Research Article

A System of Generalized Variational Inclusions Involving a New Monotone Mapping in Banach Spaces

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We introduce a new monotone mapping in Banach spaces, which is an extension of the \( C_n \)-monotone mapping studied by Nazemi (2012), and we generalize the variational inclusion involving the \( C_n \)-monotone mapping. Based on the new monotone mapping, we propose a new proximal mapping which combines the proximal mapping studied by Nazemi (2012) with the \( \eta \) mapping studied by Lan et al. (2011) and show its Lipschitz continuity. Based on the new proximal mapping, we give an iterative algorithm. Furthermore, we prove the convergence of iterative sequences generated by the algorithm under some appropriate conditions. Our results improve and extend corresponding ones announced by many others.

1. Introduction

Variational inequality theory has emerged as a powerful tool for a wide class of unrelated problems arising in various branches of physical, engineering, pure, and applied sciences in a unified and general framework. As the generalization of variational inequalities, variational inclusions have been widely studied in recent years. One of the most important problems in the theory of variational inclusions is the development of an efficient and implementable iterative algorithm. Therefore, many iterative algorithms and existence results for various variational inclusions have been studied see, for example, [1–3].

Several years ago, Xia and Huang [4] proposed the concept of general \( H \)-monotone operators in Banach spaces and studied a class of variational inclusions involving the general \( H \)-monotone operator in Banach spaces. In 2010, Luo and Huang [5] introduced a new notion of \( B \)-monotone operators in Banach spaces and gave a new proximal mapping related to these operators. Then, they used it to study a new class of variational inclusions in Banach spaces. Very recently, Nazemi [6] introduced the notion of a new class of \( C_n \)-monotone mappings which is an extension of \( B \)-monotone operators introduced in [5].

Motivated and inspired by the work going on in this direction, in this paper, we propose a new monotone mapping in Banach spaces named \( C_n \)-\( \eta \)-monotone mapping which generalizes the \( C_n \)-monotone mapping introduced in [6] from the same \( n \)-dimensional product space to different \( n \)-dimensional product space and reduces the \( C_n \) mapping from strictly monotone mapping to monotone mapping. Further, we consider a new proximal mapping which associates a \( \eta \) mapping introduced in [7] and generalizes the proximal mapping introduced in [6]. Furthermore, in the process of proving the convergence of iterative sequences generated by the algorithm, we change the condition of a uniformly smooth Banach space with \( \rho_E(t) \leq Ct^2 \) to a \( q \)-uniformly smooth Banach space, which extends the proof of the convergence of iterative sequences in [6]. The results presented in this paper generalize many known and important results in the recent literature and the references therein.

2. Preliminaries

Let \( E \) be a real Banach space, let \( E^* \) be the topological dual space of \( E \), and let \( \langle u, v \rangle \) be the dual pair between \( u \in E^* \) and \( v \in E \). Let \( CB(E) \) denote the family of all nonempty, closed,
and bounded subsets of $E$. Set $\prod_{i=1}^{n} E_i = E_1 \times E_2 \times \cdots \times E_n$. Let $H(\cdot, \cdot)$ be the Hausdorff metric on $CB(E)$ defined by

$$H(A, D) := \max \left\{ \sup_{x \in A} \inf_{y \in D} \|x - y\|, \sup_{y \in D} \inf_{x \in A} \|x - y\| \right\},$$

$A, D \in CB(E)$. \hfill (1)

We recall the following definitions and results which are needed in the sequel.

**Definition 1** (see [7]). A single-valued mapping $\eta : E \times E \to E$ is said to be $k$-Lipschitz continuous if there exists a constant $k > 0$ such that

$$\|\eta(x, y)\| \leq k \|x - y\|, \quad \forall x, y \in E. \hfill (2)$$

**Definition 2** (see [8]). A Banach space $E$ is called smooth if, for every $x \in E$ with $\|x\| = 1$, there exists a unique $f \in E'$ such that $\|f\| = f(x) = 1$. The modulus of smoothness of $E$ is the function $\rho_E : [0, \infty) \to [0, \infty)$, defined by

$$\rho_E(t) = \sup \left\{ \frac{1}{2} \left( \|x + y\| + \|x - y\| \right) - 1 : x, y \in E, \|x\| = 1, \|y\| = t \right\}. \hfill (3)$$

**Definition 3** (see [8]). The Banach space $E$ is said to be

(i) uniformly smooth if

$$\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0; \hfill (4)$$

(ii) $q$-uniformly smooth, for $q > 1$, if there exists a constant $c > 0$ such that

$$\rho_E(t) \leq ct^q, \quad t \in [0, \infty). \hfill (5)$$

It is well known (see, e.g., [9]) that

$$L_q \left( L_q \right) \text{ is } \begin{cases} q \text{-uniformly smooth,} & \text{if } 1 < q \leq 2, \\ 2 \text{-uniformly smooth,} & \text{if } q \geq 2. \end{cases} \hfill (6)$$

Note that if $E$ is uniformly smooth, $j_q$ becomes single-valued. In the study of characteristic inequalities in $q$-uniformly smooth Banach space, Xu [8] established the following lemma.

**Lemma 4** (see [8]). Let $q > 1$ be a real number and let $E$ be a smooth Banach space and $j_q : E \to 2^{E'}$ the normalized duality mapping. Then, $E$ is $q$-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for every $x, y \in E$,

$$\|x + y\|^q \leq \|x\|^q + q \left( \langle y, j_q(x) \rangle + c_q \|y\|^q \right),$$

$$\forall j_q(x) \in J_q(x). \hfill (7)$$

**Definition 5.** A single-valued mapping $g : E \to E$ is said to be $(\gamma, \mu)$-relaxed cocoercive if there exist $j_q(x - y) \in J_q(x - y)$ and $\gamma, \mu > 0$ such that

$$\langle j_q(x - y), g(x) - g(y) \rangle \geq -\gamma \|g(x) - g(y)\|^q + \mu \|x - y\|^q, \quad \forall x, y \in E. \hfill (8)$$

**Definition 6.** Let $n \geq 3$ and $M : \prod_{i=1}^{n} E_i \Rightarrow E^*$ be a multivalued mapping, $f_i : E \to E_i, i = 1, 2, \ldots, n$, and $\eta : E \times E \to E$ single-valued mappings.

(i) For each $1 \leq i \leq n$, $M(\ldots, f_i, \ldots)$ is said to be $\alpha_i$-strongly $\eta$-monotone with respect to $f_i$ (in the $i$th argument) if there exists a constant $\alpha_i > 0$ such that

$$\langle \omega_i - \omega'_i, \eta(x, y) \rangle \geq \alpha_i \|x - y\|^q,$$

$$\omega_i \in M(u_i, \ldots, u_{i-1}, f_i(x), u_{i+1}, \ldots, u_n),$$

$$\omega'_i \in M(u_i, \ldots, u_{i-1}, f_i(y), u_{i+1}, \ldots, u_n), \hfill (9)$$

$$\forall x, y \in E, u_i \in E_i, u_2 \in E_2, \ldots, \hfill (10)$$

$$u_{i-1} \in E_{i-1}, u_{i+1} \in E_{i+1}, \ldots, u_n \in E_n.$$

(ii) For each $1 \leq i \leq n$, $M(\ldots, f_i, \ldots)$ is said to be $\beta_i$-relaxed $\eta$-monotone with respect to $f_i$ (in the $i$th argument) if there exists a constant $\beta_i > 0$ such that

$$\langle \omega_i - \omega'_i, \eta(x, y) \rangle \geq -\beta_i \|x - y\|^q,$$

$$\omega_i \in M(u_i, \ldots, u_{i-1}, f_i(x), u_{i+1}, \ldots, u_n),$$

$$\omega'_i \in M(u_i, \ldots, u_{i-1}, f_i(y), u_{i+1}, \ldots, u_n), \hfill (11)$$

$$\forall x, y \in E, u_i \in E_i, u_2 \in E_2, \ldots, \hfill (12)$$

$$u_{i-1} \in E_{i-1}, u_{i+1} \in E_{i+1}, \ldots, u_n \in E_n.$$

(iii) By assumption that $n$ is an even number, $M$ is said to be $\alpha_1 \beta_2 \alpha_3 \beta_4 \cdots \alpha_{n-1} \beta_n$-symmetric $\eta$-monotone with respect to $f_1, f_2, \ldots, f_n$ if, for each $i \in \{1, 3, \ldots, n - 1\}, M(\ldots, f_i, \ldots)$ is $\alpha_i$-strongly $\eta$-monotone with respect to $f_i$ (in the $i$th argument) and for each $j \in \{2, 4, \ldots, n\}, M(\ldots, f_j, \ldots)$ is $\beta_j$-relaxed $\eta$-monotone with respect to $f_j$ (in the $j$th argument) with

$$\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1} > \beta_2 + \beta_4 + \cdots + \beta_n. \hfill (11)$$

(iv) By assumption that $n$ is an odd number, $M$ is said to be $\alpha_1 \beta_2 \alpha_3 \beta_4 \cdots \alpha_{n-1} \beta_n$-symmetric $\eta$-monotone with respect to $f_1, f_2, \ldots, f_n$ if, for each $i \in \{1, 3, \ldots, n\}, M(\ldots, f_i, \ldots)$ is $\alpha_i$-strongly $\eta$-monotone with respect to $f_i$ (in the $i$th argument) and for each $j \in \{2, 4, \ldots, n - 1\}, M(\ldots, f_j, \ldots)$ is $\beta_j$-relaxed $\eta$-monotone with respect to $f_j$ (in the $j$th argument) with

$$\alpha_1 + \alpha_3 + \cdots + \alpha_n > \beta_2 + \beta_4 + \cdots + \beta_{n-1}. \hfill (12)$$
Definition 7 (see [10]). Let $E$ be a Banach space. A multivalued mapping $A : E \ni y \mapsto C(E_i)$ is said to be $H$-Lipschitz continuous if there exists a constant $t > 0$ such that

$$H(A(x), A(y)) \leq t\|x - y\|, \quad \forall x, y \in E,$$

(13)

where $H(\cdot, \cdot)$ is the Hausdorff metric on $C(E_i)$.

Definition 8. Let, for each $i = 1, 2, \ldots, n$, $T_i : E \ni y \mapsto C(E_i)$ be a multivalued mapping. A single-valued mapping $F : \prod_{i=1}^n E_i \rightarrow E$ is said to be $F_i$-Lipschitz continuous in the $i$th argument with respect to $T_i$ ($i = 1, 2, \ldots, n$) if there exists a constant $\lambda_{F_i} > 0$ such that

$$\|F(x_1, \ldots, x_{i-1}, u_{i1}, x_{i+1}, \ldots, x_n) - F(x_1, \ldots, x_{i-1}, u_{i2}, x_{i+1}, \ldots, x_n)\|
\leq \lambda_{F_i} \|u_{i1} - u_{i2}\|, \quad \forall y_1, y_2 \in E, x_1 \in E_1, \ldots, x_{i-1} \in E_{i-1},$$

$$x_{i+1} \in E_{i+1}, \ldots, x_n \in E_n, u_{i1} \in T_i(y_1), u_{i2} \in T_i(y_2).$$

(14)

Definition 9. Let $E$ be a Banach space, $F^* \subseteq E^*$, and $\eta : E \times E \rightarrow E$ single-valued mappings; $C_\eta : E \rightarrow E^*$ is said to be $\eta$-monotone mapping if

$$\langle C_\eta(x) - C_\eta(y), \eta(x, y) \rangle \geq 0, \quad \forall x, y \in E.$$

(15)

3. $C_\eta$-$\eta$-Monotone Mapping

First, we define the notion of $C_\eta$-$\eta$-monotone mapping.

Definition 10. Let $E$ be a Banach space with the dual space $E^*$. Let $n \geq 3$ and $f_i : E \rightarrow E_i$, $i = 1, 2, \ldots, n$, $C_\eta : E \rightarrow E^*$ be single-valued mappings and $M : \prod_{i=1}^n E_i \rightarrow E^*$ a multivalued mapping.

(i) In case that $n$ is an even number, $M$ is said to be a $C_\eta$-$\eta$-monotone mapping if $M = \alpha_1 \beta_1, \alpha_2 \beta_2, \ldots, \alpha_{n-1} \beta_{n-1}$ symmetric $\eta$-monotone with respect to $f_1, f_2, \ldots, f_n$, and $C_\eta + \lambda M(f_1, f_2, \ldots, f_n) = E^*$, for every $\lambda > 0$.

(ii) In case that $n$ is an odd number, $M$ is said to be a $C_\eta$-$\eta$-monotone mapping if $M = \alpha_1 \beta_1, \alpha_2 \beta_2, \ldots, \alpha_{n-1} \beta_{n-1}$ symmetric $\eta$-monotone with respect to $f_1, f_2, \ldots, f_n$, and $C_\eta + \lambda M(f_1, f_2, \ldots, f_n) = E^*$, for every $\lambda > 0$.

Remark 11. (i) If $M(f_1, f_2, \ldots, f_n) = M, \eta(y, x) = y - x$, and $M$ is monotone, then the $C_\eta$-$\eta$-monotone mapping reduces to the general $H$-monotone mapping considered in [4].

(ii) If $M(f_1, f_2, \ldots, f_n) = M(f_1, f_2), \eta(y, x) = y - x$, then the $C_\eta$-$\eta$-monotone mapping reduces to the $H$-monotone mapping considered in [5].

(iii) If $M(f_1, f_2, \ldots, f_n) = M, \eta(y, x) = y - x$, and $M$ are $m$-relaxed monotone, then the $C_\eta$-$\eta$-monotone mapping reduces to the $A$-monotone mapping considered in [11].

(iv) If $M$ reduces to $E \times E \times E \rightarrow E^*, \eta(y, x) = y - x$, and $f_1, f_2, \ldots, f_n$ reduce to $E \rightarrow E$, then the $C_\eta$-$\eta$-monotone mapping reduces to the $C_n$-$\eta$-monotone mapping considered in [6].

Example 12. Let $E = E^0$ and then $E^* = E^0$; and assume $n$ is an even number; let $E_i \ni (\| \cdot \|_i, i = 1, 2, \ldots, n$, where $\| \cdot \|_i$ is the equivalent norm on $E_i^0$ space, $e_i = (0, \ldots, 1, 0, \ldots) \in E_i$, for $\forall x \in E$; let $f_1(x) = \alpha_1 x + e_1, f_2(x) = \beta_1 x + e_2 \in E_2$, $f_3(x) = \alpha_3 x + e_3 \in E_3, f_4(x) = \beta_3 x + e_4 \in E_4, \ldots, f_{n-1}(x) = \alpha_{n-1} x + e_{n-1} \in E_{n-1}, f_n(x) = \beta_n x + e_n \in E_n$, where $\alpha_1, \beta_2, \alpha_3, \beta_4, \ldots, \alpha_{n-1}, \beta_n > 0$ are constants such that

$$\alpha_1 - \beta_2 + \alpha_3 - \beta_4 + \cdots + \alpha_{n-1} - \beta_n = y > 0.$$

(16)

Let $M(u_1, u_2, \ldots, u_n) = (u_1 - e_1) + (u_2 - e_2) + \cdots + (u_n - e_n)$, where $u_i \in E_i, i = 1, 2, \ldots, n$; let $C_\eta(x) = x + e_n, x \in E, \eta(x, y) = y - x, \forall x, y \in E$. Then $M$ is a $C_n$-$\eta$-monotone mapping.

With no loss of generality, we may assume that $n$ is an even number in the next text.

Lemma 13. Let $\eta : E \times E \rightarrow E$, $f_i : E \rightarrow E_i$, $i = 1, 2, \ldots, n$, be single-valued mappings; $M : \prod_{i=1}^n E_i \rightarrow E^*$ a $\alpha_1 \beta_1, \alpha_2 \beta_2, \ldots, \alpha_{n-1} \beta_{n-1}$-symmetric $\eta$-monotone mapping with respect to $f_1, f_2, \ldots, f_n$. Then for $\forall x, y \in E$ one has

$$\langle u - v, \eta(y, x) \rangle \geq K_n \|x - y\|^2,$$

(17)

$u \in M(f_1, f_2, \ldots, f_n), v \in M(f_1, f_2, \ldots, f_n)$, where $K_n = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1}) - (\beta_2 + \beta_4 + \cdots + \beta_n)$.

Proof. Setting $\omega_1 \in M(f_1, f_2, \ldots, f_n), \omega_2 \in M(f_1, f_2, \ldots, f_n), \omega_{n-1} \in M(f_1, f_2, \ldots, f_n)$. From Definition 10, we have

$$\langle u - v, \eta(y, x) \rangle \geq \langle u - \omega_1, \eta(y, x) \rangle + \langle \omega_1 - \omega_2, \eta(y, x) \rangle + \cdots + \langle \omega_{n-1} - v, \eta(y, x) \rangle$$

$$\geq \alpha_1 \|x - y\|^2 - \beta_2 \|x - y\|^2 + \cdots + \alpha_{n-1} \|x - y\|^2 - \beta_n \|x - y\|^2$$

$$= K_n \|x - y\|^2,$$

(18)

where $K_n = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1}) - (\beta_2 + \beta_4 + \cdots + \beta_n)$.

This completes the proof.

Theorem 14. Let $E$ be a Banach space with the dual space $E^*$. Let $n \geq 3$ and $f_i : E \rightarrow E_i, i = 1, 2, \ldots, n, \eta : E \times E \rightarrow E$ single-valued mappings; $C_\eta : E \rightarrow E^*$ a $\eta$-monotone mapping, and $M : \prod_{i=1}^n E_i \rightarrow E^*$ a $C_\eta$-$\eta$-monotone mapping. Then, $(C_\eta + \lambda M(f_1, f_2, \ldots, f_n))^{-1}$ is a single-valued mapping.
Proof. Suppose, on the contrary, that there exists \( x_1, x_2 \in E, y^* \in E^* \), such that
\[
\begin{align*}
x_1 &= (C_n + \lambda M(f_1, f_2, \ldots, f_n))^{-1} (y^*), \\
x_2 &= (C_n + \lambda M(f_1, f_2, \ldots, f_n))^{-1} (y^*);
\end{align*}
\]
then
\[
\begin{align*}
\lambda^{-1} (y^* - C_n(x_1)) &\in M(f_1, f_2, \ldots, f_n)(x_1), \\
\lambda^{-1} (y^* - C_n(x_2)) &\in M(f_1, f_2, \ldots, f_n)(x_2).
\end{align*}
\]
Now, by using Lemma 13 and since \( C_n \) is a \( \eta \)-monotone mapping, we have
\[
0 = \lambda \langle \lambda^{-1} (y^* - C_n(x_1)) - \lambda^{-1} (y^* - C_n(x_2)), \eta(x_1, x_2) \rangle + \langle C_n(x_1) - C_n(x_2), \eta(x_1, x_2) \rangle \geq \lambda K_n \| x_1 - x_2 \|^2.
\]
Thus, we have \( x_1 = x_2 \), which implies that \( (C_n + \lambda M(f_1, f_2, \ldots, f_n))^{-1} \) is single valued. This completes the proof. \( \square \)

By Theorem 14, we can define the proximal mapping \( R_{M(f_1, f_2, \ldots, f_n)}^{C_n, \lambda} \) as follows.

Definition 15. Let \( E \) be a Banach space with the dual space \( E^* \). Let \( n \geq 3 \) and \( f_j : E \to E, i = 1, 2, \ldots, n \), be single-valued mappings, \( C_n : E \to E^* \) a \( \eta \)-monotone mapping, and \( M : \prod_{i=1}^{n} E_i \to E^* \) a \( C_n \)-\( \eta \)-monotone mapping. A proximal mapping \( R_{M(f_1, f_2, \ldots, f_n)}^{C_n, \lambda} : E^* \to E \) is defined by
\[
R_{M(f_1, f_2, \ldots, f_n)}^{C_n, \lambda, \eta} (x^*) = (C_n + \lambda M(f_1, f_2, \ldots, f_n))^{-1} (x^*), \quad \forall x^* \in E^*.
\]

Theorem 16. Let \( E \) be a Banach space with the dual space \( E^* \). Let \( \eta : E \times E \to E \) be a \( k \)-Lipschitz continuous mapping. Let \( n \geq 3 \) and \( f_j : E \to E, i = 1, 2, \ldots, n \), be single-valued mappings, \( C_n : E \to E^* \) a \( \eta \)-monotone mapping, and \( M : \prod_{i=1}^{n} E_i \to E^* \) a \( C_n \)-\( \eta \)-monotone mapping. Then, the proximal mapping \( R_{M(f_1, f_2, \ldots, f_n)}^{C_n, \lambda, \eta} : E^* \to E \) is \( k/\lambda K_n \)-Lipschitz continuous, where \( K_n = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1}) - (\beta_2 + \beta_4 + \cdots + \beta_n) \).

Proof. Let \( x^*, y^* \in E^* \) be any given points. It follows from Definition 15 that
\[
R_{M(f_1, f_2, \ldots, f_n)}^{C_n, \lambda, \eta} (x^*) = (C_n + \lambda M(f_1, f_2, \ldots, f_n))^{-1} (x^*),
\]
\[R_{M(f_1, f_2, \ldots, f_n)}^{C_n, \lambda, \eta} (y^*) = (C_n + \lambda M(f_1, f_2, \ldots, f_n))^{-1} (y^*).
\]
Setting
\[
x = R_{M(f_1, f_2, \ldots, f_n)}^{C_n, \lambda, \eta} (x^*), \quad y = R_{M(f_1, f_2, \ldots, f_n)}^{C_n, \lambda, \eta} (y^*)
\]
then
\[
x, y \in E.
\]
This implies that
\[
\frac{1}{\lambda} (x^* - C_n(x)) \in M(f_1, f_2, \ldots, f_n)(x);
\]
\[
\frac{1}{\lambda} (y^* - C_n(y)) \in M(f_1, f_2, \ldots, f_n)(y).
\]
By using Lemma 13, we have
\[
\left\| \frac{1}{\lambda} (x^* - C_n(x)) - \frac{1}{\lambda} (y^* - C_n(y)), \eta(x, y) \right\| \geq K_n \| x - y \|^2;
\]
since \( \eta \) is \( K \)-Lipschitz continuous, we have
\[
\lambda K_n \| x - y \|^2 \leq \left\langle \frac{1}{\lambda} (x^* - y^*), \eta(x, y) \right\rangle \leq k \| x^* - y^* \| \| x - y \|;
\]
thus
\[
\| x - y \| \leq k \frac{K_n}{\lambda} \| x^* - y^* \|;
\]
that is,
\[
\left\| R_{M(f_1, f_2, \ldots, f_n)}^{C_n, \lambda, \eta, \eta}(x^*) - R_{M(f_1, f_2, \ldots, f_n)}^{C_n, \lambda, \eta, \eta}(y^*) \right\|
\]
\leq k \frac{K_n}{\lambda} \| x^* - y^* \|, \quad \forall x^*, y^* \in E^*.
\]
This completes the proof. \( \square \)

4. System of Variational Inclusions: Iterative Algorithm

Let \( n \geq 3 \) and \( A : E \to E^* \), \( p : E \to E \), \( f_i : E \to E_i, i = 1, 2, \ldots, n \), \( F : \prod_{i=1}^{n} E_i \to E^* \) be single-valued mappings and \( T_i : E \to CB(E_i), i = 1, 2, \ldots, n \), \( M : \prod_{i=1}^{n} E_i \to E^* \) be multivalued mappings. We will study the following variational inclusion problem: for any given \( a \in E^* \), find \( x \in E, t_1 \in T_1(x), t_2 \in T_2(x), \ldots, t_n \in T_n(x) \), such that
\[
a \in A(x - p(x)) + M(f_1(x), f_2(x), \ldots, f_n(x))
\]
\[- F(t_1, t_2, \ldots, t_n).
\]

We remark that problem (30) includes as special cases many kinds of variational inclusion and variational inequality of [4, 5, 10, 12, 13].

Theorem 17. Let \( n \geq 3 \) and \( A : E \to E^* \), \( p : E \to E \), \( f_i : E \to E_i, i = 1, 2, \ldots, n \), \( F : \prod_{i=1}^{n} E_i \to E^* \) be single-valued mappings and let \( T_i : E \to CB(E_i), i = 1, 2, \ldots, n \), \( M : \prod_{i=1}^{n} E_i \to E^* \) be multivalued mappings. Let \( C_n : E \to E^* \) be \( \eta \)-monotone mapping and \( M : \prod_{i=1}^{n} E_i \to E^* \) a \( C_n \)-\( \eta \)-monotone mapping with respect to \( f_1, f_2, \ldots, f_n \). Then, \( (x, t_1, t_2, \ldots, t_n) \) is a solution of problem (30) \( i \) if and only if
\[
x = R_{M(f_1, f_2, \ldots, f_n)}^{C_n, \lambda} [C_n(x) - \lambda A(x - p(x))]
\]
\[+ \lambda a + \lambda F(t_1, t_2, \ldots, t_n),
\]
where \( t_1 \in T_1(x), t_2 \in T_2(x), \ldots, t_n \in T_n(x) \), and \( \lambda > 0 \) is a constant.

**Proof.** Let \((x, t_1, t_2, \ldots, t_n)\) be a solution of problem (30); then we have
\[
a \in A(x - p(x)) + M(f_1(x), f_2(x), \ldots, f_n(x)) - F(t_1, t_2, \ldots, t_n);
\]
then
\[
\lambda a + \lambda F(t_1, t_2, \ldots, t_n) - \lambda A(x - p(x)) \\
\in \lambda M(f_1, f_2, \ldots, f_n)(x), \quad \lambda > 0;
\]
thus
\[
C_n(x) - \lambda A(x - p(x)) + \lambda a + \lambda F(t_1, t_2, \ldots, t_n) \\
\in C_n(x) + \lambda M(f_1, f_2, \ldots, f_n)(x).
\]
Setting \( x^* = C_n(x) - \lambda A(x - p(x)) + \lambda a + \lambda F(t_1, t_2, \ldots, t_n) \), from the definition of \( R^{C_n,\lambda;M(f_1, f_2, \ldots, f_n)} \), we have
\[
x = R^{C_n,\lambda;M(f_1, f_2, \ldots, f_n)}(x^*) = R^{C_n,\lambda;M(f_1, f_2, \ldots, f_n)} \\
\times [C_n(x) - \lambda A(x - p(x)) + \lambda a + \lambda F(t_1, t_2, \ldots, t_n)].
\]
Conversely, let \( x = R^{C_n,\lambda;M(f_1, f_2, \ldots, f_n)}[C_n(x) - \lambda A(x - p(x)) + \lambda a + \lambda F(t_1, t_2, \ldots, t_n)] \); then
\[
C_n(x) - \lambda A(x - p(x)) + \lambda a + \lambda F(t_1, t_2, \ldots, t_n) \\
\in (C_n + \lambda M(f_1, f_2, \ldots, f_n))(x);
\]
thus we have
\[
a \in A(x - p(x)) + M(f_1(x), f_2(x), \ldots, f_n(x)) - F(t_1, t_2, \ldots, t_n).
\]
This completes the proof. \( \blacksquare \)

Based on Theorem 17, we construct the following iterative algorithm for solving problem (30).

**Iterative Algorithm 1.** For any given \( x_0 \in E \), we choose \( t_{1,0} \in T_1(x_0), t_{2,0} \in T_2(x_0), \ldots, t_{n,0} \in T_n(x_0) \) and compute \( \{x_m\}, \{t_{1,m}\}, \{t_{2,m}\}, \ldots, \{t_{n,m}\} \) by iterative schemes
\[
x_{m+1} = R^{C_n,\lambda;M(f_1, f_2, \ldots, f_n)} \left[ C_n(x_m) - \lambda A(x_m - p(x_m)) + \lambda a + \lambda F(t_{1,m}, t_{2,m}, \ldots, t_{n,m}) \right];
\]
for all \( m = 0, 1, 2, \ldots \).

Now, we give some sufficient conditions which guarantee the convergence of iterative sequences generated by Algorithm 1.

**Theorem 18.** Let \( E \) be a \( q \)-uniformly smooth Banach space with \( q > 1 \) and \( E^* \) the dual space of \( E \). Let \( \eta : E \times E \rightarrow E \) \( k \)-Lipschitz continuous. Let \( n \geq 3 \) and \( f_i : E \rightarrow E \), \( i = 1, 2, \ldots, n \), be \( \eta \)-monotone and \( \delta \)-Lipschitz continuous mappings, \( C_n : E \rightarrow E^* \) a \( \eta \)-monotone and \( \delta \)-Lipschitz continuous mapping, \( p : E \rightarrow E \) a \((y, \mu)\)-relaxed cocoercive and \( \lambda \)-Lipschitz continuous mapping, and \( M : \prod_{i=1}^{n} E_i \rightarrow E^* \) a \( C_n, \eta \)-monotone mapping. Let \( A : E \rightarrow E^* \) be \( \tau \)-Lipschitz continuous mapping and, for each \( i = 1, 2, \ldots, n \), let \( T_i : E \rightharpoonup C(E_i) \) be \( \tau \)-Lipschitz continuous with constant \( \lambda_i \). Suppose that \( F : \prod_{i=1}^{n} E_i \rightarrow E^* \) is \( \lambda \)-Lipschitz continuous in the \( i \)th argument with respect to \( T_j (j = 1, 2, \ldots, n) \) and the following condition is satisfied:
\[
0 < \frac{k}{\lambda K_n} \times \left[ \delta + \lambda \tau \left( 1 + q \gamma \lambda_p^q - q \mu + c_q \lambda_p^q \right) \right]^{1/4} + \lambda \sum_{i=1}^{n} \lambda_i \lambda_i \gamma \lambda_{p_i}^q \\
< 1,
\]
where
\[
K_n = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1}) - (\beta_2 + \beta_4 + \cdots + \beta_n).
\]
Then, the iterative sequences \( \{x_m\}, \{t_{1,m}\}, \{t_{2,m}\}, \ldots, \{t_{n,m}\} \) generated by Algorithm 1 converge strongly to \( x, t_1, t_2, \ldots, t_n \), respectively, and \( (x, t_1, t_2, \ldots, t_n) \) is a solution of problem (30).
Proof. By using Algorithm 1 and Theorem 16, we have

\[
\|x_{m+1} - x_m\| = \|R^{C_n, \lambda \eta}_{M(f_1, f_2, \ldots, f_n)} [C_n (x_m) - \lambda A (x_m - p (x_m)) + \lambda a + \lambda F (t_{1,m}, t_{2,m}, \ldots, t_{n,m})]
- \|R^{C_n, \lambda \eta}_{M(f_1, f_2, \ldots, f_n)} [C_n (x_m)] - \lambda A (x_m - p (x_m)) + \lambda a + \lambda F (t_{1,m}, t_{2,m}, \ldots, t_{n,m})]\|
\leq \frac{k}{\lambda K_n} \times \|C_n (x_m) - \lambda A (x_m - p (x_m)) + \lambda a + \lambda F (t_{1,m}, t_{2,m}, \ldots, t_{n,m}) - C_n (x_{m-1}) + \lambda A (x_{m-1} - p (x_{m-1})) - \lambda a - \lambda F (t_{1,m-1}, t_{2,m-1}, \ldots, t_{n,m-1})\|
\leq \frac{k}{\lambda K_n} \times (\|C_n (x_m) - C_n (x_{m-1})\|
+ \lambda \|A (x_m - p (x_m)) - A (x_{m-1} - p (x_{m-1}))\|
- \lambda \|F (t_{1,m}, t_{2,m}, \ldots, t_{n,m}) - F (t_{1,m-1}, t_{2,m-1}, \ldots, t_{n,m-1})\|).
\]

(41)

From the Lipschitz continuity of \(C_n\), \(p\), \(A\), and \((y, \mu)\)-relaxed cocoercivity of \(\rho\) and Lemma 4, we have

\[
\|C_n (x_m) - C_n (x_{m-1})\| \leq \delta \|x_m - x_{m-1}\|, \quad (42)
\]

\[
\|A (x_m - p (x_m)) - A (x_{m-1} - p (x_{m-1}))\| \leq \tau \|x_m - x_{m-1} - (p (x_m) - p (x_{m-1}))\|, \quad (43)
\]

\[
\|x_m - x_{m-1} - (p (x_m) - p (x_{m-1}))\|^q
\leq \|x_m - x_{m-1}\|^q
+ q \|p (x_m) - p (x_{m-1})\|^q
+ c_q \|p (x_m) - p (x_{m-1})\|^q
\leq \|x_m - x_{m-1}\|^q + q \|p (x_m) - p (x_{m-1})\|^q
+ q \|p (x_m) - p (x_{m-1})\|^q
\leq \|x_m - x_{m-1}\|^q + q \|x_m - x_{m-1}\|^q
+ q \|x_m - x_{m-1}\|^q
\leq \|x_m - x_{m-1}\|^q + q \|x_m - x_{m-1}\|^q
\]

(44)

\[
\delta + \lambda \tau (1 + q \|\lambda_p^q - \mu + c_q \lambda_p^q\|^{1/q}) + \lambda \left(1 + \frac{1}{m}\right) \sum_{i=1}^{n} \frac{\lambda F_i \lambda_i}{\lambda_i}
\]

(47)

Letting \(m \to \infty\), we obtain \(\theta_m \to \theta\), where

\[
\theta = \frac{k}{\lambda K_n} \times \left[\delta + \lambda \tau (1 + q \|\lambda_p^q - \mu + c_q \lambda_p^q\|^{1/q}) + \lambda \left(1 + \frac{1}{m}\right) \sum_{i=1}^{n} \frac{\lambda F_i \lambda_i}{\lambda_i}\right].
\]

(48)
From condition (39), we know that $0 < \theta < 1$, and hence \{x_m\} is a Cauchy sequence in $E$. Thus, there exists $x \in E$ such that $x_m \to x$, as $m \to \infty$. Now, we prove that $t_{1,m} \to t_1 \in T_1(x)$. In fact, it follows from the Lipschitz continuity of $T_1$ and Algorithm 1 that

$$
\|t_{1,m} - t_{1,m-1}\| \leq \left( 1 + \frac{1}{m} \right) H(T_1(x_m), T_1(x_{m-1})) \leq \left( 1 + \frac{1}{m} \right) \lambda_{t_1} \|x_m - x_{m-1}\|.
$$

From (49), we know that \{t_{1,m}\} is also a Cauchy sequence. In a similar way, \{t_{2,m}\}, \{t_{3,m}\}, \ldots, \{t_{n,m}\} are Cauchy sequences. Thus, there exist $t_1 \in E_1, t_2 \in E_2, \ldots, t_n \in E_n$ such that $t_{1,m} \to t_1, t_{2,m} \to t_2, \ldots, t_{n,m} \to t_n$, as $m \to \infty$. Furthermore,

$$
d(t_1, T_1(x)) \leq \|t_1 - t_{1,m}\| + d(t_{1,m}, T_1(x)) \leq \|t_1 - t_{1,m}\| + H(T_1(x_m), T_1(x)) \leq \|t_1 - t_{1,m}\| + \lambda_{t_1} \|x_m - x\| \to 0,
$$

as $m \to \infty$.

Since $T_1(x)$ is closed, we have $t_1 \in T_1(x)$. In a similar way, we can show that $t_2 \in T_2(x), t_3 \in T_3(x), \ldots, t_n \in T_n(x)$. By continuity of $A, p, C_n, F, T_1, T_2, \ldots, T_n, R^{C_0, \lambda\eta}_{M(f_1, f_2, \ldots, f_n)}$ and Algorithm 1, we have

$$
x = R^{C_0, \lambda\eta}_{M(f_1, f_2, \ldots, f_n)} \left[ C_n(x) - \lambda A(x - p(x)) + \lambda a + \lambda F(t_1, t_2, \ldots, t_n) \right].
$$

By Theorem 17, $(x, t_1, t_2, \ldots, t_n)$ is a solution of problem (30). This completes the proof. \qed

5. Conclusions

The purpose of this paper is to study a new monotone mapping in Banach spaces, which generalizes the $C_0$-monotone mapping in [6], and generalizes the concepts of many monotone mappings. Moreover, the result of Theorem 18 improves and generalizes the corresponding results of [4–6, 10, 12, 13].

References
