Regularity Result for Quasilinear Elliptic Systems with Super Quadratic Natural Growth Condition

Shuhong Chen¹ and Zhong Tan²

¹ Department of Mathematics and Information Science, Zhangzhou Normal University, Zhangzhou, Fujian 363000, China
² School of Mathematical Science, Xiamen University, Xiamen, Fujian 361005, China

Correspondence should be addressed to Shuhong Chen; shiny0320@163.com

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1. Introduction

This paper considers boundary regularity for weak solutions of quasilinear elliptic systems

\[-D_u \left( A_{ij}^{\alpha \beta} (x, u) D_{\alpha \beta} u^j \right) = B_i (x, u, D_u), \quad x \in \Omega, \quad (1)\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with boundary of class \( C^1 \), \( n \geq 2 \) and \( u \) takes value in \( \mathbb{R}^N \), \( N > 1 \). Each \( A_{ij}^{\alpha \beta} \) maps \( \Omega \times \mathbb{R}^N \) into \( \mathbb{R} \), and each \( B_i \) maps \( \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \) into \( \mathbb{R} \). A partial regularity theory of (1) must have a priori existence weak solutions. Here we assume that weak solutions exist and consider partial regularity of weak solutions directly. We further impose certain structural conditions on \( A_{ij}^{\alpha \beta} \) and \( B_i \) with \( m > 2 \) as follows.

(H1) There exists \( L > 0 \) such that

\[ A_{ij}^{\alpha \beta} (x, \xi) (v, \bar{v}) \leq L \left( 1 + |\xi|^{2} \right)^{(m-2)/2} |v| |\bar{v}| \]

for all \( (x, \xi) \in \overline{\Omega} \times \mathbb{R}^N, \ v, \bar{v} \in \mathbb{R}^{nN}. \) (2)

(H2) \( A_{ij}^{\alpha \beta} (x, \xi) \) is uniformly strongly elliptic; that is, for some \( \lambda > 0 \) we have

\[ A_{ij}^{\alpha \beta} (x, \xi) (v, v) \geq \lambda \left( 1 + |\xi|^{2} \right)^{(m-2)/2} |v|^2 \]

for all \( (x, \xi) \in \overline{\Omega} \times \mathbb{R}^N, \ v \in \mathbb{R}^{nN}. \) (3)

(H3) Assume that \( A_{ij}^{\alpha \beta} \in C^\infty (\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^{nN}) \) and further that \( A_{ij}^{\alpha \beta} \) is uniformly continuous on sets of the form \( \overline{\Omega} \times \{ \xi : |\xi| \leq M \} \), for any fixed \( M, 0 < M < \infty \).

(H4) (Natural growth condition). There exist constants \( a \) and \( b \), with \( a \) possibly depending on \( M > 0 \), such that

\[ |B_i (x, \xi, v)| \leq a (M) |v|^m + b \]

for all \( x \in \overline{\Omega}, \xi \in \mathbb{R}^N \) with \( |\xi| \leq M \) and \( v \in \mathbb{R}^{nN}. \) (4)

Further hypothesis (H3) deduces, writing \( \omega (\cdot) \) for \( \omega (M, \cdot) \), the existence of a monotone nondecreasing concave function \( \omega : [0, \infty) \rightarrow [0, \infty) \) with \( \omega (0) = 0 \), continuous at 0, such that

\[ |A_{ij}^{\alpha \beta} (x, u) - A_{ij}^{\alpha \beta} (y, v)| \leq \omega \left( |x - y|^m + |u - v|^m \right), \]

for all \( x, y \in \overline{\Omega}, u, v \in \mathbb{R}^N \) with \( |u|, |v| \leq M \) [1].
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There exist $s$ with $s > n$ and a function $g \in H^{1,s}(\Omega, R^N)$, such that

$$u|_{\partial \Omega} = g|_{\partial \Omega}. \tag{6}$$

Note that we trivially have $g \in H^{1,2}(\Omega, R^N)$. Further, by the Sobolev embedding theorem we have $g \in C^{\alpha,\beta}(\Omega)$ for any $\alpha \in (0,1)$ and $\beta \in (0,1)$. If $g|_{\partial \Omega} \equiv 0$, we will take $g \equiv 0$ on $\Omega$.

If the domain we consider is an upper half unit ball $B^*$, the boundary condition becomes as follows.

$$u|_{\partial \Omega} = g|_{\partial \Omega} \tag{7}$$

Here we write $B_\rho(x_0) = \{ x \in R^n : |x - x_0| < \rho \}$, and further $B_\rho = B_\rho(0), B = B_1$. Similarly we denote upper half balls as follows: for $x_0 \in R^{n-1} \times \{0\}$, we write $B^*_\rho(x_0)$ for $x_0 \in R^N : |x_0| < \rho$ and set $B^*_0 = B^*_\rho(0), B^* = B^*_1$. For $x_0 \in R^{n-1} \times \{0\}$ we further write $D_\rho(x_0)$ for $x_0 \in R^n : x_n = 0, |x_0| < \rho$ and set $D_\rho = D_\rho(0), D = D_1$.

**Definition 1.** By a weak solution of (1) one means a vector valued function $u \in W^{1,m}(\Omega, R^N)$ and $L^\infty(\Omega, R^N)$ such that

$$\int_\Omega A_{ij}^{ab}(x,u)(D_j u^i, D_{ab} \phi^j) dx = \int_\Omega B_i(x,u,Du) \cdot \phi^i dx \tag{8}$$

holds for all test-functions $\phi \in C_0^\infty(\Omega, R^N)$ and, by approximation, for all $\phi \in W^{1,m}_0(\Omega, R^N) \cap L^\infty(\Omega, R^N)$.

Under such assumptions, even the boundary data is smooth, one cannot expect full regularity of (1) at the boundary [2]. Then, our goal is to establish partial boundary regularity.

After the partial regularity results of the type in this paper were proved by Giusti and Miranda in [3], there are some previous partial regularity results for quasilinear systems. For example, regularity up to boundary for nonlinear and quasilinear systems [4–6] has been studied by Arkhipova. Wieger [7] established boundary regularity for systems in diagonal form first, and the proof was generalized and extended by Hildebrandt and Widman [8]. Jost and Meier [9] deduced full regularity in a neighborhood of the boundary for minima of functionals with the form $\int_\Omega A(x,u)|Du|^2 dx$. Furthermore, Duzaar et al. obtained the boundary Hausdorff dimension on the singular sets of solutions to even more general systems in [10, 11] recently. Further discussion for regularity theory can be seen in [12, 13] and their references.

Inspired by [14], in this paper, we would establish boundary regularity for quasilinear systems under natural growth condition by the method of A-harmonic approximation.

The technique of A-harmonic approximation [15–17] is a natural extension of the harmonic approximation technique, which originated from Simon’s proof of Allard’s [18] $\epsilon$-regularity theorem. In this context, using the A-harmonic approximation technique, we obtain the following regularity results.

**Theorem 2.** Consider a bounded domain $\Omega$ in $R^N$, with boundary of class $C^1$. Let $u$ be a bounded weak solution of (1) satisfying the boundary condition (H5), and $\|u\|_{L^\infty(\Omega)} \leq M < \infty$ with $\omega(2M)M < \lambda$, where the structure conditions (H1)–(H3) hold for $A_{ij}^{ab}$ and (H4) holds for $B_i$. Consider a fixed $\gamma \in (0, \sigma]$. Then there exist positive $R_0$ and $\varepsilon_0$ (depending only on $n, \lambda, L, b, M, \omega(\cdot), m, and \gamma$) with the property that

$$\int_{B_\varepsilon(x_0)} |u - u^\ast|_R^2 dx + \|g\|_{L^2(\Omega)}^2 + R^2 \leq \varepsilon_0^2 \tag{9}$$

for some $R \in (0, R_0]$ for a given $x_0 \in \partial \Omega$ implies $u \in C^{\gamma,\sigma}(\overline{B_{R/2}(x_0)} \cap \Omega, R^N)$.

Note in particular that the boundary condition (H5) means that $u^\ast_{x_0,R}$ makes sense: in fact, we have $u^\ast_{x_0,R} = g_{x_0,R}$. For $\nu \in L^1(\Omega), x_0 \in \partial \Omega$, we set $\nu^\ast_{x_0,R} = \nu^\ast_{x_0,R} \nu H^{n-1}$.

In particular, for $\nu \in L^1(D_\rho(x_0))$, $x_0 \in D$, we write $\nu^\ast_{x_0,R} = \int_{D_\rho(x_0)} \nu H^{n-1}$.

Combining this result with the analogous interior [19] and a standard covering argument allows us to obtain the following bound on the size of the singular set.

**Corollary 3.** Under the assumptions of Theorem 2 the singular set of the weak solution $u$ has $(n - 2)$-dimensional Hausdorff measure zero in $\Omega$.

If the domain of the main step in proving Theorem 2 is a half ball, the result then is the following.

**Theorem 4.** Consider a bounded weak solution of (1) on the upper half unit ball $B^*$ which satisfies the boundary condition (H5) and $\|u\|_{L^\infty(\Omega)} \leq M < \infty$ with $\omega(2M)M < \lambda$, where the structure conditions (H1)–(H3) hold for $A_{ij}^{ab}$ and (H4) holds for $B_i$. Then there exist positive $R_0$ and $\varepsilon_0$ (depending only on $n, \lambda, L, b, M, \omega(\cdot), m, and \gamma$) with the property that

$$\int_{B^*_\varepsilon(x_0)} |u - u^\ast|_R^2 dx + \|g\|_{L^2(\Omega)}^2 + R^2 \leq \varepsilon_0^2 \tag{10}$$

for some $R \in (0, R_0]$ for a given $x_0 \in D$, implies that there holds: $u \in C^{\gamma,\sigma}(\overline{B_{R/2}(x_0)}, R^N)$.

Note that analogous to the above, the boundary condition (H5) ensures that $u^\ast_{x_0,R}$ exists, and we have indeed $u^\ast_{x_0,R} = g_{x_0,R}$.

2. The A-Harmonic Approximation Technique

In this section we present the A-harmonic approximation lemma [14] and some standard results due to Companato [20].
Lemma 5 (A-harmonic approximation lemma). Consider fixed positive \( \lambda \) and \( L \), and \( n, N \in \mathbb{N} \) with \( n \geq 2 \). Then for any given \( \varepsilon > 0 \) there exists \( \delta = \delta(n, N, \lambda, L, \varepsilon) \in (0, 1] \) with the following property: for any \( A \in \text{Bil}(R^{nN}) \) satisfying

\[
A(\nu, \nu) \geq \lambda |\nu|^2 \quad \text{for all} \quad \nu \in R^{nN},
\]

\[
|A(\nu, \nu)| \leq L |\nu| |\nu| \quad \text{for all} \quad \nu, \overline{\nu} \in R^{nN}
\]

for any \( \nu \in H^{1,2}(B^*_p(x_0), R^N) \) (for some \( p > 0 \), \( x_0 \in R^n \)) satisfying

\[
\rho^{2-n} \int_{B^*_p(x_0)} |D\nu|^2 \, dx \leq 1,
\]

\[
\rho^{2-n} \int_{B^*_p(x_0)} A(D\nu, D\varphi) \, dx \leq \delta \sup_{B^*_p(x_0)} |D\varphi|,
\]

\[
\nu|_{\partial B^*_p(x_0)} = 0
\]

for all \( \varphi \in C^1_c(\overline{B^*_p(x_0)}, R^N) \), there exists an A-harmonic function

\[
v \in \overline{H} = \left\{ \overline{w} \in H^{1,2}(\overline{B^*_p(x_0)}, R^N) \right\}
\]

\[
\rho^{2-n} \int_{B^*_p(x_0)} |D\overline{w}|^2 \, dx \leq 1, \quad \overline{w} \big|_{\partial B^*_p(x_0)} \equiv 0
\]

with

\[
\rho^{-n} \int_{B^*_p(x_0)} |\nu - w|^2 \, dx \leq \varepsilon.
\]

We close this section by a standard estimate for the solutions to homogeneous second-order elliptic systems with constant coefficients [20].

Lemma 7. Consider fixed positive \( \lambda \) and \( L \), and \( n, N \in \mathbb{N} \) with \( n \geq 2 \). Then there exists \( C_0 \) depending only on \( n, L, \lambda, \) and \( L \) (without loss of generality we take \( C_0 \geq 1 \)) such that, for \( A \in \text{Bil}(R^{nN}) \) satisfying (11), any A-harmonic function \( h \) on \( B^*_p(x_0) \) with \( h|_{\partial B^*_p(x_0)} \equiv 0 \) satisfies

\[
\rho^{2-n} \sup_{B^*_p(x_0)} |Dh|^2 \leq C_0 \rho^{2-n} \int_{B^*_p(x_0)} |Dh|^2 \, dx.
\]

3. The Caccioppoli Inequality

In this section we would prove a suitable Caccioppoli inequality. First of all we recall two useful inequalities. The first is the Sobolev embedding theorem which yields the existence of a constant \( C_0 \) depending only on \( s, n, \) and \( N \) such that for \( x_0 \in D, \rho = 1 - |x_0| \) there holds

\[
\sup_{B^*_p(x_0)} \|g - g|_{x_0, \rho}|^s \leq C_p \rho^{1-(n/s)} \|g\|_{H^{1,2}(B^*_p(x_0), R^N)}.
\]

Obviously, the inequality remains true if we replace \( \|g\|_{H^{1,2}(B^*_p(x_0), R^N)} \) by \( \|g\|_{H^{1,2}(B^*_p, R^N)} \), which we will henceforth abbreviate simply as \( \|g\|_{H^{1,2}} \).

Next we note that the Poincaré inequality in this setting for \( x_0 \in D, \rho = 1 - |x_0| \) yields

\[
\int_{B^*_p(x_0)} |g - g|_{x_0, \rho}|^m \, dx \leq C_p \rho^m \int_{B^*_p(x_0)} |Dg|^m \, dx,
\]

for a constant \( C_p \) which depends only on \( n \).

Finally, we fix an exponent \( \sigma \in (0, 1) \) as follows: if \( g = 0 \), \( \sigma \) can be chosen arbitrarily (but henceforth fixed); otherwise we take \( \sigma \) fixed in \((0, 1 - (1/s))\).

Then we establish an appropriate inequality for Caccioppoli.

Theorem 8 (Caccioppoli’s inequality). Let \( u \in W^{1,4}(\Omega, R^N) \cap L^{\infty}(\overline{\Omega}, R^N) \) with \( \|u\|_{L^{\infty}} \leq M < \infty \) and \( 2a(M)M < \lambda \) be a weak solution of systems (1) under assumption conditions (H1)–(H5). Then there exists \( \rho_0(L, M, a(M), s, \|g\|_{H^{1,2}} > 0 \) such that, for all \( B^*_p(x_0) \subset B^*, \) with \( x_0 \in D^*, 0 < \rho < R < \rho_0 \), there holds

\[
\int_{B^*_p(x_0)} |Du|^2 \, dx \leq C_1 \int_{B^*_p(x_0)} \frac{|u(x) - u(x_{x_0,R})|^2}{\rho^2} \, dx + C_2 \alpha |\rho|^\alpha
\]

\[
+ C_3 (\alpha |\rho|^\alpha)^{-1/(1/2)} \|g\|_{L^\infty(B^*, R^N)}^2,
\]

where \( C_1 \) depends only on \( \lambda, L, M \), and \( C_3 \) depends on these quantities, and in addition to \( C_p, C_2 \) depends on \( \lambda, L, M, a, b, \) and \( \|g\|_{L^{\infty}(B^*, R^N)} \).
Proof. Consider a cutoff function \( \eta \in C_0^\infty(B^\infty_{\rho/2}(x_0)) \), satisfying \( 0 \leq \eta \leq 1, \eta \equiv 0 \) on \( B^\infty_{\rho/2}(x_0) \) and \( |\nabla \eta| < 4/\rho \). Then the function \( (u - g)\eta^2 \) is in \( W^{1,\infty}_0(B^\infty_{\rho/2}(x_0, R^N)) \) and thus can be taken as a test-function.

Using (H1), (H4), (H5), and Young's inequality and noting that \( 2a(M)M < \lambda \), we can get from (8) with \( \varepsilon \) positive but arbitrary (to be fixed later)

\[
\int_{B^\infty_{\rho}(x_0)} A_{ij}^{ab}(\cdot,u)(D^a u', D^b u') \eta^2 \ dx
\]

\[
\leq L \int_{B^\infty_{\rho}(x_0)} \left( 1 + |u|^2 \right)^{(m-2)/2} |Dg| |Du| \eta^2 \ dx
\]

\[
+ 2L \int_{B^\infty_{\rho}(x_0)} \left( 1 + |u|^2 \right)^{(m-2)/2} |D\eta| |Du| |u - g| \ dx
\]

\[
+ a \int_{B^\infty_{\rho}(x_0)} |Du|^m |u - g| \eta^2 \ dx + b \int_{B^\infty_{\rho}(x_0)} |u| \eta^2 \ dx
\]

\[
\leq \varepsilon \left( m^2 \int_{B^\infty_{\rho}(x_0)} \left( 1 + |u|^2 \right)^{(m-2)/2} |Du|^2 \eta^2 \ dx
\]

\[
+ a \sup_{B^\infty_{\rho}(x_0)} |u - u_{x_0}| \int_{B^\infty_{\rho}(x_0)} |Du|^m \eta^2 \ dx
\]

\[
+ a \sup_{B^\infty_{\rho}(x_0)} |g - g_{x_0}| \int_{B^\infty_{\rho}(x_0)} |Du|^m \eta^2 \ dx
\]

\[
+ \frac{L^2}{2\varepsilon} \int_{B^\infty_{\rho}(x_0)} \left( 1 + |u|^2 \right)^{(m-2)/2} |Dg|^2 \eta^2 \ dx
\]

\[
+ \frac{4L^2}{\varepsilon} \int_{B^\infty_{\rho}(x_0)} \left( 1 + |u|^2 \right)^{(m-2)/2} |D\eta|^2 |u - u_{x_0}|^2 \ dx
\]

\[
+ \frac{4L^2}{\varepsilon} \int_{B^\infty_{\rho}(x_0)} \left( 1 + |u|^2 \right)^{(m-2)/2} |D\eta|^2 |g - g_{x_0}|^2 \ dx
\]

\[
+ \frac{\varepsilon b^2}{2} \int_{B^\infty_{\rho}(x_0)} \rho^2 \eta^2 d\rho + \frac{1}{\varepsilon^2 \rho^2} \int_{B^\infty_{\rho}(x_0)} |u - u_{x_0}|^2 \ dx
\]

\[
\leq \varepsilon \int_{B^\infty_{\rho}(x_0)} \left( 1 + |u|^2 \right)^{(m-2)/2} |Du|^2 \eta^2 \ dx
\]

\[
+ a (M + \|g\|_{L^\infty(B^\infty_{\rho})}) \int_{B^\infty_{\rho}(x_0)} |Du|^m \eta^2 \ dx
\]

\[
+ \frac{4L^2}{\varepsilon} + \frac{1}{\varepsilon^2} \int_{B^\infty_{\rho}(x_0)} \left( 1 + |u|^2 \right)^{(m-2)/2} \frac{1}{\rho^2} |u - u_{x_0}|^2 \ dx
\]

\[
+ \frac{\varepsilon b^2}{4} \eta^2 \alpha_n \rho^{n+2}
\]

\[
+ \left( \frac{L^2}{2\varepsilon} + \frac{64L^2 C_p}{2\varepsilon} + \frac{4C_p}{\varepsilon} \right)
\]

\[
\leq \varepsilon \int_{B^\infty_{\rho}(x_0)} \left( 1 + |u|^2 \right)^{(m-2)/2} |Dg|^2 \eta^2 \ dx
\]

\[
+ \frac{4L^2}{\varepsilon} + \frac{1}{\varepsilon^2} \int_{B^\infty_{\rho}(x_0)} \left( 1 + |u|^2 \right)^{(m-2)/2} \frac{1}{\rho^2} |u - u_{x_0}|^2 \ dx
\]

\[
+ \frac{\varepsilon b^2}{4} \eta^2 \alpha_n \rho^{n+2}
\]

\[
+ \left( \frac{L^2}{2\varepsilon} + \frac{64L^2 C_p}{2\varepsilon} + \frac{4C_p}{\varepsilon} \right)
\]

\[
+ a (M + \|g\|_{L^\infty(B^\infty_{\rho})}) \int_{B^\infty_{\rho}(x_0)} |Du|^m \eta^2 \ dx
\]

\[
+ \frac{4L^2}{\varepsilon} + \frac{1}{\varepsilon^2} \int_{B^\infty_{\rho}(x_0)} \left( 1 + |u|^2 \right)^{(m-2)/2} \frac{1}{\rho^2} |u - u_{x_0}|^2 \ dx
\]

\[
+ \frac{\varepsilon b^2}{4} \eta^2 \alpha_n \rho^{n+2}
\]

\[
+ \left( \frac{L^2}{2\varepsilon} + \frac{64L^2 C_p}{2\varepsilon} + \frac{4C_p}{\varepsilon} \right)
\]
Lemma 9. Consider \( u \in W^{1,m}(\Omega, R^N) \cap L^{\infty}(\Omega, R^N) \) to be a weak solution of (1), \( x_0 \in D \) and \( y \in D \), \( \rho \in C^{0,\sigma}(B_{\varepsilon}(y), R^N) \) with \( \sup_{B_{\varepsilon}(y)} |D\varphi| \leq 1 \). We have

\[
\left(\frac{\rho}{2}\right)^{2-n} \int_{B_{\varepsilon}(y)} A^{ij}_{y}(y, u_y^{\rho})(D_{\rho}u^{ij}, D_{\rho}\varphi) \, dx \\
\leq C_4 \sqrt{I}\left(\sqrt{I} + \omega(I)\right) \rho \sup_{B_{\varepsilon}(x_0)} |D\varphi|.
\]

(24)

Here and hereafter, we define

\[
I(z, r) = \int_{B_r(z)} \left|u - u_x^{\rho}\right|^2 \, dx + \left\|g\right\|_{H^{1,2}B_r(z)}^{2(1-(n/2))} + r^2, \quad (25)
\]

for \( z \in D, \ r \in (0, 1 - |z|). \)

Proof. Using (8) we have

\[
\int_{B_{\varepsilon}(y)} A^{ij}_{y}(y, u_y^{\rho})(D_{\rho}u^{ij}, D_{\rho}\varphi) \, dx \\
\leq \left[ a \int_{B_{\varepsilon}(y)} \left|Du\right|^m \, dx + 2^{-n-1} \alpha_n b \rho^n \right] \cdot \rho \sup_{B_{\varepsilon}(x_0)} |D\varphi| \\
+ \int_{B_{\varepsilon}(y)} \left|A^{ij}_{y}(y, u_y^{\rho}) - A^{ij}_{y}(x, u)\right| \cdot |Du| \, dx \sup_{B_{\varepsilon}(y)} |D\varphi|.
\]

(26)

Applying in turn Young’s inequality, (H3), the Caccioppoli inequality (Theorem 8), and Jensen’s inequality, we calculate from (26)

\[
\int_{B_{\varepsilon}(y)} A^{ij}_{y}(y, u_y^{\rho})(D_{\rho}u^{ij}, D_{\rho}\varphi) \, dx \\
\leq \left[ a \int_{B_{\varepsilon}(y)} \left|Du\right|^m \, dx + 2^{-n-1} \alpha_n b \rho^n \right] \cdot \rho \\
+ \left[ \int_{B_{\varepsilon}(y)} \left|A^{ij}_{y}(y, u_y^{\rho}) - A^{ij}_{y}(x, u)\right|^2 \, dx \right]^{1/2} \cdot \left[ \int_{B_{\varepsilon}(y)} |Du|^2 \, dx \right]^{1/2}.
\]

(30)

for \( C_4 \) defined by \( C_4 = 2^{n-3} \alpha_n C_7. \)

Lemma 10. Consider \( u \) satisfying the conditions of Theorem 2 and \( \sigma \) fixed; then we can find \( \delta \) and \( s_0 \) together, with positive constants \( C_8 \) such that the smallness conditions: \( 0 < \omega(s_0) \leq \delta/2 \) and \( I(x_0, R) \leq C_8 \min\{\delta^2/4, s_0\} \), together, imply the growth condition

\[
I(y, \rho) \leq \theta^2 I(y, \rho).
\]

(31)
Proof. We now set $v = u - g$, using in turn (H1), Young’s inequality, and Hölder’s inequality. We have from (30)

$$
\left(\frac{\rho}{2}\right)^{2-n}\int_{B_{\rho u}(y)} A^{\alpha \beta}_{ij}(y, u_{j, \rho}, (D\rho w^j, D\rho \phi^i)) dx \\
\leq \left(\frac{\rho}{2}\right)^{2-n}\int_{B_{\rho u}(y)} A^{\alpha \beta}_{ij}(y, u_{j, \rho}, (D\rho u^j, D\rho \phi^i)) dx \\
+ \left(\frac{\rho}{2}\right)^{2-n}\int_{B_{\rho u}(y)} A^{\alpha \beta}_{ij}(y, u_{j, \rho}, (D\rho g^j, D\rho \phi^i)) dx \\
\leq C_9 \sqrt{I} \left(\sqrt{I} + \omega(I)\right) \rho \sup_{B_{\rho u}(x)} |D\rho^i|,
$$

(32)

for $C_9 = \max \{C_4, (\alpha \rho^2) (1 - \eta /n)\}$.

We now set $v = w / \gamma$, for $\gamma = C_9 \sqrt{I}$. From (32) we then have

$$
\left(\frac{\rho}{2}\right)^{2-n}\int_{B_{\rho u}(y)} A^{\alpha \beta}_{ij}(y, u_{j, \rho}, (D\rho v^j, D\rho \phi^i)) dx \\
\leq \left(\sqrt{I} + \omega(I)\right) \rho \sup_{B_{\rho u}(x)} |D\phi|,
$$

(33)

and from (32) we observe from the definition of $C_9$ (recalling also the definition of $\gamma$)

$$
\left(\frac{\rho}{2}\right)^{2-n}\int_{B_{\rho u}(y)} |Dv^2| dx < 1.
$$

(34)

Further we note

$$
v \big|_{D_{\rho u}(y)} = \frac{1}{\gamma} w \big|_{D_{\rho u}(y)} = \frac{1}{\gamma} (u - g) \big|_{D_{\rho u}(y)} = 0.
$$

(35)

For $\varepsilon > 0$ we take $\delta = \delta(n, N, \lambda, L, \varepsilon)$ to be the corresponding $\delta$ from the A-harmonic approximation lemma. Suppose that we could ensure that the smallness condition

$$
\sqrt{I} + \omega(I) \leq \delta
$$

(36)

holds. Then in view of (33), (34), and (35) we would be able to apply Lemma 5 to conclude the existence of a function $h \in H^{1,2}(B_{\rho / 2}(y), R^N)$ which is $A^{\alpha \beta}_{ij}(y, u_{j, \rho}, \rho^i)$-harmonic, with $h \big|_{D_{\rho / 2}(y)} = 0$ such that

$$
\left(\frac{\rho}{2}\right)^{2-n}\int_{B_{\rho u}(y)} |Dh|^2 dx \leq 1,
$$

(37)

$$
\left(\frac{\rho}{2}\right)^{2-n}\int_{B_{\rho u}(y)} |v - h|^2 dx \leq \varepsilon.
$$

(38)

For $\theta \in (0, 1/4]$ arbitrary (to be fixed later), we have from the Campanato theorem, noting (37) and recalling also that $h(y) = 0$, $w = u - g$

$$
\left(\frac{\rho}{2}\right)^{2-n}\int_{B_{\rho u}(y)} |v|^2 dx \leq \theta^2 \rho^2 \sup_{B_{\rho u}(x)} |D\rho^i| \\
\leq 4C_9 \theta^2.
$$

(39)

Using (38) and (39) we observe

$$
(\theta \rho)^{-n}\int_{B_{\rho u}(y)} |v|^2 dx \\
\leq 2(\theta \rho)^{-n} \left[\int_{B_{\rho u}(y)} |v - h|^2 dx + \int_{B_{\rho u}(y)} |h|^2 dx\right] \\
\leq 2(\theta \rho)^{-n} \left[\left(\frac{\rho}{2}\right)^{2-n} (\varepsilon + 1) \alpha_n (\theta \rho)^n \sup_{B_{\rho u}(x)} |h|^2\right] \\
\leq 2\theta^2 \nu^{-n} \varepsilon + 4\alpha_n C_9 \theta^2,
$$

(40)

and, hence, on multiplying this through by $\theta^2$, we obtain the estimate

$$
(\theta \rho)^{-n}\int_{B_{\rho u}(y)} |w|^2 dx \leq C_9 \left(2\theta^2 \nu^{-n} \varepsilon + 4\alpha_n C_9 \theta^2\right) I.
$$

(41)

For the time being, we restrict to the case that $g$ does not vanish identically. Recalling that $v = u - g$, using in turn Poincaré’s, Sobolev’s, and then Hölder’s inequalities, and noting also that $u_{j, \rho} = g_{j, \rho}$, thus from (41) we get

$$
(\theta \rho)^{-n}\int_{B_{\rho u}(y)} |u - u_{j, \rho}|^2 dx \\\n\leq 2(\theta \rho)^{-n} \left[\int_{B_{\rho u}(y)} |u - g|^2 dx + \int_{B_{\rho u}(y)} |g - g_{j, \rho}|^2 dx\right] \\
\leq 2C^2 \left(2\theta^2 \nu^{-n} \varepsilon + 4\alpha_n C_9 \theta^2\right) I \\
+ 2(\theta \rho)^{-n} \left[\alpha_n (\theta \rho)^n \right]^{-\nu^{-n}} \left|\alpha_n (\theta \rho)^n \right|^{-\nu^{-n}} I \\
\leq C_{10} \left(\theta^2 \nu^{-n} + \theta^2\right) I + C_{10} \theta^2\nu^{-n} I,
$$

(42)

for $C_{10} = \max \{8\alpha_n C_9 C_{\alpha_n}^2, 2^{1/2} C_{\alpha_n} \alpha_n^{-1}(\nu^{-n})\}$, and provided $\varepsilon = \theta^2 \nu^{-n}$, we have

$$
(\theta \rho)^{-n}\int_{B_{\rho u}(y)} |u - u_{j, \rho}|^2 dx \leq 3C_{10} \theta^2\nu^{-n} I.
$$

(43)

Note that fix $\varepsilon = \theta^2 \nu^{-n}$, which is also fixed $\delta$. Since $\rho \leq 1$, we see from the definition of $I$

$$
\|g\|^2_{L^2(\theta \rho)} \leq \theta^2 I,
$$

(44)

and further

$$
(\theta \rho)^2 \theta \rho \leq \theta^2 I.
$$

(45)

Combining these estimates with (43), we can get

$$
I(\theta \rho) \leq 3(C_{10} + 1) \theta^2\nu^{-n} I.
$$

(46)

Choose $\theta \in (0, 1/4]$ sufficiently small that there holds: $3(C_{10} + 1) \theta^2\nu^{-n} I \leq \theta^2 \nu^{-n}$. 

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We can see from (46)
\[
I(y, \theta \rho) \leq \theta^{2\alpha} I.
\] (47)
We now choose \( s_0 > 0 \) such that \( 0 < \omega(s_0) < (\delta/2) \) and define \( C_8 \) by
\[
C_8 = \max \left\{ 2^{n-1}, 2C_9^2 + 1, 2C_9^2 + 1 \right\}.
\] (48)
Suppose that we have
\[
I(x_0, R) \leq C_8^{-1} \min \left\{ \frac{\delta^2}{4}, s_0 \right\},
\] (49)
for some \( R \in (0, R_0) \), where \( R_0 = \min \{ \sqrt{2s_0}, 1-|x_0| \} \).

For any \( y \in D_{R/2}(x_0) \) we use the Sobolev inequality to calculate
\[
\frac{\alpha_n R^n}{2\pi^{n-1}} |u'_{x,R} - u'_{y,R/2}|^2 \\
= \int_{B_{R/2}(y)} |u'_{x,R} - u'_{y,R/2}|^2 \, dx \\
= \int_{B_{R/2}(y)} |g_{x,R} - g_{y,R/2}|^2 \, dx \\
\leq 2 \int_{B_{R/2}(y)} |g - g'_{y,R/2}|^2 \, dx + 2 \int_{B_{R/2}(y)} |g - g'_{y,R/2}|^2 \, dx \\
\leq 2\alpha_n C_9^2 \|g\|_{H^1,R}^2 R^{n(1-(n/2))}.
\] (50)

Then we can calculate
\[
I\left(y, \frac{1}{2} R\right) \\
\leq 2^{n-1} \int_{B_{R/2}(y)} |u - u'_{y,R/2}|^2 \, dx \\
+ (2C_9^2 + 1) \|g\|_{H^1,R}^2 R^{n(1-(n/2))} + \frac{1}{4} R^2 \\
\leq C_8 I(x_0, R).
\] (51)

Then we have
\[
\sqrt{I\left(y, \frac{1}{2} R\right)} + \omega\left(I\left(y, \frac{1}{2} R\right)\right) \\
\leq C_8 I(x_0, R) + \sqrt{\omega(C_8 I(x_0, R))} \\
\leq \frac{1}{2} \delta + \omega(s_0) \leq \delta,
\] (52)
which means that the condition (49) is sufficient to guarantee the smallness condition (37) for \( \rho = R/2 \), for all \( y \in D_{R/2}(x_0) \).

We can thus conclude that (46) holds in this situation. From (46) we thus have
\[
\sqrt{I\left(y, \frac{\theta \rho}{2}\right)} + \sqrt{\omega\left(I\left(y, \frac{\theta \rho}{2}\right)\right)} \\
\leq \sqrt{I\left(y, \frac{1}{2} R\right)} + \sqrt{\omega\left(I\left(y, \frac{1}{2} R\right)\right)} \leq \delta,
\] (53)
meaning that we can apply (46) on \( B_{\theta \rho/2}(y) \) as well, yielding
\[
I\left(y, \frac{\theta^2 R}{2}\right) \leq \theta^{2\alpha} I\left(y, \frac{R}{2}\right),
\] (54)
and inductively
\[
I\left(y, \frac{\theta^k R}{2}\right) \leq \theta^{2\alpha k} I\left(y, \frac{R}{2}\right).
\] (55)

The next step is to go from a discrete to a continuous version of the decay estimate. Given \( \rho \in (0, R/2] \), we can find \( k \in N_0 \) such that \( \theta^{k+1} R/2 < \rho \leq \theta^k R/2 \). Firstly we use the Sobolev inequality, to see
\[
\int_{B_{\rho/2}(y)} |u_{\rho} - u_{y,\theta\rho R/2}|^2 \, dx \\
\leq 2\alpha_n \left(\frac{1}{2\theta^k R}\right)^n C_9^2 \|g\|_{H^1,R}^2 \left(\frac{1}{2\theta^k R}\right)^{2(1-(n/2))},
\] (56)
which allows us to deduce
\[
\int_{B_{\rho/2}(y)} |u - u_{\rho,y}|^2 \, dx \\
\leq 2 \int_{B_{\rho/2}(y)} |u - u_{y,\theta\rho R/2}|^2 \, dx \\
+ 4\alpha_n \left(\frac{1}{2\theta^k R}\right)^n C_9^2 \|g\|_{H^1,R}^2 \left(\frac{1}{2\theta^k R}\right)^{2(1-(n/2))},
\] (57)
and, hence,
\[
I\left(y, \rho\right) \leq C_{11} I\left(y, \frac{\theta^k R}{2}\right),
\] (58)
for \( C_{11} = 8\theta^{-n} C_9^2 + 1 \). Combining this with (55) and (51), we have
\[
I\left(y, \rho\right) \\
\leq C_{11}^{2\alpha k} I\left(y, \frac{R}{2}\right) \leq C_8 C_{11}^{2\alpha} \left(\frac{2\rho}{R}\right)^{2\alpha} I\left(x_0, R\right),
\] (59)
and more particularly
\[
\inf_{\mu \in R^N} \int_{B_{\mu/2}(y)} |u - \mu|^2 \, dx \leq C_{12} I\left(x_0, R\right) \left(\frac{\rho}{R}\right)^{2\alpha},
\] (60)
for \( C_{12} = C_8 C_{11}(2/\theta)^{2\alpha} \). Recall that this estimate is valid for all \( y \in D \) and \( \rho \) with \( D_{\rho/2}(y) \subseteq D_{R/2}(x_0) \); assume only the condition (49) on \( I(x_0, R) \). This yields after replacing \( R \) with \( 6R \) the boundary estimate (13) which requires to apply Lemma 6.

Combining the boundary and interior estimates [19] we can derive the desired result. As the argument for combining the boundary and interior regularity results is relatively standard, we omit it. Hence we can apply Lemma 6 and conclude the desired Hölder continuity.
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References
