Research Article

Viscosity Approximation Methods for Two Accretive Operators in Banach Spaces

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We introduced a viscosity iterative scheme for approximating the common zero of two accretive operators in a strictly convex Banach space which has a uniformly Gâteaux differentiable norm. Some strong convergence theorems are proved, which improve and extend the results of Ceng et al. (2009) and some others.

1. Introduction and Preliminaries

Let E be a real Banach space, C a nonempty closed convex subset of E, and T : C → C a mapping. Recall that T is nonexpansive if ||Tx − Ty|| ≤ ||x − y||, for all x, y ∈ C. A point x ∈ C is a fixed point of T provided that Tx = x. Denote by FT the set of fixed points of T; that is, FT = {x ∈ C, Tx = x}. Throughout this paper, we assume that T is a contraction with constant α ∈ (0, 1) such that ||f(x) − f(y)|| ≤ α||x − y||, for all x, y ∈ C. Let ΣC = {f : C → C | f is a contraction with constant α}. The normalized duality mapping J from E into 2E* is given by J(x) = {f ∈ E* : ⟨x, f⟩ = ||x||² = ||f||²}, x ∈ E, where E* denotes the dual space of E and ⟨·, ·⟩ denotes the generalised duality pairing.

A Banach space E is said to be strictly convex if ||(x + y)/2|| < 1, for all x ≠ y ∈ E with ||x|| = ||y|| = 1. It is said to be uniformly convex if limn→∞||xn − yn|| = 0, for any two sequences {xn}, {yn} in E such that ||xn|| = ||yn|| = 1 and limn→∞(||xn + yn/2||/2) = 1.

The norm of E is said to be Gâteaux differentiable if

\[ \lim_{t \to 0} \left( \frac{||x + ty|| - ||x||}{t} \right) \]

exists for each x, y in its unit sphere U = {x ∈ E, ||x|| = 1}. Such an E is called a smooth Banach space. The norm is said to be uniformly Gâteaux differentiable if, for each y ∈ U, the limit is attained uniformly for x ∈ U. It is well known that E is smooth if and only if the duality mapping J is single valued and that, if E has a uniformly Gâteaux differentiable norm, J is uniformly norm to weak* continuous on each bounded subset of E (cf. [1]).

Let D be a subset of C. Then Q : C → D is called a retraction from C onto D if Q(x) = x for all x ∈ D. A retraction Q : C → D is said to be sunny if Q(Qx + t(x − Qx)) = Qx for all x ∈ C and t ≥ 0 whenever Qx + t(x − Qx) ∈ C. A subset D of C is said to be a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction of C onto D. In a smooth Banach space E, it is known that Q : C → D is a sunny nonexpansive retraction if and only if the following condition holds (cf. [2, page 48]):

\[ ⟨x − Qx, J(z − Qx)⟩ ≤ 0, \quad x ∈ C, \quad z ∈ D. \]  \hfill (2)

Recall that an operator A with D(A) and R(A) in E is said to be accretive if, for each x1 ∈ D(A) and y1 ∈ Ax1, i = 1, 2, there exists a j ∈ J(x2 − x1) such that ⟨y2 − y1, j⟩ ≥ 0. An accretive operator A is m-accretive if R(I + λA) = E, for all λ > 0. Denote by A−10 the set of zeros of A; that is, A−10 = {x ∈ DA, Ax = 0}.

Denote by Jλ (r > 0) the resolvent of A; that is, Jλ = (I + rA)−1. It is well known that F(Jλ) = A−10, for all r > 0. And if D(A) is convex, then Jλ is a nonexpansive mapping from E to
If $E$ is a Hilbert space, then $A$ is a maximal monotone operator if and only if $A$ is an $m$-accretive operator.

Recently, the approximation of zeros of accretive operators has been studied extensively (see, e.g., [3–9]). Specially, Ceng et al. [10] studied the following composite iterative scheme in uniformly smooth Banach spaces:

\begin{align}
y_n &= \alpha_n u + (1 - \alpha_n) J_{\tau_n} x_n, \\
x_{n+1} &= \beta_n y_n + (1 - \beta_n) J_{\tau_n} y_n,
\end{align}

where $u \in D(A)$ is an arbitrary (but fixed) element. They proved that $\{x_n\}$ generated by (3) converges strongly to a zero of $m$-accretive operator $A$ under certain appropriate conditions.

Very recently, Chen et al. [11] considered the following viscosity iteration scheme in a reflexive Banach space having a weakly sequentially continuous duality mapping:

\begin{align}
y_n &= \alpha_n f(x_n) + (1 - \alpha_n) S_{\tau_n} x_n, \\
x_{n+1} &= \beta_n y_n + (1 - \beta_n) y_n,
\end{align}

where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$. Under some conditions, they showed that $\{x_n\}$ generated by (4) converges strongly to a zero of $m$-accretive operator $A$.

In this paper, motivated by [10–14], we will consider the following composite iteration scheme:

\begin{align}
y_n &= \alpha_n f(x_n) + (1 - \alpha_n) S_{\tau_n} x_n, \\
x_{n+1} &= \beta_n y_n + (1 - \beta_n) y_n,
\end{align}

where $A$ and $B$ are $m$-accretive operators, $S_{\tau_n} = (1 - \lambda_n) I^A + \lambda_n I^B$ such that $F(S_n) = A^{-1} 0 \cap B^{-1} 0 \neq \emptyset$, $C = D(A) = D(B)$, $f \in \Sigma_C$, and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$. Under some conditions, we will prove that $\{x_n\}$ generated by (5) converges strongly to a common zero of $A$ and $B$ in a strictly convex and reflexive Banach space having a uniformly Gâteaux differentiable norm, which improve the corresponding results in [10–13].

**Lemma 1** (see [10]). In a Banach space $E$, the following inequality holds:

\[ \|x + y\| \leq \|x\|^2 + 2 \langle y, f(x + y) \rangle, \quad \forall x, y \in E, \]

where $\|f(x + y)\| \leq \|x\|y + g(x, y)$.

**Lemma 2** (see [10, 13]). Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the condition

\[ \alpha_{n+1} \leq (1 - \gamma_n) \alpha_n + \sigma_n y_n, \quad \forall n \geq 0, \]

where $\{y_n\} \subset (0, 1)$ and $\{\sigma_n\}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = 0$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) either $\limsup_{n \to \infty} \sigma_n = 0$ or $\sum_{n=1}^{\infty} |y_n \sigma_n| < \infty$.

Then $\lim_{n \to \infty} \alpha_n = 0$.

**Lemma 3** (the resolvent identity [10]). For $\lambda > 0, \mu > 0$ and $x \in E$,

\[ J_{\lambda} x = J_{\mu} \left( \frac{\mu}{\lambda} x + \left( 1 - \frac{\mu}{\lambda} \right) J_{\lambda} x \right). \]

**Lemma 4** (see [3, Theorem 4.1, page 287]). Let $E$ be a uniformly smooth Banach space, $C$ be a closed convex subset of $E$, $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$, and $f \in \Sigma_C$. Then $\{z_n\}$ defined by the following

\[ z_n = tf(z_n) + (1 - t) T z_n, \quad z_n \in C, \]

converges strongly to a point in $\text{Fix}(T)$. If, moreover, one defines

\[ Q(f) := \lim_{n \to 0} z_n, \quad f \in \Sigma_C, \]

then $Q(f)$ solves the variational inequality

\[ \langle (I - f) Q(f), J(Q(f) - p) \rangle \leq 0, \quad \forall f \in \Sigma_C, \quad p \in F(T). \]

Recall that a mapping $g : C \to C$ is said to be weakly contractive [15, 16] if

\[ \|g(x) - g(y)\| \leq \|x - y\| - \psi \|x - y\|, \quad \forall x, y \in C, \]

where $\psi : [0, +\infty) \to [0, +\infty)$ is a continuous and strictly increasing function such that $\psi$ is positive on $(0, +\infty)$ and $\psi(0) = 0$. As a special case, if $\psi(t) = (1 - k)t$ for $t \in [0, +\infty)$, where $k \in (0, 1)$, then the weakly contractive mapping $g$ is a contraction with constant $k$. Rhodes [17] obtained the following result for weakly contractive mapping (see also [16]).

**Lemma 5** (see [17, Theorem 2]). Let $(X, d)$ be a complete metric space and $g$ a weakly contractive mapping on $X$. Then $g$ has a unique fixed point $p \in X$.

**Lemma 6.** Let $\{s_n\}$ and $\{y_n\}$ be two sequences of nonnegative real numbers and $\{\lambda_n\}$ a sequence of positive numbers satisfying the conditions

(i) $\sum_{n=1}^{\infty} \lambda_n = +\infty$,

(ii) $\lim_{n \to +\infty} (y_n / \lambda_n) = 0$.

Let the recursive inequality

\[ s_{n+1} \leq s_n - \lambda_n \psi(s_n) + \lambda_n y_n, \quad n = 0, 1, 2, \ldots, \]

be given where $\psi(t)$ is a continuous and strict increasing function on $[0, +\infty)$ with $\psi(0) = 0$. Then $\lim_{n \to +\infty} s_n = 0$.

2. Main Results

Throughout this section, we assume the following:

(i) $E$ is a strictly convex Banach space which has a uniformly Gâteaux differentiable norm, and $C$ is a nonempty closed convex subset of $E$. 


(ii) Take $S_r = (1-\lambda)J^A_r + \lambda J^B_r$, $0 < \lambda < 1$. Obviously $S_r$ is nonexpansive mapping and $F(S_r) = A^{-1}0 \cap B^{-1}0$, if $E$ is a strictly convex Banach space. Indeed, it is easy to see that $F(S_r) > A^{-1}0 \cap B^{-1}0$. Let $q \in F(S_r)$, $p \in A^{-1}0 \cap B^{-1}0$; then

$$
\|q - p\| \leq (1-\lambda)\|J^A_r q - p\| + \lambda \|J^B_r q - p\|
$$

$$
\leq (1-\lambda)\|J^A_r q - p\| + \lambda \|q - p\| \leq \|q - p\|. \quad (14)
$$

From the above formula, we obtain

$$
(1-\lambda)\|J^A_r q - p\| + \lambda \|q - p\| = \|q - p\|, \quad \text{so} \quad \|J^A_r q - p\| = \|q - p\|. \quad \text{Similarly,} \quad \|J^B_r q - p\| = \|q - p\|.
$$

But

$$
\|q - p\| = \|(1-\lambda)\left(J^A_r q - p + \lambda \left(J^B_r q - p\right)\right)\|. \quad (15)
$$

Then the strict convexity of $E$ implies that $q - p = J^A_r q - p = J^B_r q - p$, that is, $q = J^A_r q = J^B_r q$, or $q \in A^{-1}0 \cap B^{-1}0$.

**Theorem 7.** Let $E$ be a strictly convex Banach space which has a uniformly Gâteaux differentiable norm, $A$, $B$ two m-accretive maps in $E$ such that $C = D(A) = D(B)$ is convex and $A^{-1}0 \cap B^{-1}0 \neq \emptyset$, and $f : C \to A$ a fixed contraction mapping with contract constant $\alpha$. Suppose that $\alpha_n \in (0, 1)$, $\beta_n \in (0, 1)$, and $r_n > 0$ satisfy the following conditions:

(i) $\sum_{n=1}^{\infty} \alpha_n = \alpha_0$, $\alpha_n \to 0$, as $n \to \infty$;

(ii) $\beta_n \to 0$, and $r_n \to r > 0$, as $n \to \infty$;

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Let $\{x_n\}$ be the composite viscosity process defined by

$$
y_n = \alpha_n f(x_n) + (1 - \alpha_n) S_{r_n} x_n, \quad x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) y_n. \quad (16)
$$

Then $\{x_n\}$ converges strongly to $p \in A^{-1}0 \cap B^{-1}0$, where $p$ is the unique solution of the following variational inequality:

$$
\left\langle (1-f)p, J(p-q) \right\rangle \leq 0, \quad f \in \Sigma_C, \quad q \in A^{-1}0 \cap B^{-1}0. \quad (17)
$$

**Proof.** First, by using Lemma 4, we know that there exists the unique solution $p$ of a variational inequality

$$
\left\langle (1-f)p, J(p-q) \right\rangle \leq 0, \quad f \in \Sigma_C, \quad q \in A^{-1}0 \cap B^{-1}0, \quad (18)
$$

where $p = \lim_{t \to 0} z_t$ and $z_t$ is defined by $z_t = tf(z_t) + (1-t)S_{r}\{z_t\}$ for each $r > 0$ and $0 < t < 1$.

Next, we will divide our discussion into the following steps.

**Step 1.** We will show that $\{x_n\}$ is bounded. In fact, take $p \in A^{-1}0 \cap B^{-1}0$. Then

$$
\|y_n - p\| = \|\alpha_n f(x_n) + (1 - \alpha_n) S_{r_n} x_n - p\|
$$

$$
\leq (1 - \alpha_n) \|x_n - p\| + \alpha_n \|f(p) - p\|. \quad (19)
$$

Therefore,

$$
\|x_{n+1} - p\| = \|\beta_n f(x_n) + (1 - \beta_n) y_n - p\|
$$

$$
\leq (1 - \beta_n) \|x_n - p\| + \beta_n \|f(p) - p\|
$$

$$
\leq (1 - \beta_n) (\alpha_n + \alpha_n \alpha_n \alpha_n) \|x_n - p\|
$$

$$
+ (\beta_n + \alpha_n - \alpha_n \alpha_n \alpha_n) \|f(p) - p\|
$$

$$
\leq \max \left\{ \frac{1}{1 - \alpha} \|f(p) - p\|, \|x_n - p\| \right\}. \quad (20)
$$

Using the induction method, we have

$$
\|x_n - p\| \leq \max \left\{ \frac{1}{1 - \alpha} \|f(p) - p\|, \|x_0 - p\| \right\}, \quad n \geq 0, \quad (21)
$$

which implies that $\{x_n\}, \{f(x_n)\}, \{y_n\}$, and $\{f(y_n)\}$ are all bounded. Since $\|S_{r_n} x_n - p\| \leq \|x_n - p\|$, then $\{S_{r_n} x_n\}$ is bounded. Following the conditions of (i) and (ii), we obtain that

$$
\|x_{n+1} - y_n\| = \beta_n \|f(x_n) - y_n\| \to 0, \quad \text{as} \quad n \to \infty, \quad (22)
$$

$$
\|y_n - S_{r_n} x_n\| = \alpha_n \|f(x_n) - S_{r_n} x_n\| \to 0, \quad \text{as} \quad n \to \infty. \quad (23)
$$

**Step 2.** We show that $\|x_{n+1} - x_n\| \to 0$.

For this, we estimate $y_{n+1} - y_n$ first. From (16), we know that

$$
y_n = \alpha_n f(x_n) + (1 - \alpha_n) S_{r_n} x_n, \quad y_{n-1} = \alpha_n f(x_{n-1}) + (1 - \alpha_n) S_{r_{n-1}} x_{n-1}. \quad (24)
$$

Then simple calculations show that

$$
y_n - y_{n-1} = (1 - \alpha_n) \left(S_{r_n} x_n - S_{r_{n-1}} x_{n-1}\right)
$$

$$
+ \alpha_n \left(f(x_n) - f(x_{n-1})\right)
$$

$$
+ (\alpha_n - \alpha_n) \left(f(x_{n-1}) - S_{r_{n-1}} x_{n-1}\right). \quad (25)
$$

It follows from (25) that

$$
\|y_n - y_{n-1}\| \leq (1 - \alpha_n) \|S_{r_n} x_n - S_{r_{n-1}} x_{n-1}\| + \alpha_n \alpha_n \|x_n - x_{n-1}\|
$$

$$
+ \|\alpha_n - \alpha_n\| \|f(x_{n-1}) - S_{r_{n-1}} x_{n-1}\|. \quad (26)
$$

In view of Lemma 3, we have

$$
J^{A}_{r_n} x_n = J^{A}_{r_{n-1}} \left( \frac{r_{n-1}}{r_n} x_n + \left(1 - \frac{r_{n-1}}{r_n}\right) J^{A}_{r_n} x_n \right). \quad (27)
$$
If $r_{n-1} \leq r_n$, then
\[
\|J_{r_n}^A x_n - J_{r_{n-1}}^A x_{n-1}\| \\
= \left\| J_{r_n}^A \left( \frac{r_{n-1}}{r_n} x_n + \left(1 - \frac{r_{n-1}}{r_n} \right) f_{r_n}^A x_n \right) - J_{r_{n-1}}^A x_{n-1}\right\| \\
\leq \frac{r_{n-1}}{r_n} \|x_n - x_{n-1}\| + \left(1 - \frac{r_{n-1}}{r_n} \right) \|f_{r_n}^A x_n - x_{n-1}\|. \tag{28}
\]
Similarly,
\[
\|J_{r_n}^B x_n - J_{r_{n-1}}^B x_{n-1}\| \leq \frac{r_{n-1}}{r_n} \|x_n - x_{n-1}\| + \frac{r_{n-1} - r_n}{\epsilon} \|f_{r_n}^A x_n - x_{n-1}\|. \tag{29}
\]
Thus, let $M = \sup\{ (1/\epsilon) \|f_{r_n}^A x_n - x_{n-1}\|, (1/\epsilon) \|f_{r_n}^B x_n - x_{n-1}\| \}$; we have
\[
\|S_n x_n - S_{n-1} x_{n-1}\| \\
\leq (1 - \lambda) \|J_{r_n}^A x_n - J_{r_{n-1}}^A x_{n-1}\| + \lambda \|J_{r_n}^B x_n - J_{r_{n-1}}^B x_{n-1}\|. \tag{30}
\]
Substituting (30) into (26) we get
\[
\|y_n - y_{n-1}\| \leq (1 - \alpha_n) \left( \|x_n - x_{n-1}\| + (r_n - r_{n-1}) M \right) \\
+ \alpha_n \epsilon \|x_n - x_{n-1}\| \\
+ \|S_n x_n - S_{n-1} x_{n-1}\| \leq (1 - \alpha_n (1 - \alpha)) \|x_n - x_{n-1}\| \\
+ (1 - \alpha_n) (r_n - r_{n-1}) M \\
+ \|S_n x_n - S_{n-1} x_{n-1}\| \leq (1 - \alpha_n (1 - \alpha)) \|x_n - x_{n-1}\| \\
+ M_1 \left( \|x_n - x_{n-1}\| + |r_n - r_{n-1}| \right), \tag{31}
\]
where $M_1$ is a constant such that
\[
M_1 > \max \{ M, \|f(x_n - 1) - S_{r_n} x_{n-1}\| \}. \tag{32}
\]
On the other hand, we have
\[
x_{n+1} = \beta_n f(y_n) + (1 - \beta_n) y_n, \\
x_n = \beta_n f(y_{n-1}) + (1 - \beta_n) y_{n-1}. \tag{33}
\]
Simple calculations show that
\[
x_{n+1} - x_n = (1 - \beta_n) (y_n - y_{n-1}) + \beta_n (f(y_n) - f(y_{n-1})) \\
+ (\beta_n - \beta_{n-1}) (f(y_{n-1}) - y_{n-1}). \tag{34}
\]
It follows that
\[
\|x_{n+1} - x_n\| \leq (1 - \beta_n (1 - \alpha)) \|x_n - x_{n-1}\| \\
+ \beta_n \|x_n - y_{n-1}\| \\
+ |\beta_n - \beta_{n-1}| \|f(y_{n-1}) - y_{n-1}\| \tag{35}
\]
Substituting (31) into (35) we get
\[
\|x_{n+1} - x_n\| \\
\leq (1 - \beta_n (1 - \alpha)) \left( (1 - \alpha_n (1 - \alpha)) \|x_n - x_{n-1}\| \\
+ M_1 \left( \|x_n - x_{n-1}\| + |r_n - r_{n-1}| \right) \right) \\
+ \beta_n \|x_n - y_{n-1}\| \\
\leq (1 - \alpha_n (1 - \alpha)) \|x_n - x_{n-1}\| \\
+ M_2 \left( \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| + |r_n - r_{n-1}| \right), \tag{36}
\]
where $M_2$ is a constant such that
\[
M_2 > \max \{ \|f(y_{n-1}) - y_{n-1}\|, M_1 \}. \tag{37}
\]
From conditions (i)–(iii), we have that
\[
\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \alpha_n \to 0, \text{ as } n \to \infty, \tag{38}
\]
Hence, noticing (36) and applying Lemma 2, we obtain $\|x_{n+1} - x_n\| \to 0$. Then by (22) we obtain
\[
\|x_n - y_n\| \leq \|x_{n+1} - x_n\| + \|x_{n+1} - y_{n-1}\| \to 0, \quad (n \to \infty). \tag{39}
\]
Step 3. We prove that $\|x_n - S_n x_n\| \to 0, \|y_n - S_n y_n\| \to 0$. In fact, since
\[
\|x_n - S_n x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - S_n x_n\|, \tag{40}
\]
from (22) and (23), we have \( \|x_n - S_r x_n\| \to 0 \). Then

\[
\begin{align*}
\|S_r x_n - S_r x_n\| &\leq (1 - \lambda) \|J^A_{r_n} x_n - J^A_{r_n} x_n\| + \lambda \|J^B_{r_n} x_n - J^B_{r_n} x_n\| \\
&\leq (1 - \lambda) \left\| \left( \frac{r}{r_n} x_n + \left( 1 - \frac{r}{r_n} \right) J^A_{r_n} x_n \right) - J^A_{r_n} x_n \right\| + \lambda \left\| \left( \frac{r}{r_n} x_n + \left( 1 - \frac{r}{r_n} \right) J^B_{r_n} x_n \right) - J^B_{r_n} x_n \right\| \\
&\leq (1 - \lambda) \left\| \left( \frac{r}{r_n} x_n + \left( 1 - \frac{r}{r_n} \right) J^A_{r_n} x_n \right) - x_n \right\| + \lambda \left\| \left( \frac{r}{r_n} x_n + \left( 1 - \frac{r}{r_n} \right) J^B_{r_n} x_n \right) - x_n \right\| \\
&\leq \left| 1 - \frac{r}{r_n} \right| \max \{ \|J^A_{r_n} x_n - x_n\|, \|J^B_{r_n} x_n - x_n\| \} \longrightarrow 0.
\end{align*}
\]

Hence, we have

\[
\begin{align*}
\|x_n - S_r x_n\| &\leq \|x_n - S_r x_n\| + \|S_r x_n - S_r x_n\| \to 0, \quad n \to \infty, \\
\|y_n - S_r y\| &\leq \|x_n - y\| + \|x_n - S_r x_n\| + \|S_r y - S_r x_n\| \\
&\leq 2 \|x_n - y\| + \|x_n - S_r x_n\| \to 0, \quad n \to \infty.
\end{align*}
\]

Step 4. We show that \( \limsup_{n \to \infty} \langle (I - f)p, J(y_n - p) \rangle \leq 0 \), \( \limsup_{n \to \infty} \langle (I - f)p, f(x_{n+1}) - p \rangle \leq 0 \).

To prove this, let \( \{y_n\} \) be a subsequence of \( \{y_n\} \) such that

\[
\limsup_{n \to \infty} \langle (I - f)p, J(y_n - p) \rangle = \lim_{j \to \infty} \left\langle (I - f)p, J(y_{n_j} - p) \right\rangle.
\]

By Lemma 4, \( \lim_{t \to 0^+} \zeta_t = p \in F(S_r), \) where \( \zeta_t = tf(z_t) + (1 - t)S_r(z_t) \). Then

\[
z_t - y_{n_j} = t_n \left( f(z_{n_j}) - y_{n_j} \right) + (1 - t_n) \left( S_r(z_{n_j}) - y_{n_j} \right).
\]

For each integer \( n \geq 0 \), let \( t_n \in (0, 1) \) such that

\[
t_n \to 0, \quad \frac{\|S_r y_{n_j} - y_{n_j}\|}{t_n} \to 0, \quad n \to \infty.
\]

Using Lemma 1, we get

\[
\begin{align*}
\|z_t - y_{n_j}\|^2 &\leq (1 - t_n)^2 \|S_r(z_{n_j}) - y_{n_j}\|^2 \\
&\quad + 2t_n \left\langle f(z_{n_j}) - y_{n_j}, f(z_{n_j} - y_{n_j}) \right\rangle \\
&\leq (1 - 2t_n + t_n^2) \left\langle S_r(z_{n_j}) - S_r y_{n_j}, \|y_{n_j} - S_r y_{n_j}\|^2 \right\rangle \\
&\quad + 2t_n \left\langle f(z_{n_j}) - z_{n_j}, f(z_{n_j} - y_{n_j}) \right\rangle \\
&\leq (1 + t_n^2) \|z_{n_j} - y_{n_j}\|^2 + (1 + t_n)^2 \|S_r y_{n_j} - y_{n_j}\| \\
&\quad \times \left( 2 \|z_{n_j} - y_{n_j}\| + \|S_r y_{n_j} - y_{n_j}\| \right) \\
&\quad + 2t_n \left\langle f(z_{n_j}) - z_{n_j}, f(z_{n_j} - y_{n_j}) \right\rangle.
\end{align*}
\]

and hence

\[
\begin{align*}
\left\langle z_t - f(z_{n_j}), f(z_{n_j} - y_{n_j}) \right\rangle &\leq \frac{t_n}{2} \|z_{n_j} - y_{n_j}\|^2 + \frac{(1 + t_n)^2 \|S_r y_{n_j} - y_{n_j}\|}{2t_n} \\
&\quad \times \left( 2 \|z_{n_j} - y_{n_j}\| + \|S_r y_{n_j} - y_{n_j}\| \right).
\end{align*}
\]

Since \( \{x_n\}, \{z_{n_j}\}, \) and \( \{S_r y_{n_j}\} \) are bounded, then \( \|S_r y_{n_j} - y_{n_j}\|/2t_n \to 0 \). Therefore,

\[
\limsup_{j \to \infty} \left\langle z_t - f(z_{n_j}), f(z_{n_j} - y_{n_j}) \right\rangle \leq 0.
\]

We also know that

\[
\begin{align*}
\left\langle p - f(z_{n_j}), f(y_{n_j} - p) \right\rangle &= \left\langle p - f(z_{n_j}), f(y_{n_j} - p) - f(z_{n_j} - y_{n_j}) \right\rangle \\
&\quad + \left\langle p - z_{n_j}, f(z_{n_j} - y_{n_j}) \right\rangle \\
&\quad + \left\langle z_{n_j} - f(z_{n_j}), f(z_{n_j} - y_{n_j}) \right\rangle.
\end{align*}
\]

Notice that \( z_{n_j} \to p, p \in F(S_r), n \to \infty \), and \( f \) is norm to weak* uniformly continuous on bounded subset of \( E \); then we obtain

\[
\begin{align*}
\left\langle p - z_{n_j}, f(z_{n_j} - y_{n_j}) \right\rangle &\to 0, \quad n \to \infty, \\
\left\langle p - f(z_{n_j}), f(y_{n_j} - p) - f(z_{n_j} - y_{n_j}) \right\rangle &\to 0, \quad n \to \infty.
\end{align*}
\]
From (48), (49), and the two results mentioned above, we have
\[
\limsup_{n \to \infty} \langle (I - f)p, J(y_n - p) \rangle \leq 0.
\] (51)

Using (22) and the property of \(J\), we obtain the result that
\[
\limsup_{n \to \infty} \langle (I - f)p, J(y_n - p) \rangle = \limsup_{n \to \infty} \langle (I - f)p, J(x_{n+1} - p) \rangle \leq 0.
\] (52)

Step 5. \(\lim_{n \to \infty} \|x_n - p\| = 0\).

Using (16), we have
\[
\|y_n - p\|^2 = \left\| \alpha_n (f(x_n) - p) + (1 - \alpha_n) (Sx_n - p) \right\|^2
= \left\| (1 - \alpha_n) (Sx_n - p) + \alpha_n (f(x_n) - f(p)) \right\|^2 + \alpha_n \langle f(p) - p, J(x_{n+1} - p) \rangle
\leq (1 - \alpha_n (1 - \alpha)) \|x_n - p\|^2 + 2\alpha_n \langle f(p) - p, J(y_n - p) \rangle.
\] (53)

Applying Lemma 1, we obtain
\[
\|x_{n+1} - p\|^2 = \left\| \beta_n (f(y_n) - p) + (1 - \beta_n) (y_n - p) \right\|^2
= \left\| (1 - \beta_n) (y_n - p) \right\|^2 + \beta_n \langle f(y_n) - f(p), J(y_{n+1} - p) \rangle
\leq (1 - \beta_n (1 - \alpha)) \|y_n - p\|^2 + 2\beta_n \langle f(p) - p, J(x_{n+1} - p) \rangle
\leq (1 - \alpha_n (1 - \alpha)) \|x_n - p\|^2 + 2\alpha_n \langle f(p) - p, J(y_{n+1} - p) \rangle
\leq \left[ 1 - (\alpha_n + \beta_n) (1 - \alpha) \right] \|x_n - p\|^2 + \alpha_n \beta_n (1 - \alpha)^2 \|x_n - p\|^2
- 2\alpha_n \beta_n (1 - \alpha) \langle f(p) - p, J(y_n - p) \rangle
+ 2(\alpha_n + \beta_n) \langle f(p) - p, J(x_{n+1} - p) \rangle
+ 2\alpha_n \langle f(p) - p, J(y_n - p) - J(x_{n+1} - p) \rangle\]
\leq [1 - (\alpha_n + \beta_n) (1 - \alpha)] \|x_n - p\|^2
+ (\alpha_n + \beta_n) \frac{\alpha_n \beta_n}{\alpha_n + \beta_n}(1 - \alpha)^2 L^2
+ 2(\alpha_n + \beta_n) \frac{\alpha_n \beta_n}{\alpha_n + \beta_n} (1 - \alpha) L^2
+ 2(\alpha_n + \beta_n) \{f(p) - p, J(x_{n+1} - p)\}
\] (54)

where \(L = \sup_{n \geq 0} \|x_n - p\|, \|y_n - p\|, \|f(p) - p\|\). Put
\[
y_n = (\alpha_n + \beta_n) (1 - \alpha),
\]
\[
\sigma_n = \frac{\alpha_n \beta_n}{\alpha_n + \beta_n} (1 - \alpha)^2 L^2 + \frac{2\alpha_n \beta_n}{\alpha_n + \beta_n} L^2
+ \frac{2}{1 - \alpha} \{f(p) - p, J(x_{n+1} - p)\}
\] (55)
\[
+ \frac{\alpha_n}{\alpha_n + \beta_n} \frac{2}{1 - \alpha} \times \{f(p) - p, J(y_n - p) - J(x_{n+1} - p)\}.
\]

From the conditions (i) and (ii), the result of Step 4, and the facts that \(\alpha_n \beta_n/(\alpha_n + \beta_n) \to 0\) and \(\{f(p) - p, J(y_n - p) - J(x_{n+1} - p)\} \to 0\), we know that \(y_n \to 0, \sum_{n=1}^{\infty} y_n = +\infty\) and \(\limsup_{n \to \infty} \sigma_n \leq 0\). In view of Lemma 2, (54) reduces to
\[
\|x_{n+1} - p\| \leq (1 - \gamma_n) \|x_n - p\| + y_n \sigma_n;
\] (56)
then we know that
\[
\lim_{n \to \infty} \|x_n - p\| = 0.
\] (57)

This completes the proof. \(\square\)

Remark 8. If we modify (16) as follows:
\[
y_n = \alpha_n f(x_n) + (1 - \alpha_n) Sx_n,
\]
\[
x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) y_n,
\]
\[
y_n = \alpha_n u + (1 - \alpha_n) S_{x_n}x_n,
\]
\[
x_{n+1} = \beta_n f(y_n) + (1 - \beta_n) y_n,
\] (58) (59)
or
\[
y_n = \alpha_n u + (1 - \alpha_n) S_{x_n}x_n,
\]
\[
x_{n+1} = \beta_n f(y_n) + (1 - \beta_n) y_n.
\] (60)

Then, imitating the proof of Theorem 7, we can also get the result of Theorem 7. Therefore, from the compare of iterative scheme, the conclusions of [10, 11] are special cases of Theorem 7.
Example 9. Next we study the following optimization problem:

$$\min_{x \in C} h(x), \quad \min_{x \in C} k(x),$$  \hspace{1cm} (61)

where $C$ is an interior nonempty closed convex subset of a Hilbert space and $h, k : C \to \mathbb{R}$ are two proper convex and lower semicontinuous functionals. To solve optimization problem (61), we will list the following well known results.

Proposition 10 (see [18]). Let $\varphi : C \to \mathbb{R}$ be a proper convex and lower semicontinuous functional. Then

(i) $\partial \varphi : C \to E^*$ ($\partial$ denotes the subdifferential in the sense of convex analysis) is a maximal monotone mapping; 
(ii) $\varphi(x_0) = \min_{x \in C} \varphi(x)$ if and only if $0 \in \partial \varphi(x_0)$.

In Hilbert space $\partial \varphi$ is an $m$-accretive mapping. Thus $A = \partial h, B = \partial k$ are two $m$-accretive mappings. Solving optimization problem (61) is equivalent to finding a common zero of $A$ and $B$.

Let

$$f_A^r = (I + rA)^{-1}, \quad f_B^r = (I + rB)^{-1},$$

$$S_r = (1 - \lambda) f_A^r + \lambda f_B^r, \quad \alpha_n = \beta_n = \frac{1}{n}, \quad r_n = \frac{r n}{n + 1}, \quad (r > \varepsilon). \quad (62)$$

Then the conditions (i), (ii), and (iii) of Theorem 7 are satisfied. For arbitrary $f \in \Sigma_C$ the sequence $\{x_n\}$ generated by (16) converges strongly to a common zero of $A$ and $B$, which is also the solution of the optimization problem (60).

Theorem 11. Let $A, B$ be two accretive maps in $E$ with $A^{-1} 0 \cap B^{-1} 0 \neq \emptyset$ and satisfy the following range conditions: $D(A) \subseteq C \subseteq R(I + rA) \cap R(I + rB), D(B) \subseteq C \subseteq R(I + rA) \cap R(I + rB)$ which are convex. Let $f, \{\alpha_n\}, \{\beta_n\}, \{r_n\},$ and $\{\gamma_n\}$ be the same as those in Theorem 7. Let $\{x_n\}$ be a sequence generated by (60). Then $\{x_n\}$ converges strongly to $p \in A^{-1} 0 \cap B^{-1} 0$, where $p$ is the unique solution of the following variational inequality:

$$\langle (1 - f) p, J (p - q) \rangle \leq 0, \quad f \in \Sigma_C, \quad q \in A^{-1} 0 \cap B^{-1} 0. \quad (63)$$

Theorem 12. Let $A, B$ be two accretive maps in $E$ with $F = A^{-1} 0 \cap B^{-1} 0 \neq \emptyset$ and satisfy the following range conditions: $D(A) \subseteq C \subseteq R(I + rA) \cap R(I + rB), D(B) \subseteq C \subseteq R(I + rA) \cap R(I + rB)$ which are convex. Let $f, \{\alpha_n\}, \{\beta_n\},$ and $\{r_n\}$ be the same as those in Theorem 7. Let $g : C \to C$ be a weakly contractive mapping with the function $\psi$. Let $\{x_n\}$ be a sequence generated by

$$y_n = \alpha_n g(x_n) + (1 - \alpha_n) S_{r_n} x_n,$$

$$x_{n+1} = \beta_n f(y_n) + (1 - \beta_n) y_n. \quad (64)$$

Then $\{x_n\}$ converges strongly to $p = Q(g(p)) \in A^{-1} 0 \cap B^{-1} 0$, where $Q$ is a sunny nonexpansive retraction from $C$ onto $C$.

Proof. Since $E$ is a uniformly smooth Banach space, then there is a sunny nonexpansive retraction $Q$ from $C$ onto $F$. Then $Q \circ g$ is a weakly contractive mapping of $C$ into itself. Indeed, for all $x, y \in C$,

$$\|Q(g(x)) - Q(g(y))\| \leq \|g(x) - g(y)\| \leq \|x - y\| - \psi(\|x - y\|). \quad (65)$$

Lemma 5 assures that there exists a unique element $p \in C$ such that $p = Q(g(p))$. Such $p \in C$ is an element of $A^{-1} 0 \cap B^{-1} 0$. Now we define an iterative scheme as follows:

$$z_n = \alpha_n g(p) + (1 - \alpha_n) S_{r_n} w_n,$$

$$w_{n+1} = \beta_n f(z_n) + (1 - \beta_n) z_n. \quad (66)$$

Let $w_n$ be the sequence generated by (66). Then Remark 8 (59) assures that $w_n$ converges strongly to $p = Q(g(p))$ as $n \to \infty$. For any $n$, we have

$$\|x_{n+1} - w_{n+1}\| \leq \beta_n \|f(z_n) - f(z_n)\| + (1 - \beta_n) \|y_n - z_n\| \leq \|y_n - z_n\| \leq \alpha_n \|g(x_n) - g(p)\| + (1 - \alpha_n) \|S_{r_n} x_n - S_{r_n} w_n\| \leq \alpha_n \|g(x_n) - g(w_n)\| + \alpha_n \|g(w_n) - g(p)\|$$

$$\leq (1 - \alpha_n) \|x_n - w_n\| \leq \alpha_n \psi(\|x_n - w_n\|) + \alpha_n \|w_n - p\|.$$ 

Thus, we obtain $s_n = \|x_n - w_n\|$ the following recursive inequality:

$$s_{n+1} = s_n - \alpha_n \psi(s_n) + \alpha_n \|w_n - p\|. \quad (68)$$

Since $\|w_n - p\| \to 0$, it follows from Lemma 6 that $\lim_{n \to \infty} \|x_n - w_n\| = 0$. Hence

$$\lim_{n \to \infty} \|x_n - p\| \leq \lim_{n \to \infty} \|x_n - w_n\| + \|w_n - p\| = 0. \quad (69)$$

This completes the proof. \(\square\)

In virtue of the weakly contractive mapping $g$ being a contraction, using Theorem 12 we may obtain the following.

Corollary 13. Let $A, B$ be two accretive maps in $E$ with $F = A^{-1} 0 \cap B^{-1} 0 \neq \emptyset$ and satisfy the following range conditions: $D(A) \subseteq C \subseteq R(I + rA) \cap R(I + rB), D(B) \subseteq C \subseteq R(I + rA) \cap R(I + rB)$ which are convex. Let $f, g \in \Sigma_C, \{\alpha_n\}, \{\beta_n\},$ and $\{r_n\}$ be the same as those in Theorem 7. Let $\{x_n\}$ be a sequence generated by

$$y_n = \alpha_n g(x_n) + (1 - \alpha_n) S_{r_n} x_n,$$

$$x_{n+1} = \beta_n f(y_n) + (1 - \beta_n) y_n. \quad (70)$$
Then \( \{x_n\} \) converges strongly to \( p = Q(g(p)) \in A^{-1}0 \cap B^{-1}0, \)
where \( Q \) is a sunny nonexpansive retraction from \( C \) onto \( F. \)

**Example 14.** Next we give an essential example.

Let \( \Omega \) be a bounded domain in a Euclidean space \( \mathbb{R}^N \)
with Lipschitz boundary \( \Gamma. \) Let \( \phi : \Gamma \times \mathbb{R} \to \mathbb{R} \) be a given
function such that for each \( x \in \Gamma \)

(i) \( \phi_x = \phi(x, \cdot) : \mathbb{R} \to \mathbb{R} \) is a proper, convex, lower
-semicontinuous function with \( \phi_0(0) = 0. \)

(ii) \( \beta_x = \partial \phi_x \) (subdifferential of \( \phi_x \)) is the maximal
monotone mapping on \( \mathbb{R} \) with \( 0 \in \beta_0(0) \) and for each
t \( \in \mathbb{R}, \) the function \( x \in \Gamma \to (I + \lambda \beta_x)^{-1}(t) \in \mathbb{R} \)
is measurable for \( \lambda > 0. \)

Let \( \alpha : \mathbb{R}^N \to \mathbb{R}^N \) be a continuous, monotone function
such that there exist constants \( k_1, k_2 \) satisfying (i) \( |\alpha(\xi)| \leq k_1|\xi| \)
and (ii) \( |\alpha(\xi)| \leq k_2|\xi|^2 \) for each \( \xi \in \mathbb{R}^N. \)

**Definition 15** (see [19]). One first defines a mapping \( A^\alpha : \)
\( H^1(\Omega) \to (H^1(\Omega))^* \) (\( H^1(\Omega) \) is a Sobolev space) by
\[
(A^\alpha u, v) = \int_\Omega \langle \alpha(\text{grad} u), \text{grad} v \rangle \, dx \tag{71}
\]
for \( u, v \in H^1(\Omega). \) Clearly \( A^\alpha \) is an everywhere defined, mono-
tone, demi-continuous operator from \( H^1(\Omega) \) into \((H^1(\Omega))^*\).

Second one defines an operator \( A_p^\alpha : L^p(\Omega) \to 2^{L^p(\Omega)} \) for
\( 1 < p < +\infty \)

(i) For \( p \geq 2 \) one defines the domain of \( A_p^\alpha \) by
\[
D(A_p^\alpha) = \{ u \in L^p(\Omega) \colon \text{there exists an } f \in L^p(\Omega) \text{ such that } A^\alpha u + \partial \Phi (u) \ni f \}. \tag{72}
\]

Here \( \Phi(u) = \int_\Omega \phi_0(u(x))dI^1(x) \) is the proper, convex, l.s.c.
function (see [19, Lemma 3.1]). For \( u \in D(A_p^\alpha) \) we set
\[
A_p^\alpha (u) = \{ f \in L^p(\Omega) \mid A^\alpha u + \partial \Phi (u) \ni f \} \tag{73}
\]

(ii) For \( 1 < p < 2 \), one defines \( A_p^\alpha \) as the \( L^p \)-closure of \( A_2^\alpha \)
defined in (i) above.

For the operator \( A_p^\alpha \) one has following results.

**Lemma 16** (see [19, Lemma 3.4]). \( A_p^\alpha : L^p(\Omega) \to 2^{L^p(\Omega)} \) is
\( m \)-accretive operator \( (1 < p < +\infty). \)

**Lemma 17** (see [19, Proposition 3.2]). Let \( f \in L^p(\Omega), u \in L^p(\Omega) \) such that \( f \in A_p^\alpha u. \) Then

(i) \( \text{div}(\alpha(\text{grad} u)) = f, \text{ a.e. on } \Omega \) and

(ii) \( \langle n, \alpha(\text{grad} u) \rangle \ni \beta_x(\alpha(u(x))) \) for a.e. \( x \in \Gamma. \)

**Lemma 18** (see [19, Proposition 3.3]). \( \beta_x \equiv 0 \) for \( x \in \Gamma. \)
Then \( (A_p^\alpha)^{-1}0 = \{ u \in L^p(\Omega) \mid u = a \text{ constant function} \}. \)

Clearly for different \( \alpha_1, \alpha_2, A_{p_1}^\alpha, A_{p_2}^\alpha \) are two \( m \)-accretive operators. The above results show that
\[
\emptyset \neq \{ u \in L^p(\Omega) \mid u = a \text{ constant function} \} \tag{74}
\]

For the sake of finding a common zero, Theorems 7, 11, and
12 provided three different iterative algorithms. Therefore the
study of a common zero of two accretive operators makes sense.

**Remark 19.** The results presented in this paper substantially
improve and extend the results of Ceng et al. [10] from the
following aspects.

(1) Theorems 7 and 12 extend the result on the iterative
construction of the zero for a single accretive operator
to the case of that for common zeros of two accretive
operators. If we modify two accretive operators as
finite accretive operators, then, imitating the proof of
Theorem 7, we can also get the result of Theorem 7.

(2) Our results include one or two different viscosity
items. Remark 8 shows that the conclusions of Ceng
et al. are special cases of this paper.

(3) The viscosity item is changed from a contractive
mapping \( f \) to weakly contractive mapping \( g \) in
Theorem 12.

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