Research Article

Adaptive Exponential Stabilization for a Class of Stochastic Nonholonomic Systems

Xiaoyan Qin

College of Mathematics and Statistics, Zaozhuang University, Zaozhuang 277160, China

Correspondence should be addressed to Xiaoyan Qin; qin-xiaoyan@163.com

Received 21 July 2013; Revised 22 October 2013; Accepted 22 October 2013

1. Introduction

The nonholonomic systems cannot be stabilized by stationary continuous state feedback, although it is controllable, due to Brockett's theorem [1]. So the well-developed smooth nonlinear control theory and the method cannot be directly used in these systems. Many researchers have studied the control and stabilization of nonholonomic systems in the nonlinear control field and obtained some success [2–6]. It should be mentioned that many literatures consider the asymptotic stabilization of nonholonomic systems; the exponential convergence is also an important topic theme, which is demanded in many practical applications. However, the exponential regulation problem, particularly the systems with parameterization, has received less attention. Recently, [3] firstly introduced a class of nonholonomic systems with strong nonlinear uncertainties and obtained global exponential regulation. References [4, 5] studied a class of nonholonomic systems with output feedback control. Reference [6] combined the idea of combined input-state-scaling and backstepping technology, achieving the asymptotic stabilization for nonholonomic systems with nonlinear parameterization.

It is well known that when the backstepping designs were firstly introduced, the stochastic nonlinear control had obtained a breakthrough [7]. Based on quartic Lyapunov functions, the asymptotical stabilization control in the large of the open-loop system was discussed in [8]. Further research was developed by the recent work [9–16]. [17–19] studied a class of nonholonomic systems with stochastic unknown covariance disturbance. Since stochastic signals are very prevalent in practical engineering, the study of nonholonomic systems with stochastic disturbances is very significant. So, there exists a natural problem that is how to design an adaptive exponential stabilization for a class of nonholonomic systems with stochastic drift and diffusion terms. Inspired by these papers, we will study the exponential regulation problem with nonlinear parameterization for a class of stochastic nonholonomic systems. We use the input-state-scaling, the backstepping technique, and the switching scheme to design a dynamic state-feedback controller with $\sum^T \sum \neq I$; the closed-loop system is globally exponentially regulated to zero in probability.

This paper is organized as follows. In Section 2, we give the mathematical preliminaries. In Section 3, we construct the new controller and offer the main result. In the last section, we present the conclusions.

2. Problem Statement and Preliminaries

In this paper, we consider a class of stochastic nonholonomic systems as follows:
\[ dx_0 = d_0(t)u_0 dt + f_0(t, x_0) dt \]
\[ dx_i = d_i(t)u_0 dt + f_i(t, x_0, x_i) dt + \varphi_i(\xi) \sum \langle t \rangle d\omega, \]
\[ i = 1, \ldots, n - 1, \]
\[ dx_n = d_n(t)u_1 dt + f_n(t, x_0, x_n) dt + \varphi_n(\xi) \sum \langle t \rangle d\omega, \] (1)

where \( x_0 \in R \) and \( x = [x_1, \ldots, x_n]^T \in R^n \) are the system states and \( u_0 \in R \) and \( u_1 \in R \) are the control inputs, respectively. \( \Phi_i = [x_1, x_2, \ldots, x_n]^T \in R^i \), \( i = 1, 2, \ldots, n \), and \( \Phi_n = x \); \( \omega \in R^n \) is an \( r \)-dimensional standard Wiener process defined on the complete probability space \( (\Omega, F, P) \) with \( \Omega \) being a sample space, \( F \) being a filtration, and \( P \) being measure. The drift and diffusion terms \( f_i(\cdot), \varphi_i(\cdot) \) are assumed to be smooth, vanishing at the origin \((x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)\); \( \sum(t) : R^n \rightarrow R^{n^2} \) is the Borel bounded measurable functions and is nonnegative definite for each \( t \geq 0 \). \( d(t) \) are disturbed virtual control coefficients, where \( i = 1, \ldots, n \).

Next we introduce several technical lemmas which will play an important role in our later control design.

Consider the following stochastic nonlinear system:

\[ dx = f(x, t) dt + g(x, t) d\omega, \quad x(0) = x_0 \in R^n, \] (2)

where \( x \in R^n \) is the state of system (2), the Borel measurable functions: \( f : R^{n+1} \rightarrow R^n \) and \( g : R^{n+1} \rightarrow R^{n \times r} \) are assumed to be \( C^3 \) in their arguments, and \( \omega \in R^n \) is an \( r \)-dimensional standard Wiener process defined on the complete probability space \( (\Omega, F, P) \).

**Definition 1** (see [8]). Given any \( V(x, t) \in C^{1,2} \), for stochastic nonlinear system (2), the differential operator \( L \) is defined as follows:

\[ LV(x, t) = \frac{\partial V}{\partial t} + \sum \sum \frac{\partial^2 V}{\partial x^i \partial x^j} \sum \sum \frac{\partial f}{\partial x^{i1}} g^{ij} \sum \sum \frac{\partial f}{\partial x^{j2}} g^{ij} \] (3)

where \( C^{1,2}(R^n \times R_+; R) \) denotes all nonnegative functions \( V(x, t) \) on \( R^n \times R_+ \), which are \( C^1 \) in \( t \) and \( C^2 \) in \( x \), and for simplicity, the smooth function \( f(\cdot) \) is denoted by \( f \).

**Lemma 2** (see [8]). Let \( x \) and \( y \) be real variables. Then, for any positive integers \( m, n \), and any real number \( \varepsilon > 0 \), the following inequality holds:

\[ a(x) x^m y^n \leq \varepsilon |x|^m |y|^n + \frac{n}{m+n} \left( \frac{m+n}{m} \right)^{m+n} \times a(\frac{m+n}{m}) \varepsilon^{-m/n} |x|^{m+n}. \] (4)

**Lemma 3** (see [7]). Considering the stochastic nonlinear system (2), if there exist a \( C^{1,2} \) function \( V(x, t) \), \( K_{\infty} \) class functions \( a \) and \( \overline{a} \), constant \( \overline{\xi} \), and a nonnegative functions \( W(x, t) \) such that

\[ a(|x|) \leq V(x) \leq \overline{a}(|x|), \quad LV(x) \leq -W(x, t) + \overline{\xi}, \] (5)

then for each \( x_0 \in R^n \). (1) For (2), there exists an almost surely unique solution on \([0, \infty)\). (2) When \( \overline{\xi} = 0, f(0, t) = 0, g(0, t) = 0, \) and \( W(x, t) = W(x) \) is continuous, the equilibrium \( x = 0 \) is globally stable in probability, and the solution \( x(t) \) satisfies \( P\{\lim_{t \to \infty} W(x(t) = 0) = 1 \}. \) (3) For any given \( \varepsilon > 0 \), there exist a class KL function \( \beta_\varepsilon(\cdot, \cdot) \) and K function \( \gamma(\cdot) \) such that \( P\{(x(t)) < \beta_\varepsilon(|x_0|, t) + \gamma(\varepsilon) \} \geq 1 - \varepsilon \) for any \( t \geq 0 \).

**Lemma 4** (see [20]). For any real-valued continuous function \( f(x, y) \), \( x \in R^m, y \in R^n \), there exist smooth scalar-value functions \( a(x) \geq 0, b(y) \geq 0, c(x) > 1, \) and \( d(y) \geq 1, \) such that \( |f(x, y)| \leq a(x) + b(y) \), and \( |f(x, y)| \leq c(x)d(y) \).

**3. Controller Design and Analysis**

The purpose of this paper is to construct a smooth state-feedback control law such that the solution process of system (1) is bounded in probability. For clarity, the case that \( x_0(t_0) \neq 0 \) is firstly considered. Then, the case where the initial \( x_0(t_0) = 0 \) is dealt with later. The triangular structure of system (1) suggests that we should design the control inputs \( u_0 \) and \( u_1 \) in two separate stages.

To design the controller for system (1), the following assumptions are needed.

**Assumption 5.** For \( 0 \leq i \leq n \), there are some positive constants \( \lambda_1 \) and \( \lambda_2 \) that satisfy the inequality \( \lambda_1 \leq d_i(t) \leq \lambda_2 \).

**Assumption 6.** For \( f_0(t, x_0) \), there exists a nonnegative smooth function \( \gamma_0(t, x_0) \), such that \( |f_0(x_0, t)| \leq |x_0|y_0(t, x_0) \).

For each \( f_i(t, x_0, \overline{\xi}) \), \( \varphi_i(\overline{\xi}) \), there exist nonnegative smooth functions \( \gamma_i(t, x_0, \overline{\xi}) \) and \( \rho_i(\overline{\xi}) \), such that \( |f_i(t, x_0, \overline{\xi})| \leq (\sum_{k=1}^n |x_k|) \gamma_i(t, x_0, \overline{\xi}), |\varphi_i(\overline{\xi})| \leq (\sum_{k=1}^n |x_k|) \rho_i(\overline{\xi}) \).

3.1. Designing \( u_0 \) for \( x_0 \)-Subsystem. For \( x_0 \)-subsystem, the control \( u_0 \) can be chosen as

\[ u_0 = -\lambda_0 x_0, \] (6)

where \( \lambda_0 = (k_0 + y_0)/\lambda_0 \) and \( k_0 \) is a positive design parameter.

Consider the Lyapunov function candidate \( V_0 = x_0^2/2 \).

From (6) and Assumptions 5 and 6, we have

\[ LV_0 = x_0 (d_0u_0 + f_0(t, x_0)) \leq d_0u_0x_0 + x_0^2y_0 \leq -k_0x_0^2 = -2k_0V_0. \] (7)

So, we obtain the first result of this paper.

**Theorem 7.** The \( x_0 \)-subsystem, under the control law (6) with an appropriate choice of the parameters \( k_0, \lambda_{01}, \lambda_{02} \), is globally exponentially stable.

**Proof.** Clearly, from (7), \( LV_0 \leq 0 \), which implies that \( |x_0(t)| \leq |x_0(0)|e^{-k_0(t-t_0)} \). Therefore, \( x_0 \) is globally exponentially convergent. Consequently, \( x_0 \) can be zero only at \( t = t_0 \), when
Abstract and Applied Analysis

\(x(t_0) = 0\) or \(t = \infty\). It is concluded that \(x_0\) does not cross zero for all \(t \in (t_0, \infty)\) provided that \(x(t_0) \neq 0\).

**Remark 8.** If \(x(t_0) \neq 0, u_0\) exists and does not cross zero for all \(t \in (t_0, \infty)\) independent of the \(x\)-subsystem from (6).

### 3.2 Backstepping Design for \(u_1\)

From the above analysis, the \(x_0\)-state in (1) can be globally exponentially regulated to zero as \(t \to \infty\), obviously. In this subsection, we consider the control law \(u_1\), for the \(x\)-subsystem by using backstepping technique. To design a state-feedback controller, one first introduces the following discontinuous input-state-scaling transformation:

\[
\eta_i = \frac{e^{\alpha t} x_i}{\xi_i}, \quad i = 1 \ldots, n, \quad u = e^{\alpha t} u_1.
\]

Under the new \(x\)-coordinates, \(x\)-subsystems is transformed into

\[
d\eta_i = d\eta_i dt + f_i^T \sum(t) d\omega, \quad i = 1 \ldots, n - 1,
\]

\[
d\eta_n = d\eta_n dt + \phi_n^T \sum(t) d\omega,
\]

where

\[
f_i = \frac{e^{\alpha t} f_i}{\xi_i^{n-i}}, \quad \phi_i = \frac{e^{\alpha t} \phi_i}{\xi_i^{n-i}},
\]

(10)

In order to obtain the estimations for the nonlinear functions \(f_i\) and \(\phi_i\), the following Lemma can be derived by Assumption 6.

**Lemma 9.** For \(i = 1, 2 \ldots, n\), there exist nonnegative smooth functions \(\bar{f}_i(\cdot), \bar{\phi}_i(\cdot)\), such that

\[
|f_i| \leq \left( \sum_{k=1}^{i} |\eta_k| \right) \bar{f}_i(x_0, \bar{x}_i), \quad i = 1 \ldots, n - 1,
\]

(11)

\[
|\phi_i| \leq \left( \sum_{k=1}^{i} |\eta_k| \right) \bar{\phi}_i(\bar{x}_i).
\]

(12)

**Proof.** We only prove (11). The proof of (12) is similar to that of (11). In view of (6), (8), (10) and Assumption 6, one obtains

\[
|f_i| = \left| \frac{e^{\alpha t} f_i}{\xi_i^{n-i}} - \frac{(n-i) \eta_i}{\xi_i} \frac{d\eta_i}{\xi_i} (d_0 x_0 + f_0) \right|
\]

\[
\leq |\alpha \eta_i| + \left( \sum_{k=1}^{i} |\eta_k| \right) \left| \frac{e^{\alpha t} x_k}{\xi_i^{n-k}} \right| \eta_i
\]

\[
+ (n-i) (\lambda_0 \lambda_{i+2} + \gamma_0) |\eta_i|
\]

\[
\leq |\alpha \eta_i| + \left( \sum_{k=1}^{i} |\eta_k| \right) \left| \frac{e^{\alpha t} x_i}{\xi_i^{n-i}} \right| \eta_i
\]

\[
+ (n-i) (\lambda_0 \lambda_{i+2} + \gamma_0) |\eta_i|
\]

\[
\leq |\alpha \eta_i| + \left( \sum_{k=1}^{i} |\eta_k| \right) \left| \frac{e^{\alpha t} x_i}{\xi_i^{n-i}} \right| \eta_i
\]

\[
+ (n-i) (\lambda_0 \lambda_{i+2} + \gamma_0) |\eta_i|
\]

\[
\leq |\alpha \eta_i| + \left( \sum_{k=1}^{i} |\eta_k| \right) \left| \frac{e^{\alpha t} x_i}{\xi_i^{n-i}} \right| \eta_i
\]

\[
+ (n-i) (\lambda_0 \lambda_{i+2} + \gamma_0) |\eta_i|
\]

\[
\bar{f}_i(x_0, \bar{x}_i),
\]

(13)

where \(\bar{f}_i(x_0, \bar{x}_i) \geq |\alpha | + |\lambda_{i+2} x_i| \eta_i + (n-i) (\lambda_0 \lambda_{i+2} + \gamma_0)\).

To design a state-feedback controller, one introduces the coordinate transformation

\[
z_i = \eta_i, \quad i = 1, 2 \ldots, n,
\]

(14)

where \(\alpha_1, \ldots, \alpha_n\) are smooth virtual control laws and will be designed later and \(\alpha_1 = 0\). \(\theta\) denotes the estimate of \(\theta\), where

\[
\theta = \sup_{t \geq 0} \left\{ \max \left( \left\| \sum(t) \sum(t) \right\|, \left\| \sum(t) \sum(t) \right\| \right) \right\}
\]

(15)

Then using (9), (10), (14) and Itô differentiation rule, one has

\[
dz_i = d(\eta_i - \alpha_i)
\]

\[- \left( d_i \eta_{i+1} + F_i \eta_i, x_0 \right) dt + G_i (\eta_i) \sum(t) d\omega
\]

\[
- \frac{1}{2} \sum_{k=m+1}^{i-1} \frac{\partial^2 \alpha_i}{\partial \eta_k \partial \eta_m} \left( \sum(t) \sum(t) \phi_m \eta_m dt \right),
\]

(16)

where \(\eta_{i+1} = u, F_i(\eta_i, x_0) = f_i + \sum_{k=1}^{i} (\partial \alpha_i / \partial \eta_k) (d_0 \eta_{i+1} + f_k)\), and \(G_i(\eta_i, x_0) = \phi_i + \sum_{k=1}^{i} (\partial \alpha_i / \partial \eta_k) \phi_k\), where \(i = 1, 2 \ldots, n\). Using Lemmas 2, 4, and 9 and (14), we easily obtain the following lemma.

**Lemma 10.** For \(1 \leq i \leq n\), there exist nonnegative smooth functions \(y_i(\bar{x}_i, x_0), \rho_i(\bar{x}_i), \bar{p}_i(\bar{x}_i)\), such that

\[
|F_i| \leq \left( \sum_{k=1}^{i} |\bar{x}_k| \right) y_i (\bar{x}_i, x_0),
\]

\[
|G_i| \leq \left( \sum_{k=1}^{i} |\bar{x}_k| \right) \rho_i (\bar{x}_i),
\]

\[
|\Phi_i| \leq \left( \sum_{k=1}^{i} |\bar{x}_k| \right) \bar{p}_i (\bar{x}_i).
\]

(17)
The proof of Lemma 10 is similar to that of Lemma 9, so we omitted it.

We now give the design process of the controller.

Step 1. Consider the first Lyapunov function $V_i(\xi, \hat{\theta}) = (1/4)z_i^4 + (1/2)(\hat{\theta} - \theta)^2$. By (14), (15), and (16), we have

$$LV_1 = z_i^4(d_i \eta_2 + F_i) + \frac{3z_i^2}{2} \text{Tr} \left( G_i^T \left( \sum_t (t) G_i \right) \right)$$
$$+ \left( \hat{\theta} - \theta \right)^2 .$$

Using Lemma 10 and Lemma 4, we have

$$\left| z_i^4 F_i \right| \leq z_i^4 \gamma_{11}(z_i, x_0)$$
$$\left| \frac{3z_i^2}{2} \text{Tr} \left( G_i^T \left( \sum_t (t) G_i \right) \right) \right| \leq z_i^4 \rho_{11}(z_i, x_0) \theta.$$  

Substituting (19) into (18) and using (14), we have

$$LV_1 \leq d_i z_i^4(\eta_2 - \alpha_2) + d_i z_i^2 \alpha_2 + z_i^4 \rho_{11}(z_i, x_0) \theta$$
$$+ z_i^4 \gamma_{11}(z_i, x_0) + \left( \hat{\theta} - \theta \right)^2$$
$$\leq d_i z_i^4 \gamma_{11}(z_i, x_0) + \left( \hat{\theta} - \theta \right)^2,$$  

where $\alpha_2 = -z_i \beta_3 = -z_i((c_i + \gamma_{11} + p_{11}^2 \hat{\theta})/\lambda_{11})$. Substituting $\alpha_2$ into (20), we have

$$LV_1 \leq d_i z_i^4 \gamma_{11}(z_i, x_0) + \left( \hat{\theta} - \theta \right)^2 .$$  

Then, define the $i$th Lyapunov candidate function $V_i(\xi, \hat{\theta}) = V_{i-1} + (1/4)z_i^4$. From (16) and (22), it follows that

$$LV_i \leq \sum_{j=1}^{i-1} \left( c_j - e_j - e_j \right) z_j^4 - c_{i-1} z_{i-1}^4 + d_i z_i^4 \gamma_{11}(z_i, x_0)$$
$$+ \left( \hat{\theta} - \theta \right)^2$$
$$+ \left( \hat{\theta} - \theta \right)^2 .$$

Using Lemmas 9 and 4, there are always known nonnegative smooth functions $\psi_{i1}(\xi, \psi_{i1}(\xi), \psi_{i2}(\xi), \psi_{i3}(\xi), \psi_{i4}(\xi)$ and constant $\epsilon_i > 0, \epsilon_{ij} > 0$, where $i = 1, \ldots, n$ and $j = 1, 2, 3, 4$.

Consider

$$z_i^4 F_i \leq \left| \sum_{k=1}^{i-1} \left| z_k \right| \right| \gamma_{11}(\xi, x_0)$$
$$\leq \gamma_{i1} z_i^4 + \left| \sum_{k=1}^{i-1} \left( \epsilon_{k1} z_k^4 + \frac{3}{4} \epsilon_{k2} z_k^4 + \frac{3}{4} \epsilon_{k3} z_k^4 + \frac{3}{4} \epsilon_{k4} z_k^4 \right) \right|.$$  

Using Lemmas 9 and 4, there are always known nonnegative smooth functions $\psi_{i1}(\xi, \psi_{i1}(\xi), \psi_{i2}(\xi), \psi_{i3}(\xi), \psi_{i4}(\xi)$ and constant $\epsilon_i > 0, \epsilon_{ij} > 0$, where $i = 1, \ldots, n$ and $j = 1, 2, 3, 4$.

Step 2. Assume that at step $i - 1$, there exists a smooth state-feedback virtual control $\alpha_i = \gamma_{11}(\xi, x_0) + \left( \hat{\theta} - \theta \right)^2,$ such that

$$LV_{i-1} \leq -\sum_{j=1}^{i-1} \left( c_j - e_j - e_j \right) z_j^4 - c_{i-1} z_{i-1}^4 + d_i z_i^3 \gamma_{i-1}(z_i, x_0)$$
$$+ \left( \hat{\theta} - \theta - \epsilon_{i-1} \hat{\theta} \right)^2.$$  

where $\lambda_{i-1}$, $\gamma_{i1}$, $\gamma_{i2}$, $\gamma_{i3}$, and $\gamma_{i4}$ are positive constants, and $\epsilon_j = \sum_{k=1}^j (\epsilon_{k1} + \epsilon_{k2} + \epsilon_{k3} + \epsilon_{k4})$, where $j = 1, \ldots, n$. Then, define the $i$th Lyapunov candidate function $V_i(\xi, \hat{\theta}) = V_{i-1} + (1/4)z_i^4$. From (16) and (22), it follows that

$$LV_i \leq \sum_{j=1}^{i-1} \left( c_j - e_j - e_j \right) z_j^4 - c_{i-1} z_{i-1}^4 + d_i z_i^3 \gamma_{i-1}(z_i, x_0)$$
$$+ \left( \hat{\theta} - \theta - \epsilon_{i-1} \hat{\theta} \right)^2.$$  

where $\lambda_{i-1}$, $\gamma_{i1}$, $\gamma_{i2}$, $\gamma_{i3}$, and $\gamma_{i4}$ are positive constants, and $\epsilon_j = \sum_{k=1}^j (\epsilon_{k1} + \epsilon_{k2} + \epsilon_{k3} + \epsilon_{k4})$, where $j = 1, \ldots, n$.
\[
\begin{align*}
\sum_{k=1}^{i-1} e_k z_k^4 + \psi_{i4} z_i^4, \\
\leq \sum_{k=1}^{i-1} e_k z_k^4 + \psi_{i4} z_i^4,
\end{align*}
\]

where \(\psi_{i4} \geq (3/4)(4\epsilon_{i4})^{-1/3} \sqrt{1 + (\partial \alpha_i / \partial \theta) z_i^4 P_{i11}^{1/2}}\).

Theorem 11. Under Assumption 5, if the proposed adaptive controller (31) together with the above switching control strategy is used in (1), then for any initial condition \((x_0, \tilde{\theta}) \in R^n\),
the closed-loop system has an almost surely unique solution on \([0,\infty)\), the solution process is bounded in probability, and \(P\{\lim_{t\to\infty} \hat{\theta}(t) \text{ exists and is finite} \} = 1\).

**Proof.** According to the above analysis, it suffices to prove in the case \(x_0(0) \neq 0\). Since we have already proven that \(x_0\) can be globally exponentially convergent to zero in probability in Section 3.1, we only need prove that \(x(t)\) is convergent to zero in probability also. In this case, we choose the Lyapunov function \(V = V_\eta\), and \(c_i > \varepsilon_i + \varepsilon_i\) from (32) and Lemma 3, we know that the closed-loop system has an almost surely unique solution on \([0,\infty)\), and the solution process is bounded in probability. \(\Box\)

### 4. Conclusions

This paper investigates the globally exponential stabilization problem for a class of stochastic nonholonomic systems in chained form. To deal with the nonlinear parametrization problem, a parameter separation technique is introduced. With the help of backstepping technique, a smooth adaptive controller is constructed which ensures that the closed-loop system is globally asymptotically stable in probability. A further work is how to design the output-feedback tracking control for more high-order stochastic nonholonomic systems.

### Acknowledgments

This work was supported by the university research projects of Department of Education in Shandong Province, China (J13L03). The author would like to thank the reviewers for their helpful comments.

### References


