx \\ & \leq C\beta^{-1} \|\mu_\lambda^{*,\rho}(F_1)\|_{L_w^2}^2 \\ & \leq C\beta^{-1} \|F_1\|_{L_w^2}^2 \\ & \leq C\|f\|_{WH_w^1}. \end{aligned} \quad (54)$$

Below, we will give the estimate of J_2 . If we set

$$A_{k_0} = \bigcup_{k=k_0+1}^{\infty} \bigcup_i \bar{Q}_i^k, \quad (55)$$

where $\bar{Q}_i^k = Q(x_i^k, 8\tau^{(k-k_0)/n}r_i^k)$, τ is a fixed positive number such that $1 < \tau < 2$; thus,

$$\begin{aligned} J_2 & = \beta w\left(\left\{x \in 2A_{k_0} : |\mu_\lambda^{*,\rho}(F_2)(x)| > \frac{\beta}{2}\right\}\right) \\ & \quad + \beta w\left(\left\{x \in (2A_{k_0})^c : |\mu_\lambda^{*,\rho}(F_2)(x)| > \frac{\beta}{2}\right\}\right) \\ & =: J'_2 + J''_2. \end{aligned} \quad (56)$$

Noting that $w \in A_2$, then by Lemmas 4 and 5, we have

$$\begin{aligned} J'_2 & \leq 2\beta \sum_{k=k_0+1}^{\infty} \sum_i w(\bar{Q}_i^k) \leq C\beta \sum_{k=k_0+1}^{\infty} \tau^{k-k_0} \sum_i w(Q_i^k) \leq C \sum_{k=k_0+1}^{\infty} \left(\frac{\tau}{2}\right)^{k-k_0} \|f\|_{WH_w^1} \leq C\|f\|_{WH_w^1}. \\ J''_2 & \leq \beta \frac{2}{\beta} \int_{(2A_{k_0})^c} \mu_\lambda^{*,\rho}(F_2)(x) w(x) dx \\ & = C \int_{(2A_{k_0})^c} \left(\int \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left| \sum_{k=k_0+1}^{\infty} \sum_i \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z) dz \right|^2 \frac{dt dy}{t^{n+2\rho+1}} \right)^{1/2} w(x) dx \\ & \leq C \sum_{k=k_0+1}^{\infty} \sum_i \int_{(2A_{k_0})^c} \left(\int \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z) dz \right|^2 \frac{dt dy}{t^{n+2\rho+1}} \right)^{1/2} w(x) dx \\ & \leq C \sum_{k=k_0+1}^{\infty} \sum_i \int_{(2A_{k_0})^c} (J_3 + J_4) w(x) dx, \end{aligned} \quad (57)$$

where

$$\begin{aligned} J_3 & = \left(\int \int_{|y-x|<t} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z) dz \right|^2 \frac{dt dy}{t^{n+2\rho+1}} \right)^{1/2}, \\ J_4 & = \left(\int \int_{|y-x|\geq t} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z) dz \right|^2 \frac{dt dy}{t^{n+2\rho+1}} \right)^{1/2}. \end{aligned} \quad (58)$$

Similarly as in the proof of Theorem 1, for $x \in (2A_{k_0})^c$, if the integration domains of J_4 is $y \in 4Q_i^k, |y-x| \geq t$, then we

denote it by J_{41} . If $y \in (4Q_i^k)^c, |y-x| \geq t$, we denote J_4 by J_{42} . Moreover, if $t \leq |y-x_i^k| + 2r_i^k$ in the integration domains

of J_{42} , we denote it by J_5 , otherwise denote it by J_6 . Further, we divide J_5 again by the integration domains; namely, if $|x - x_i^k| \leq 2|y - x_i^k|$, we denote it by J_{51} , otherwise denote it by J_{52} . Now, we are in a position to give the estimates of J_3 , J_{41} , J_{51} , J_{52} , J_6 , respectively. First, we take $0 < \varepsilon < \min\{1/2, \rho - n/2, \alpha, (\lambda - 2)n/2\}$ in the whole proof of Theorem 2. Obviously,

$$\begin{aligned} & \int_{(2A_{k_0})^c} J_3 w(x) dx \\ & \leq \int_{(2A_{k_0})^c} \left(\int \int_{|y-x|<t} \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \right. \right. \\ & \quad \times b_i^k(z) dz \left. \left. \frac{dt dy}{t^{n+2\rho+1}} \right)^{1/2} w(x) dx. \end{aligned} \quad (59)$$

By the proof of Theorem 1, we have

$$\sum_{k=k_0+1}^{\infty} \sum_i \int_{(2A_{k_0})^c} J_3 w(x) dx \leq C \|f\|_{WH_w^1}. \quad (60)$$

Notice that if $y \in 4Q_i^k$, $x \in (2A_{k_0})^c$, $|y - x| \geq t$, $z \in Q_i^k$, it is easy to check that

$$(a') \quad |y - x| \geq |x - x_i^k| - |y - x_i^k| \geq |x - x_i^k|/2, |y - z| < 8r_i^k.$$

If $y \in (4Q_i^k)^c$, $x \in (2A_{k_0})^c$, $|y - x| \geq t$, $z \in Q_i^k$, then

$$(b') \quad |y - z| \sim |y - x_i^k|, |y - x_i^k| - 2r_i^k \leq t \leq |y - x_i^k| + 2r_i^k, \text{ for } t \leq |y - x_i^k| + 2r_i^k;$$

$$(c') \quad |1/((|y - x_i^k| - 2r_i^k)^{n+2\rho} - 1/((|y - x_i^k| + 2r_i^k)^{n+2\rho})| \leq C(t_i^k/|y - x_i^k|^{n+2\rho+1});$$

$$(d') \quad |y - x| \geq |x - x_i^k| - |y - x_i^k| \geq |x - x_i^k|/2, \text{ for } |x - x_i^k| \geq 2|y - x_i^k|;$$

$$(e') \quad t + |x - y| \geq t + |x - x_i^k| - |y - x_i^k| \geq |x - x_i^k| + 2r_i^k, \text{ for } t > |y - x_i^k| + 2r_i^k.$$

Now, let us estimate J_{41} . By the fact (a') and the Minkowski inequality, we have

$$\begin{aligned} J_{41} & \leq \left(\int \int_{\substack{y \in 4Q_i^k \\ |y-x| > |x-x_i^k|/2 \\ |y-x| \geq t}} \left(\frac{t}{t + |x - y|} \right)^{2n+2\varepsilon} \left| \int_{\substack{|y-z| < 8r_i^k \\ |y-z| < t}} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_i^k(z) dz \right|^2 \frac{dt dy}{t^{n+2\rho+1}} \right)^{1/2} \\ & \leq C 2^k \int_{Q_i^k} \left(\int \int_{\substack{y \in 4Q_i^k \\ |y-x| > |x-x_i^k|/2 \\ |y-z| < 8r_i^k, |y-z| < t \\ |y-x| \geq t}} \left(\frac{t}{t + |x - x_i^k|/2} \right)^{2n+2\varepsilon} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dt dy}{t^{n+2\rho+1}} \right)^{1/2} dz \\ & \leq C 2^k \left(\frac{1}{|x - x_i^k|/2} \right)^{n+\varepsilon/2} \int_{Q_i^k} \left(\int_{\substack{y \in 4Q_i^k \\ |y-x| > |x-x_i^k|/2 \\ |y-z| < 8r_i^k}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \int_0^{|y-x|} \frac{t^{2n+2\varepsilon}}{|x - x_i^k|^\varepsilon |y-z|^{2\rho-n-\varepsilon} t^{2n+\varepsilon+1}} dt dy \right)^{1/2} dz \\ & \leq C 2^k \frac{1}{|x - x_i^k|^{n+\varepsilon/2}} \int_{Q_i^k} \left(\int_{\substack{y \in 4Q_i^k \\ |y-x| > |x-x_i^k|/2 \\ |y-z| < 8r_i^k}} \frac{|\Omega(y-z)|^2 |y-x|^\varepsilon}{|y-z|^{n-\varepsilon} |x - x_i^k|^\varepsilon} dy \right)^{1/2} dz \\ & \leq C 2^k \frac{1}{|x - x_i^k|^{n+\varepsilon/2}} \int_{Q_i^k} \left(\int_{|y-z| < 8r_i^k} \frac{|\Omega(y-z)|^2}{|y-z|^{n-\varepsilon}} dy \right)^{1/2} dz \\ & \leq C 2^k \frac{1}{|x - x_i^k|^{n+\varepsilon/2}} \int_{Q_i^k} \left(\int_{S^{n-1}} \int_0^{8r_i^k} \frac{|\Omega(y')|^2}{|s|^{n-\varepsilon}} s^{n-1} ds d\sigma(y') \right)^{1/2} dz \end{aligned}$$

$$\begin{aligned}
&\leq C2^k \frac{1}{|x - x_i^k|^{n+\varepsilon/2}} \int_{Q_i^k} (8r_i^k)^{\varepsilon/2} dz \\
&\leq C2^k \frac{(r_i^k)^{n+\varepsilon/2}}{|x - x_i^k|^{n+\varepsilon/2}}.
\end{aligned} \tag{61}$$

As for J_{51} , notice that $y \in (4Q_i^k)^c$, $x \in (2A_{k_0})^c$, $|y - x| \geq t$, $z \in Q_i^k$, using the Minkowski inequality and the previous facts (b') and (c'), we have

$$\begin{aligned}
J_{51} &\leq \int_{Q_i^k} |b_i^k(z)| \\
&\quad \times \left[\int_{\substack{y \in (4Q_i^k)^c \\ |x - x_i^k| \leq 2|y - x_i^k|}} \left(\int_{|y - x_i^k| - 2r_i^k}^{|y - x_i^k| + 2r_i^k} \frac{dt}{t^{n+2\rho+1}} \right) \right. \\
&\quad \times \left. \frac{|\Omega(y - z)|^2}{|y - z|^{2n-2\rho}} dy \right]^{1/2} dz \\
&\leq C2^k \int_{Q_i^k} \left(\int_{\substack{y \in (4Q_i^k)^c \\ |x - x_i^k| \leq 2|y - x_i^k|}} \frac{|\Omega(y - z)|^2}{|y - z|^{2n-2\rho}} \right. \\
&\quad \times \left. \frac{r_i^k}{|y - x_i^k|^{n+2\rho+1}} dy \right)^{1/2} dz \\
&\leq C2^k \\
&\quad \times \int_{Q_i^k} \left(\int_{\substack{y \in (4Q_i^k)^c \\ |x - x_i^k| \leq 2|y - x_i^k|}} \frac{|\Omega(y - z)|^2}{|y - z|^{2n-2\rho}} \right. \\
&\quad \times \left. \frac{r_i^k}{|y - x_i^k|^{2\rho-n+1-2\varepsilon}} dy \right)^{1/2} dz \\
&\leq C2^k \frac{(r_i^k)^{n+1/2}}{|x - x_i^k|^{n+\varepsilon}} \\
&\quad \times \left(\int_{S^{n-1}} \int_{4r_i^k}^\infty \frac{|\Omega(y')|^2}{s^{2-2\varepsilon}} ds d\sigma(y') \right)^{1/2} \\
&\leq C2^k \frac{(r_i^k)^{n+\varepsilon}}{|x - x_i^k|^{n+\varepsilon}}.
\end{aligned} \tag{62}$$

Now we consider J_{52} . By the fact (d') and $t/(t + |x - y|) < t$, we have

$$\begin{aligned}
J_{52} &\leq \int_{Q_i^k} |b_i^k(z)| \left[\int_{\substack{y \in (4Q_i^k)^c \\ |x - x_i^k| \geq 2|y - x_i^k|}} \left(\int_{|y - x_i^k| - 2r_i^k}^{|y - x_i^k| + 2r_i^k} \left(\frac{t}{t + |x - y|} \right)^{2n+2\varepsilon} \frac{dt}{t^{n+2\rho+1}} \right) \frac{|\Omega(y - z)|^2}{|y - z|^{2n-2\rho}} dy \right]^{1/2} dz \\
&\leq C2^k \int_{Q_i^k} \left(\int_{\substack{y \in (4Q_i^k)^c \\ |x - x_i^k| \geq 2|y - x_i^k|}} \left(\int_{|y - x_i^k| - 2r_i^k}^{|y - x_i^k| + 2r_i^k} \frac{t^{n+2\varepsilon-2\rho-1}}{|x - y|^{2n+2\varepsilon}} dt \right) \frac{|\Omega(y - z)|^2}{|y - z|^{2n-2\rho}} dy \right)^{1/2} dz \\
&\leq C2^k \int_{Q_i^k} \left(\int_{\substack{y \in (4Q_i^k)^c \\ |x - x_i^k| \geq 2|y - x_i^k|}} \frac{|\Omega(y - z)|^2}{|x - y|^{2n+2\varepsilon} |y - z|^{2n-2\rho}} \frac{r_i^k}{|y - x_i^k|^{2\rho-n-2\varepsilon+1}} dy \right)^{1/2} dz
\end{aligned}$$

$$\begin{aligned}
&\leq C 2^k \int_{Q_i^k} \left(\int_{y \in (4Q_i^k)^c} \frac{|\Omega(y-z)|^2}{|y-z|^{n-2\varepsilon+1}} \frac{r_i^k}{|x-x_i^k|^{2n+2\varepsilon}} dy \right)^{1/2} dz \\
&\leq C 2^k \frac{(r_i^k)^{n+\varepsilon}}{|x-x_i^k|^{n+\varepsilon}}.
\end{aligned} \tag{63}$$

For $y \in (4Q_i^k)^c$, since $t > |y-x_i^k| + 2r_i^k$, then we have $Q_i^k \subset \{z : |y-z| < t\}$. Thus, by the cancellation of b_i^k , we have $\int_{|y-z|<t} b_i^k(z) dz = 0$. By using the fact (e'), we get

$$\begin{aligned}
J_6 &= \left(\int \int_{\substack{y \in (4Q_i^k)^c \\ |y-x_i^k|+2r_i^k < t \\ |y-x| \geq t}} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left| \int_{|y-z|<t} \left(\frac{|\Omega(y-z)|}{|y-z|^{n-\rho}} - \frac{|\Omega(y-x_i^k)|}{|y-x_i^k|^{n-\rho}} \right) b_i^k(z) dz \right|^2 \frac{dy dt}{t^{n+2\rho+1}} \right)^{1/2} \\
&\leq \int_{\mathbb{R}^n} |b_i^k(z)| \left(\int \int_{\substack{y \in (4Q_i^k)^c \\ |y-x_i^k|+2r_i^k < t \\ |y-x| \geq t \\ |y-z| < t}} \frac{t^{\lambda n}}{(t+|x-y|)^{2n+2\varepsilon}} \frac{1}{(t+|x-y|)^{\lambda n-2n-2\varepsilon}} \left| \frac{|\Omega(y-z)|}{|y-z|^{n-\rho}} - \frac{|\Omega(y-x_i^k)|}{|y-x_i^k|^{n-\rho}} \right|^2 \frac{dy dt}{t^{n+2\rho+1}} \right)^{1/2} dz \\
&\leq C \int_{Q_i^k} \frac{|b_i^k(z)|}{(|x-x_i^k|+2r_i^k)^{n+\varepsilon}} \left(\int \int_{\substack{y \in (4Q_i^k)^c \\ |y-x_i^k|+2r_i^k < t \\ |y-x| \geq t \\ |y-z| < t}} \frac{t^{\lambda n}}{(t+|x-y|)^{\lambda n-2n-2\varepsilon}} \left| \frac{|\Omega(y-z)|}{|y-z|^{n-\rho}} - \frac{|\Omega(y-x_i^k)|}{|y-x_i^k|^{n-\rho}} \right|^2 \frac{dy dt}{t^{n+2\rho+1}} \right)^{1/2} dz \\
&\leq C \int_{Q_i^k} \frac{|b_i^k(z)|}{(|x-x_i^k|+2r_i^k)^{n+\varepsilon}} \\
&\quad \times \left[\int_{\substack{y \in (4Q_i^k)^c \\ |y-x| > |y-x_i^k|+2r_i^k}} \left| \frac{|\Omega(y-z)|}{|y-z|^{n-\rho}} - \frac{|\Omega(y-x_i^k)|}{|y-x_i^k|^{n-\rho}} \right|^2 \left(\int_{|y-x_i^k|+2r_i^k}^{|y-x|} \frac{t^{\lambda n-2n-2\varepsilon}}{(t+|x-y|)^{\lambda n-2n-2\varepsilon} t^{2\rho-n+1-2\varepsilon}} dt \right) dy \right]^{1/2} dz.
\end{aligned} \tag{64}$$

Noting that $0 < \varepsilon < \rho - n/2$, we have $2\rho - n - 2\varepsilon > 0$. Hence

$$\begin{aligned}
&\int_{|y-x_i^k|+2r_i^k}^{|y-x|} \frac{1}{t^{2\rho-n+1-2\varepsilon}} dt \\
&\leq \int_{|y-x_i^k|+2r_i^k}^{\infty} \frac{1}{t^{2\rho-n+1-2\varepsilon}} dt \\
&\leq C \frac{1}{(|y-x_i^k|+2r_i^k)^{2\rho-n-2\varepsilon}}.
\end{aligned} \tag{65}$$

Thus,

$$\begin{aligned}
J_6 &\leq C \int_{Q_i^k} \frac{|b_i^k(z)|}{(|x-x_i^k|+2r_i^k)^{n+\varepsilon}} \\
&\quad \times \left(\int_{y \in (4Q_i^k)^c} \left| \frac{|\Omega(y-z)|}{|y-z|^{n-\rho}} - \frac{|\Omega(y-x_i^k)|}{|y-x_i^k|^{n-\rho}} \right|^2 \right. \\
&\quad \times \left. \frac{1}{|y-x_i^k|^{2\rho-n-2\varepsilon}} dy \right)^{1/2} dz
\end{aligned}$$

$$\begin{aligned}
&\leq C \int_{Q_i^k} \frac{|b_i^k(z)|}{(|x - x_i^k| + 2r_i^k)^{n+\varepsilon}} \\
&\quad \times \left(\sum_{j=2}^{\infty} \int_{2^j r_i^k \leq |y - x_i^k| < 2^{j+1} r_i^k} \left| \frac{|\Omega(y - z)|}{|y - z|^{n-\rho}} - \frac{|\Omega(y - x_i^k)|}{|y - x_i^k|^{n-\rho}} \right|^2 \right. \\
&\quad \left. \times \frac{1}{|y - x_i^k|^{2\rho - n - 2\varepsilon}} dy \right)^{1/2} dz \\
&\leq C \int_{Q_i^k} \frac{|b_i^k(z)|}{(|x - x_i^k| + 2r_i^k)^{n+\varepsilon}} \\
&\quad \times \sum_{j=2}^{\infty} (2^j r_i^k)^{\rho - n/2 - \varepsilon} \\
&\quad \times \left(\int_{2^j r_i^k \leq |y - x_i^k| < 2^{j+1} r_i^k} \left| \frac{|\Omega(y - z)|}{|y - z|^{n-\rho}} - \frac{|\Omega(y - x_i^k)|}{|y - x_i^k|^{n-\rho}} \right|^2 dy \right)^{1/2} dz \\
&\leq C \int_{Q_i^k} \frac{|b_i^k(z)|}{(|x - x_i^k| + 2r_i^k)^{n+\varepsilon}} \sum_{j=2}^{\infty} (2^j r_i^k)^{\varepsilon} \\
&\quad \times \left\{ \frac{1}{2^j} + \frac{1}{2^{\alpha}} \int_{|z - x_i^k|/2^{j+1} r_i^k}^{|z - x_i^k|/2^j r_i^k} \frac{w_2(\delta)}{\delta^{1+\alpha}} d\delta \right\} dz \\
&\leq C \int_{Q_i^k} \frac{|b_i^k(z)| (r_i^k)^{\varepsilon}}{(|x - x_i^k| + 2r_i^k)^{n+\varepsilon}} dz \\
&\leq C 2^k \frac{(r_i^k)^{n+\varepsilon}}{|x - x_i^k|^{n+\varepsilon}}.
\end{aligned} \tag{66}$$

From (61) to (66), we can obtain

$$\begin{aligned}
&\sum_{k=k_0+1}^{\infty} \sum_i \int_{(2A_{k_0})^c} J_4 w(x) dx \\
&\leq C \sum_{k=k_0+1}^{\infty} \sum_i \int_{(2A_{k_0})^c} (J_{41} + J_{42}) w(x) dx \\
&\leq C \sum_{k=k_0+1}^{\infty} \sum_i \int_{(2A_{k_0})^c} (J_{41} + J_5 + J_6) w(x) dx \\
&\leq C \sum_{k=k_0+1}^{\infty} \sum_i \int_{(2A_{k_0})^c} (J_{41} + J_{51} + J_{52} + J_6) w(x) dx \\
&\leq C \sum_{k=k_0+1}^{\infty} \sum_i \int_{(2A_{k_0})^c} 2^k \\
&\quad \times \left(\frac{(r_i^k)^{n+\varepsilon/2}}{|x - x_i^k|^{n+\varepsilon/2}} + \frac{(r_i^k)^{n+\varepsilon}}{|x - x_i^k|^{n+\varepsilon}} \right) w(x) dx
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{k=k_0+1}^{\infty} 2^k \sum_i (r_i^k)^{\varepsilon/2+n} \\
&\quad \times \sum_{j=3}^{\infty} \int_{2^j \tau^{(k-k_0)/n} r_i^k < |x - x_i^k| < 2^{j+1} \tau^{(k-k_0)/n} r_i^k} \frac{w(x)}{|x - x_i^k|^{n+\varepsilon/2}} dx \\
&\quad + C \sum_{k=k_0+1}^{\infty} 2^k \sum_i (r_i^k)^{\varepsilon+n} \\
&\quad \times \sum_{j=3}^{\infty} \int_{2^j \tau^{(k-k_0)/n} r_i^k < |x - x_i^k| < 2^{j+1} \tau^{(k-k_0)/n} r_i^k} \frac{w(x)}{|x - x_i^k|^{n+\varepsilon}} dx \\
&\leq C \sum_{k=k_0+1}^{\infty} 2^k \sum_i (r_i^k)^{\varepsilon/2+n} \\
&\quad \times \sum_{j=3}^{\infty} \frac{1}{(2^j r_i^k)^{\varepsilon/2+n}} \int_{|x - x_i^k| < 2^{j+1} \tau^{(k-k_0)/n} r_i^k} w(x) dx \\
&\quad + \sum_{k=k_0+1}^{\infty} 2^k \sum_i (r_i^k)^{\varepsilon+n} \\
&\quad \times \sum_{j=3}^{\infty} \frac{1}{(2^j r_i^k)^{\varepsilon+n}} \int_{|x - x_i^k| < 2^{j+1} \tau^{(k-k_0)/n} r_i^k} w(x) dx \\
&\leq C \sum_{k=k_0+1}^{\infty} 2^k \sum_i \sum_{j=3}^{\infty} 2^{-(j+1)(\varepsilon/2)} \tau^{-\varepsilon(k-k_0)/2n} w(Q_i^k) \\
&\quad + C \sum_{k=k_0+1}^{\infty} 2^k \sum_i \sum_{j=3}^{\infty} 2^{-(j+1)\varepsilon} \tau^{-\varepsilon(k-k_0)/n} w(Q_i^k) \\
&\leq C \sum_{k=k_0+1}^{\infty} \tau^{-\varepsilon(k-k_0)/2n} \|f\|_{WH_w^1} \\
&\quad + C \sum_{k=k_0+1}^{\infty} \tau^{-\varepsilon(k-k_0)/n} \|f\|_{WH_w^1} \\
&\leq C \|f\|_{WH_w^1}.
\end{aligned} \tag{67}$$

We conclude the proof of Theorem 3. $\square$

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