Research Article

On the Largest Disc Mapped by Sum of Convex and Starlike Functions

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For a normalized analytic function f defined on the unit disc $\mathbb{D}$, let $\phi(f, f', f''; z)$ be a function of positive real part in $\mathbb{D}$, $\psi(f, f', f''; z)$ need not have that property in $\mathbb{D}$, and $\chi = \phi + \psi$. For certain choices of ϕ and ψ , a sharp radius constant ρ is determined, $0 < \rho < 1$, so that $\chi(\rho z)/\rho$ maps $\mathbb{D}$ onto a specified region in the right half-plane.

1. Introduction

Let $\mathcal{A}$ be the class of functions f analytic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}$ be its subclass consisting of univalent functions. For two analytic functions f and g , the function f is subordinate to g , written $f(z) \prec g(z)$, if there is an analytic self-map $w : \mathbb{D} \rightarrow \mathbb{D}$ with $w(0) = 0$ satisfying $f(z) = g(w(z))$. Given an analytic function p with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$, denote by $\mathcal{ST}(p)$ and $\mathcal{CV}(p)$ the subclasses of $\mathcal{A}$ consisting, respectively, of f satisfying $zf'(z)/f(z) \prec p(z)$ and $1 + zf''(z)/f'(z) \prec p(z)$.

For various choices of p , these classes reduce to well-known subclasses of starlike and convex functions. For instance, with $p(z) = (1 + (1 - 2\alpha)z)/(1 - z)$, $0 \leq \alpha < 1$, then $\mathcal{ST}(\alpha)$ and $\mathcal{CV}(\alpha)$ are, respectively, the subclasses consisting of *starlike functions of order α* and *convex functions of order α* . The classes $\mathcal{ST} := \mathcal{ST}(0)$ and $\mathcal{CV} := \mathcal{CV}(0)$ are the familiar subclasses of $\mathcal{S}$ of starlike and convex functions. For $p(z) = (1 + (1 - 2\beta)z)/(1 - z)$, $\beta > 1$, $\mathcal{M}(\beta) = \mathcal{ST}(p)$ is the class of functions $f \in \mathcal{A}$ satisfying

$$\mathcal{M}(\beta) := \left\{ f \in \mathcal{A} : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) < \beta \quad (z \in \mathbb{D}) \right\} \quad (1)$$

studied by Uralegaddi et al. [1]. Various subclasses of $\mathcal{M}(\beta)$ have been investigated in [2–5]. For $p(z) =$

$((1+z)/(1-z))^\alpha$, $0 < \alpha \leq 1$, the class $\mathcal{SSST}(\gamma) := \mathcal{ST}(p)$ is the class of *strongly starlike functions of order γ* . The class $\mathcal{S}_{\mathcal{ST}} := \mathcal{ST}(\sqrt{1+z})$ introduced by Sokół and Stankiewicz [6] consists of functions $f \in \mathcal{A}$ satisfying

$$\left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1 \quad (z \in \mathbb{D}). \quad (2)$$

Thus, a function $f \in \mathcal{A}$ is in the class $\mathcal{S}_{\mathcal{ST}}$ if $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$. Results related to the class $\mathcal{S}_{\mathcal{ST}}$ can be found in [3, 7–9].

In investigating the class $\mathcal{UCV}$ of uniformly convex functions, Rønning [10] introduced a class $\mathcal{S}_{\mathcal{P}}$ of *parabolic starlike functions*. These are functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (z \in \mathbb{D}). \quad (3)$$

It is important to keep in mind that the qualifier “parabolic” refers to the geometry of the image of $\mathbb{D}$ under the map $zf'(z)/f(z)$; that is, the domain necessarily lies in a parabolic region of the w -plane. It does not convey the interpretation that the function f maps the disk $\mathbb{D}$ onto a parabolic region. This terminology of *parabolic starlike functions* is however widely accepted and used by authors. Ali

and Ravichandran [11] recently surveyed works on uniformly convex and parabolic starlike functions.

This paper finds radius estimates for classes of functions in $\mathcal{A}$. The radius of a property P in a given set of functions $\mathcal{M}$ [12, page 119] is the largest number R such that every function in the set $\mathcal{M}$ has the property P in each disc $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ for every $r < R$. For example, the Koebe function $k(z) = z/(1-z)^2$, which maps $\mathbb{D}$ onto the domain $\mathbb{C} \setminus \{w \in \mathbb{R} : w \leq -1/4\}$, is starlike but not convex. However, k maps the disc $\mathbb{D}_r$ onto a convex domain for every $r \leq 2 - \sqrt{3}$. Indeed, every univalent function $f \in \mathcal{S}$ maps $\mathbb{D}_r$ onto a convex domain for $r \leq 2 - \sqrt{3}$ [13, Theorem 2.13, page 44]. This number is known as the radius of convexity for $\mathcal{S}$.

It is known that $\mathcal{CV} \subseteq \mathcal{ST}(1/2) \subseteq \{f \in \mathcal{A} : \operatorname{Re}(f(z)/z) > 1/2, z \in \mathbb{D}\}$. The function $g(z) = z/(1-z)$ is convex and therefore starlike of order $1/2$; it is clear that the function

$$\phi(z) := \frac{zg'(z)}{g(z)} \quad (4)$$

has real part greater than $1/2$. Now the function

$$\psi(z) := \frac{z^2 g''(z)}{g(z)} \quad (5)$$

takes values in $\mathbb{C} \setminus \{w \in \mathbb{R} : w \leq -1/2\}$, and therefore does not have positive real part for all $z \in \mathbb{D}$. Their sum

$$\begin{aligned} \phi(z) + \psi(z) &= \frac{zg'(z)}{g(z)} + \frac{z^2 g''(z)}{g(z)} \\ &= \frac{zg'(z)}{g(z)} \left(1 + \frac{zg''(z)}{g'(z)} \right) \end{aligned} \quad (6)$$

takes values in $\{w := x + iy \in \mathbb{C} : y^2 > -x/2\}$ and therefore the sum $\phi + \psi$ does not have positive real part in $\mathbb{D}$. This motivates us to determine the largest radius ρ such that

$$\operatorname{Re} \left(\frac{z^2 g''(z)}{g(z)} + \frac{zg'(z)}{g(z)} \right) > \alpha \quad (|z| \leq \rho). \quad (7)$$

More generally, let $\phi = \phi(f, f', f''; z)$ and $\psi = \psi(f, f', f''; z)$ be functions satisfying $\operatorname{Re} \phi > 0$ in $\mathbb{D}$, while $\operatorname{Re} \psi$ need not necessarily be positive in the whole unit disc $\mathbb{D}$. For certain choices of ϕ and ψ , a sharp radius constant ρ is determined, $0 \leq \rho < 1$, so that whenever $|z| < \rho$, the sum $\phi + \psi$ takes values in specified regions in the complex plane. The results obtained are shown to reduce those of Singh and Paul [14] in certain special cases.

2. Main Results

For $\phi(z) := \phi(f, f', f''; z) = zf'(z)/f(z)$ and $\psi(z) := \psi(f, f', f''; z) = z^2 f''(z)/f(z)$, with $f \in \mathcal{ST}(1/2)$, several radius results for the sum $\phi + \psi$ to be in certain regions in the complex plane are obtained in the following result.

Theorem 1. Let $f \in \mathcal{ST}(1/2)$; let $\chi : \mathbb{D} \rightarrow \mathbb{C}$ be defined by

$$\chi(z) = \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}, \quad (8)$$

$$\chi_i(z) = \chi(\rho_i z), \quad i = 1, 2, \dots, 6.$$

Then

(a) $\operatorname{Re} \chi_1(z) > \alpha$, $0 \leq \alpha < 1$, where ρ_1 is given by

$$\rho_1 = \begin{cases} \sqrt{\frac{7-16\alpha}{11-16\alpha+8\sqrt{2-4\alpha}}}, & 0 \leq \alpha \leq \frac{2+\sqrt{13}}{18} \\ \frac{2(1-\alpha)}{1+2\alpha+\sqrt{1+8\alpha}}, & \frac{2+\sqrt{13}}{18} < \alpha < 1. \end{cases} \quad (9)$$

(b) $|\chi_2^2(z) - 1| < 1$, where ρ_2 is given by

$$\rho_2 = \frac{2(\sqrt{2}-1)}{1+2\sqrt{2}+\sqrt{1+8\sqrt{2}}} \approx 0.112903. \quad (10)$$

(c) $\operatorname{Re} \chi_3(z) < \beta$, $\beta > 1$, where ρ_3 is given by

$$\rho_3 = \frac{2(\beta-1)}{1+2\beta+\sqrt{1+8\beta}}. \quad (11)$$

(d) $|\chi_4(z) - 1| < 1 - \alpha$, $0 \leq \alpha < 1$, where ρ_4 is given by

$$\rho_4 = \frac{2(1-\alpha)}{5-2\alpha+\sqrt{17-8\alpha}}. \quad (12)$$

(e) $|\arg(\chi_5(z))| < \gamma\pi/2$, $0 < \gamma \leq 1$ where $\rho_5 = \rho_5(\gamma) \in (0, 1)$ is the root of the equation in r :

$$\begin{aligned} & (1+2(1-r^2)t_0) \sqrt{4t_0 - (1+t_0-r^2t_0)^2} \\ & - \left(1+t_0(-1-3r^2+2(1-r^2)^2t_0) \right) \tan\left(\frac{\pi\gamma}{2}\right) = 0, \\ & t_0 = \frac{5-r^2+\sqrt{9-10r^2+17r^4}}{8(1-r^4)}. \end{aligned} \quad (13)$$

In particular,

$$\begin{aligned} \rho_5\left(\frac{1}{4}\right) &\approx 0.131522, & \rho_5\left(\frac{1}{2}\right) &\approx 0.266747, \\ \rho_5\left(\frac{3}{4}\right) &\approx 0.409049, & \rho_5(1) &\approx 0.560097. \end{aligned} \quad (14)$$

(f) Also $|\chi_6(z) - 1| < \operatorname{Re} \chi_6(z)$, where $\rho_6 \approx 0.23605 \in (0, 1)$ is the root of the following equation in r :

$$\begin{aligned} & 2(1+9r^2)t_0 - 1 + (-5+26r^2-21r^4)t_0^2 \\ & + 4(1-r^2)^2(1+3r^2)t_0^3 - 4(1-r^2)^4t_0^4 = 0, \end{aligned} \quad (15)$$

and $t_0 \in (1/(1+r)^2, 1/(1-r)^2)$ is the root of the equation in t :

$$2(1+9r^2) + 2(-5+26r^2-21r^4)t + 12(1-r^2)^2(1+3r^2)t^2 - 16(1-r^2)^4t^3 = 0. \quad (16)$$

Each radius constant ρ_i is sharp.

For two analytic functions $f, g \in \mathcal{A}$, their convolution or Hadamard product, denoted by $f * g$, is defined by $(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n$. The following results are needed in the sequel.

Lemma 2 ([15, Lemma 2.7, page 126; Lemma 3.5, page 130]). If $f \in \mathcal{CV}$ and $g \in \mathcal{ST}$, or f and g belong to $\mathcal{ST}(1/2)$, then

$$\frac{f * g^F}{f * g}(\mathbb{D}) \subset \overline{\text{co}}(F(\mathbb{D})) \quad (17)$$

for any function F analytic in $\mathbb{D}$, where $\overline{\text{co}}(F(\mathbb{D}))$ denotes the closed convex hull of $F(\mathbb{D})$.

Lemma 3 ([7, Lemma 2.2, page 6559]). For $0 < a < \sqrt{2}$, let r_a be given by

$$r_a = \begin{cases} (\sqrt{1-a^2} - (1-a^2))^{1/2}, & 0 < a \leq 2\sqrt{2}/3 \\ \sqrt{2}-a, & 2\sqrt{2}/3 \leq a < \sqrt{2}, \end{cases} \quad (18)$$

and for $a > 0$, let R_a be given by

$$R_a = \begin{cases} \sqrt{2}-a, & 0 < a \leq \frac{1}{\sqrt{2}}, \\ a, & \frac{1}{\sqrt{2}} \leq a. \end{cases} \quad (19)$$

Then,

$$\{w : |w-a| < r_a\} \subseteq \{w : |w^2-1| < 1\} \subseteq \{w : |w-a| < R_a\}. \quad (20)$$

Proof of Theorem 1. Let $h : \mathbb{D} \rightarrow \mathbb{C}$ be defined by

$$h(z) = \frac{2}{(1-z)^2} - \frac{1}{1-z}. \quad (21)$$

First, for each $i = 1, 2, \dots, 6$, $h_i(z) = h(\rho_i z)$ will be shown to, respectively, satisfy $\text{Re } h_1(z) > \alpha$, $|h_2(z) - 1| < 1$, $\text{Re } h_3(z) < \beta$, $|h_4(z) - 1| < 1 - \alpha$, $|\arg h_5(z)| < \gamma\pi/2$, and $|h_6(z) - 1| < \text{Re } h_6(z)$. Then, using Lemma 2, χ_i is deduced to satisfy the required condition.

As in [14], let

$$\frac{1}{1-z} = Re^{i\theta}, \quad (22)$$

so that

$$\begin{aligned} \frac{1}{1+r} \leq R \leq \frac{1}{1-r} \quad (|z|=r), \\ \cos \theta = \frac{1+R^2-r^2R^2}{2R}. \end{aligned} \quad (23)$$

(a) By (21), (22), and (23), it follows that

$$\begin{aligned} \text{Re } h(z) &= 2R^2 \cos 2\theta - R \cos \theta \\ &= \frac{1}{2} - \frac{1}{2}(1+3r^2)t + (1-r^2)^2t^2 \quad (t := R^2) \\ &=: \phi(t). \end{aligned} \quad (24)$$

Case (i). Suppose that $0 \leq \alpha \leq (2 + \sqrt{13})/18$. We assert that $\min \phi(t) > \alpha$ for $|z| < \rho_1$, where the minimum is taken over all $t \in (1/(1+r)^2, 1/(1-r)^2)$. Let $r < \rho_1$. Then

$$\frac{\partial \phi(t)}{\partial t} = -\frac{1}{2}(1+3r^2) + 2(1-r^2)^2t = 0 \quad (25)$$

if $t = t_0 := (1+3r^2)/(4(1-r^2)^2)$, $\partial^2 \phi(t_0)/\partial t^2 > 0$, and that for $r \geq 4 - \sqrt{13}$,

$$\frac{1}{(1+r)^2} \leq t_0 \leq \frac{1}{(1-r)^2}. \quad (26)$$

Thus, for $4 - \sqrt{13} \leq r < \rho_1$,

$$\min \phi(t) = \phi(t_0) = \frac{7-22r^2-r^4}{16(1-r^2)^2} > \alpha. \quad (27)$$

On the other hand, if $r < 4 - \sqrt{13}$, then it can be shown that

$$\begin{aligned} \min \phi(t) &= \phi\left(\frac{1}{(1+r)^2}\right) \\ &= \frac{1-r}{(1+r)^2} > \frac{\sqrt{13}-3}{(5-\sqrt{13})^2} \\ &= \frac{2+\sqrt{13}}{18} > \alpha. \end{aligned} \quad (28)$$

Case (ii). For $(2 + \sqrt{13})/18 < \alpha < 1$, then $\min \phi(t) > \alpha$ in $|z| < \rho_1$, $t \in (1/(1+r)^2, 1/(1-r)^2)$. Indeed for $r < \rho_1 < 4 - \sqrt{13}$, as in the case (i),

$$\min \phi(t) = \phi\left(\frac{1}{(1+r)^2}\right) = \frac{1-r}{(1+r)^2} > \alpha. \quad (29)$$

The previously mentioned two cases show that $\text{Re } h_1(z) > \alpha$ in $\mathbb{D}$. Figure 1 illustrates sharpness of the radius $\rho_1 = \sqrt{5} - 2$ in the case $\alpha = 0.5$.

(b) For h given by (21), a calculation shows that

$$\begin{aligned} |h(z) - 1| &= \left| \frac{2}{(1-z)^2} - \frac{1}{1-z} - 1 \right| \\ &= \left| \frac{z}{1-z} + \frac{2z}{(1-z)^2} \right| \\ &\leq \frac{r}{1-r} + \frac{2r}{(1-r)^2}. \end{aligned} \quad (30)$$

FIGURE 1: Image of $|z| \leq \sqrt{5} - 2$ touches $\operatorname{Re} w = 0.5$.

By Lemma 3, the function h satisfies

$$|h^2(z) - 1| < 1 \quad (31)$$

provided

$$\frac{r}{1-r} + \frac{2r}{(1-r)^2} \leq \sqrt{2} - 1; \quad (32)$$

that is,

$$\sqrt{2}r^2 - (1 + 2\sqrt{2})r + (\sqrt{2} - 1) \geq 0. \quad (33)$$

This inequality holds if $r \leq \rho_2$. Figure 2 illustrates sharpness of the radius $\rho_2 \approx 0.1129$.

(c) From (30), it follows that

$$\operatorname{Re} h(z) \leq 1 + \frac{r}{1-r} + \frac{2r}{(1-r)^2} \leq \beta \quad (34)$$

provided

$$\beta r^2 - (1 + 2\beta)r - (1 - \beta) \geq 0 \quad (35)$$

holds, which occurs whenever $r \leq \rho_3$. Sharpness of the radius $\rho_3 = (4 - \sqrt{13})/3$ in the case $\beta = 1.5$ is illustrated in Figure 3.

(d) Inequality (30) also yields

$$|h(z) - 1| \leq \frac{r}{1-r} + \frac{2r}{(1-r)^2} \leq 1 - \alpha \quad (36)$$

provided

$$r^2(\alpha - 2) + r(5 - 2\alpha) + \alpha - 1 \leq 0, \quad (37)$$

that is, when $r \leq \rho_4$. Figure 4 illustrates sharpness of the radius $\rho_4 = (4 - \sqrt{13})/3$ in the case $\alpha = 0.5$.

FIGURE 2: Image of $|z| \leq 0.1129$ touches $|w^2 - 1| = 1$.

(e) For the function h given by (21), it follows from (22) and (23) that

$\arg h(z)$

$$\begin{aligned} &= \arctan \left(\frac{2R \sin 2\theta - \sin \theta}{2R \cos 2\theta - \cos \theta} \right) \\ &= \arctan \left(\frac{(4R \cos \theta - 1) \sin \theta}{2R(2\cos^2 \theta - 1) - \cos \theta} \right) \\ &= \arctan \left(\frac{(1 + 2(1 - r^2)R^2) \sqrt{4R^2 - (1 + (1 - r^2)R^2)^2}}{1 - (1 + 3r^2)R^2 + 2(1 - r^2)^2 R^4} \right) \\ &= \arctan \left(\frac{(1 + 2(1 - r^2)t) \sqrt{4t - (1 + (1 - r^2)t)^2}}{1 - (1 + 3r^2)t + 2(1 - r^2)^2 t^2} \right) \\ &:= \arctan(\phi(t)), \end{aligned} \quad (38)$$

$t \in (1/(1+r)^2, 1/(1-r)^2)$. A calculation shows that $\phi'(t) = 0$ where

$$\begin{aligned} t &= t_0 \\ &= \frac{5 - r^2 + \sqrt{9 - 10r^2 + 17r^4}}{8(1 - r^4)} \in \left(\frac{1}{(1+r)^2}, \frac{1}{(1-r)^2} \right). \end{aligned} \quad (39)$$

Now $\phi'(t) > 0$ for $t < t_0$, $\phi'(t) < 0$ if $t > t_0$, and

$$\phi\left(\frac{1}{(1+r)^2}\right) = \phi\left(\frac{1}{(1-r)^2}\right) = 0. \quad (40)$$

Thus

$$\begin{aligned} \max \phi(t) &= \phi(t_0) \\ &= \frac{(1 + 2(1 - r^2)t_0) \sqrt{4t_0 - (1 + (1 - r^2)t_0)^2}}{1 - (1 + 3r^2)t_0 + 2(1 - r^2)^2 t_0^2}. \end{aligned} \quad (41)$$

FIGURE 3: Image of $|z| \leq (4 - \sqrt{13})/3$ touches $\operatorname{Re} w = 1.5$.

FIGURE 4: Image of $|z| \leq (4 - \sqrt{13})/3$ touches $|w - 1| = 0.5$.

Evidently, (38) and (41) give

$$\begin{aligned} & |\arg h(z)| \\ & \leq \left| \arctan \left(\frac{(1 + 2(1 - r^2)t_0) \sqrt{4t_0 - (1 + (1 - r^2)t_0)^2}}{1 - (1 + 3r^2)t_0 + 2(1 - r^2)^2 t_0^2} \right) \right| \\ & \leq \frac{\gamma\pi}{2} \end{aligned} \quad (42)$$

provided

$$\begin{aligned} & (1 + 2(1 - r^2)t_0) \sqrt{4t_0 - (1 + (1 - r^2)t_0)^2} \\ & - \left(1 + t_0(-1 - 3r^2 + 2(1 - r^2)^2 t_0) \right) \tan \left(\frac{\pi\gamma}{2} \right) \\ & \leq 0. \end{aligned} \quad (43)$$

Figure 5 illustrates sharpness of the radius $\rho_5 \approx 0.266747$ in the case $\gamma = 0.5$.

(f) The inequality

$$|h(z) - 1| < \operatorname{Re} h(z) \quad (44)$$

holds if

$$\begin{aligned} & (2R^2 \cos 2\theta - R \cos \theta - 1)^2 + (2R^2 \sin 2\theta - R \sin \theta)^2 \\ & < (2R^2 \cos 2\theta - R \cos \theta)^2, \end{aligned} \quad (45)$$

or, with $t = R^2$,

$$\begin{aligned} \phi(t) &:= 2(1 + 9r^2)t - 1 + (-5 + 26r^2 - 21r^4)t^2 \\ &+ 4(1 - r^2)^2(1 + 3r^2)t^3 - 4(1 - r^2)^4 t^4 < 0, \end{aligned} \quad (46)$$

FIGURE 5: Image of $|z| \leq 0.266747$ touches $|\arg w| = \pi/4$.

$t \in (1/(1 + r)^2, 1/(1 - r)^2)$. Then,

$$\begin{aligned} \phi'(t) &= 2(1 + 9r^2) + 2(-5 + 26r^2 - 21r^4)t \\ &+ 12(1 - r^2)^2(1 + 3r^2)t^2 - 16(1 - r^2)^4 t^3. \end{aligned} \quad (47)$$

FIGURE 6: Image of $|z| \leq 0.23605$ touches $|w - 1| = \operatorname{Re} w$.

Let $r < \rho_6$. Since

$$\begin{aligned} \phi' \left(\frac{1}{(1+r)^2} \right) &= \frac{4(-3 + 11r + 6r^2 + 7r^3 - r^4)}{(1+r)^2} > 0 \quad \text{if } r > 0.234722, \\ \phi' \left(\frac{1}{(1-r)^2} \right) &= -\frac{4(3 + 11r - 6r^2 + 7r^3 + r^4)}{(1-r)^2} < 0, \end{aligned} \quad (48)$$

there exists a unique $t_0 \in (1/(1+r)^2, 1/(1-r)^2)$ such that $\phi'(t_0) = 0$ and $\max \phi(t) = \phi(t_0)$.

Thus, $\max \phi(t) = \phi(t_0) < 0$ for $0.234722 < r < \rho_6$. When $r \leq 0.234722$,

$$\max \phi(t) = \phi \left(\frac{1}{(1+r)^2} \right) = \frac{4(4r - 1 + r^2)}{(1+r)^2} < 0; \quad (49)$$

hence, $\phi(t) < 0$ for $r < \rho_6$. Figure 6 illustrates sharpness of the radius $\rho_6 \approx 0.23605$.

Next, consider $g_i(z) = z/(1 - \rho_i z) \in \mathcal{CV} \subset \mathcal{ST}(1/2)$, $i = 1, 2, \dots, 6$. Then,

$$\begin{aligned} \frac{f(z) * g_i(z) h_i(z)}{f(z) * g_i(z)} &= \frac{f(z) * (z/(1 - \rho_i z)) (2\rho_i z/(1 - \rho_i z)^2 + 1/(1 - \rho_i z))}{f(z) * (z/(1 - \rho_i z))} \end{aligned}$$

$$\begin{aligned} &= \frac{f(z) * (2\rho_i z^2/(1 - \rho_i z)^3 + z/(1 - \rho_i z)^2)}{f(z) * (z/(1 - \rho_i z))} \\ &= \frac{\rho_i^2 z^2 f''(\rho_i z) + \rho_i z f'(\rho_i z)}{f(\rho_i z)} \\ &= \chi(\rho_i z) = \chi_i(z). \end{aligned} \quad (50)$$

Lemma 2, together with (50) and the corresponding inequality for the function h_i , shows that each function χ_i satisfies the required condition. For sharpness, consider the function $f_0(z) = z/(1 - z) \in \mathcal{CV} \subset \mathcal{ST}(1/2)$. Then,

$$\frac{z^2 f_0''(z) + z f_0'(z)}{f_0(z)} = \frac{2}{(1-z)^2} - \frac{1}{1-z} = h(z). \quad (51)$$

Sharpness of the numbers ρ_i is now evident in view of the definition h . $\square$

For $\alpha = 0$, Theorem 1(a) reduces to the following corollary.

Corollary 4 ([14, Theorem 5, page 724]). *If $f \in \mathcal{ST}(1/2)$, then*

$$\operatorname{Re} \left(\frac{z^2 f''(z)}{f(z)} + \frac{z f'(z)}{f(z)} \right) > 0 \quad (52)$$

in $|z| < \rho = \sqrt{8\sqrt{2} - 11} = 0.56$. The number ρ is sharp.

Theorem 5. *Let $f \in \mathcal{ST}(1/2)$ and $\chi : \mathbb{D} \rightarrow \mathbb{C}$ be defined by*

$$\chi(z) = \frac{f(z)}{z} + f'(z), \quad \chi_i(z) = \chi(\rho_i z), \quad i = 1, 2, 3. \quad (53)$$

Then

(a) $\operatorname{Re} \chi_1(z) > \alpha$, $0 \leq \alpha < 1$, where ρ_1 is given by

$$\rho_1 = \begin{cases} \sqrt{\frac{7-8\alpha}{5-8\alpha+4\sqrt{2}\sqrt{1-\alpha}}}, & 0 \leq \alpha < \frac{7+4\sqrt{7}}{18}, \\ \frac{4-2\alpha}{2\alpha-1+\sqrt{1+4\alpha}}, & \frac{7+4\sqrt{7}}{18} \leq \alpha < 1. \end{cases} \quad (54)$$

(b) $|\arg \chi_2(z)| < \gamma\pi/2$, $0 < \gamma \leq 1$, where $\rho_2 = \rho_2(\gamma) \in (0, 1)$ is the root of the equation

$$\begin{aligned} & \left(\sqrt{9+5r^2}(-2+r^2) - 2r^2 \right) \\ & \times \sqrt{16r^2 + \left(3+3r^2 - \sqrt{9+5r^2}(r^2-2) \right)^2} \\ & + \left(\sqrt{9+5r^2}(r^2-2) - 11 - 11r^4 + 5r^2 \right) \\ & \times \left(2 + \sqrt{9+5r^2}(r^2-2) \right) \tan \left(\frac{\pi\gamma}{2} \right) = 0. \end{aligned} \quad (55)$$

In particular,

$$\begin{aligned}\rho_2\left(\frac{1}{4}\right) &\approx 0.257136, & \rho_2\left(\frac{1}{2}\right) &\approx 0.487998, \\ \rho_2\left(\frac{3}{4}\right) &\approx 0.674274, & \rho_2(1) &= \sqrt{\frac{7}{5+4\sqrt{2}}} \approx 0.810465.\end{aligned}\quad (56)$$

(c) Also $|\chi_3(z) - 1| < \operatorname{Re} \chi_3(z)$, where $\rho_3 \approx 0.44915$ is given by the equation in r

$$\begin{aligned}24r^2t_0 - 8 + (-1 + 18r^2 - 17r^4)t_0^2 \\ + 2(1 - r^2)^2(-1 + 3r^2)t_0^3 - (1 - r^2)^4t_0^4 = 0\end{aligned}\quad (57)$$

and t_0 is given by the equation in t

$$\begin{aligned}24r^2 + 2(-1 + 18r^2 - 17r^4)t \\ + 6(1 - r^2)^2(-1 + 3r^2)t^2 - 4(1 - r^2)^4t^3 = 0.\end{aligned}\quad (58)$$

Each radius constant ρ_i is sharp.

Proof. Let

$$h(z) = \frac{1}{1-z} + \frac{1}{(1-z)^2} \quad (z \in \mathbb{D}). \quad (59)$$

Each $h_i(z) = h(\rho_i z)$, $i = 1, 2, 3$, is shown to, respectively, satisfy $\operatorname{Re} h_1(z) > \alpha$, $|\arg h_2(z)| < \gamma\pi/2$, and $|h_3(z) - 1| < \operatorname{Re} h_3(z)$. Then, it follows from Lemma 2 that χ_i satisfies the required condition.

(a) We claim that $\operatorname{Re} h(z) > \alpha$ in $|z| < \rho_1$. By (22) and (23),

$$\begin{aligned}\operatorname{Re} h(z) &= R \cos \theta + R^2 \cos 2\theta \\ &= 1 + \frac{1}{2} \left((1 - 3r^2)t + (1 - r^2)^2 t^2 \right) \quad (t := R^2) \\ &:= \phi(t).\end{aligned}\quad (60)$$

Case (i). Suppose $0 \leq \alpha < (7 + 4\sqrt{7})/18$. In this case, it is shown that $\min \phi(t) > \alpha$ for $|z| < \rho_1$ over all t in $(1/(1+r)^2, 1/(1-r)^2)$. Let $r < \rho_1$. It can be verified that

$$\frac{\partial \phi(t)}{\partial t} = \frac{1}{2} (1 - 3r^2 + 2(1 - r^2)^2 t) = 0 \quad (61)$$

if $t = t_0 = (3r^2 - 1)/(2(1 - r^2)^2)$, $\partial^2 \phi(t_0)/\partial t^2 = (1 - r^2)^2 > 0$, and that for $r \geq \sqrt{7} - 2$,

$$\frac{1}{(1+r)^2} \leq t_0 \leq \frac{1}{(1-r)^2}. \quad (62)$$

Thus for $\sqrt{7} - 2 \leq r < \rho_1$,

$$\min \phi(t) = \phi(t_0) = \frac{7 - 10r^2 - r^4}{8(1 - r^2)^2} > \alpha. \quad (63)$$

FIGURE 7: Image of $|z| \leq \sqrt{3}/5$ touches $\operatorname{Re} w = 0.5$.

On the other hand, if $r < \sqrt{7} - 2$, then

$$\min \phi(t) = \phi\left(\frac{1}{(1+r)^2}\right) = \frac{2+r}{(1+r)^2}. \quad (64)$$

Since $g(r) = (2+r)/(1+r)^2$ is a decreasing function in $(0, \sqrt{7} - 2)$,

$$\begin{aligned}\min \phi(t) &= \frac{(2+r)}{(1+r)^2} > \frac{2+\sqrt{7}-2}{(1+\sqrt{7}-2)^2} \\ &= \frac{\sqrt{7}}{(\sqrt{7}-1)^2} = \frac{4\sqrt{7}+7}{18} > \alpha.\end{aligned}\quad (65)$$

Case (ii). For $(7+4\sqrt{7})/18 \leq \alpha < 1$, we prove that $\min \phi(t) > \alpha$ in $|z| < \rho_1$, $t \in (1/(1+r)^2, 1/(1-r)^2)$. Let $r < \rho_1 < \sqrt{7} - 2$. As in Case (i), then

$$\min \phi(t) = \phi\left(\frac{1}{(1+r)^2}\right) = \frac{(2+r)}{(1+r)^2} > \alpha. \quad (66)$$

It is evident from the previous two cases that $\operatorname{Re} h_1(z) > \alpha$ in $\mathbb{D}$. Figure 7 shows that, for $\alpha = 0.5$, the radius $\rho_1 = \sqrt{3}/5$ is sharp.

(b) Let $h(re^{it}) = u + iv$. Then,

$$\begin{aligned}u &= \frac{2(1+r^2+r^2 \cos 2t) - r(5+r^2) \cos t}{(1+r^2-2r \cos t)^2}, \\ v &= \frac{r(3+r^2-4r \cos t) \sin t}{(1+r^2-2r \cos t)^2}.\end{aligned}\quad (67)$$

By (67), it follows that

$$\begin{aligned} \arg h(re^{it}) \\ = \arctan \left(\frac{r(3+r^2-4r\cos t)\sin t}{2(1+r^2+r^2\cos 2t)-r(5+r^2)\cos t} \right). \end{aligned} \quad (68)$$

Let $g : [-1, 1] \rightarrow \mathbb{R}$ be defined by

$$g(x) = \frac{r(3+r^2-4rx)\sqrt{1-x^2}}{2-r(5+r^2)x+4r^2x^2}. \quad (69)$$

(The case $-\sqrt{1-x^2}$ is similar.) A calculation shows that

$$g'(x) = \frac{r(1+r^2-2rx)(r(7+r^2)-6x(1+r^2)+4rx^2)}{\sqrt{1-x^2}(2-r(5+r^2)x+4r^2x^2)^2}. \quad (70)$$

Let

$$x_0 = \frac{3+3r^2-\sqrt{9-10r^2+5r^4}}{4r} \in (0, 1]. \quad (71)$$

Then, $g'(x_0) = 0$, $g'(x) > 0$ for $x < x_0$, and $g'(x) < 0$ for $x > x_0$. Thus,

$$g(x) \leq \max_{x \in [-1, 1]} g(x) = g(x_0). \quad (72)$$

Now (68) and (72) show that

$$|\arg h(re^{it})| \leq \frac{\gamma\pi}{2} \quad (73)$$

provided

$$\frac{r(3+r^2-4rx_0)\sqrt{1-x_0^2}}{2-r(5+r^2)x_0+4r^2x_0^2} \leq \tan\left(\frac{\gamma\pi}{2}\right); \quad (74)$$

that is,

$$\tan\left(\frac{\gamma\pi}{2}\right) - \frac{r(3+r^2-4rx_0)\sqrt{1-x_0^2}}{2-r(5+r^2)x_0+4r^2x_0^2} \geq 0. \quad (75)$$

FIGURE 8: Image of $|z| \leq 0.487998$ touches $|\arg w| = \pi/4$.

Thus, $|\arg h_2(z)| < \gamma\pi/2$ in $\mathbb{D}$. Figure 8 shows that, for $\gamma = 0.5$, the radius $\rho_2 = 0.487998$ is sharp.

(c) Proceeding similarly as in part (a),

$$|h(z) - 1| < \operatorname{Re} h(z), \quad (76)$$

provided

$$\begin{aligned} 24r^2R^2 - 8 + (18r^2 - 1 - 17r^4)R^4 \\ - 2(1-r^2)^2(1-3r^2)R^6 - (1-r^2)^4R^8 < 0. \end{aligned} \quad (77)$$

Let $\phi : (1/(1+r)^2, 1/(1-r)^2) \rightarrow \mathbb{R}$ be defined by

$$\begin{aligned} \phi(t) = 24r^2t - 8 + (18r^2 - 1 - 17r^4)t^2 \\ - 2(1-r^2)^2(1-3r^2)t^3 - (1-r^2)^4t^4. \end{aligned} \quad (78)$$

Now

$$\begin{aligned} \phi'(t) = 24r^2 + 2(18r^2 - 1 - 17r^4)t \\ - 6(1-r^2)^2(1-3r^2)t^2 - 4(1-r^2)^4t^3. \end{aligned} \quad (79)$$

FIGURE 9: Image of $|z| \leq 0.44915$ touches $|w - 1| = \operatorname{Re} w$.

Let $r < \rho_3$. Since

$$\begin{aligned} \phi' \left(\frac{1}{(1+r)^2} \right) &= \frac{4(-3+7r+12r^2+7r^3+r^4)}{(1+r)^2} > 0 \quad \text{if } r > 0.27606, \\ \phi' \left(\frac{1}{(1-r)^2} \right) &= \frac{4(-3-7r+12r^2-7r^3+r^4)}{(1-r)^2} < 0, \end{aligned} \quad (80)$$

there exists a unique $t_0 \in (1/(1+r)^2, 1/(1-r)^2)$ such that $\phi'(t_0) = 0$ and $\max \phi(t) = \phi(t_0)$.

Then, for $0.27606 < r < \rho_3$,

$$\max \phi(t) = \phi(t_0) < 0. \quad (81)$$

When $r \leq 0.27606$,

$$\max \phi(t) = \phi \left(\frac{1}{(1+r)^2} \right) = \frac{4(-3+r^2)}{(1+r)^2} < 0. \quad (82)$$

Evidently, $\phi(t) < 0$ for $r < \rho_3$ and hence $|h_3(z) - 1| < \operatorname{Re} h_3(z)$ in $\mathbb{D}$. Figure 9 shows that the radius $\rho_3 \approx 0.44915$ is sharp.

Now, with $g(z) = z \in \mathcal{ST}(1/2)$, $i = 1, 2, 3$,

$$\begin{aligned} \frac{f(z) * zh_i(z)}{f(z) * z} &= \frac{f(z) * z \left(\frac{1}{1-\rho_i z} + \frac{1}{(1-\rho_i z)^2} \right)}{f(z) * z} \\ &= \frac{f(\rho_i z)}{\rho_i z} + f'(\rho_i z) \\ &= \chi(\rho_i z) = \chi_i(z). \end{aligned} \quad (83)$$

Lemma 2, together with (83) and the corresponding inequality for the function h_i , shows that each function χ_i satisfies the required condition. For sharpness, consider the function $f_0(z) = z/(1-z) \in \mathcal{CV} \subset \mathcal{ST}(1/2)$. Clearly

$$\frac{f_0(z)}{z} + f'_0(z) = \frac{1}{1-z} + \frac{1}{(1-z)^2} = h(z); \quad (84)$$

hence the fact that the number ρ_i is sharp follows from the definition of h . $\square$

For $\alpha = 0$, Theorem 5(a) reduces to the following corollary.

Corollary 6 ([14, Theorem 3, page 722]). *If $f \in \mathcal{ST}(1/2)$, then*

$$\operatorname{Re} \left(\frac{f(z)}{z} + f'(z) \right) > 0 \quad (85)$$

in $|z| < \rho = \sqrt{4\sqrt{2}-5} \approx 0.81$. The number ρ is sharp.

Theorem 7. *Let $f \in \mathcal{CV}$, $\chi : \mathbb{D} \rightarrow \mathbb{C}$ be defined by*

$$\begin{aligned} \chi(z) &= \left(1 + \frac{zf''(z)}{f'(z)} \right) + \frac{1}{f'(z)}, \\ \chi_i(z) &= \chi(\rho_i z), \quad i = 1, 2. \end{aligned} \quad (86)$$

Then,

- (a) $\operatorname{Re} \chi_1(z) > \alpha$, $0 \leq \alpha < 1$, where $\rho_1 = \sqrt{3 - \sqrt{5 + 2\alpha}}$;
- (b) $|\arg \chi_2(z)| < \gamma\pi/2$, $0 < \gamma \leq 1$, where $\rho_2 = \rho(\gamma) \in (0, 1)$ is the root of the equation in r

$$\begin{aligned} &2r^2 \sqrt{1-x_0^2} (r - (3+r^2)x_0 + 2rx_0^2) + \tan \left(\frac{\pi\gamma}{2} \right) \\ &\times (2 - r^2 - r^4 - 4rx_0 + (6r^2 + 2r^4)x_0^2 - 4r^3x_0^3) = 0, \end{aligned} \quad (87)$$

and $x_0 \in [-1, 1]$ is the root of the equation

$$\begin{aligned} &6 - 5r^2 - 4r^4 - r^6 + (-6r + 15r^3 + 7r^5)x_0 \\ &+ (-12 - 8r^2 - 16r^4)x_0^2 + (24r + 16r^3)x_0^3 \\ &- 16r^2x_0^4 = 0. \end{aligned} \quad (88)$$

In particular,

$$\rho_2\left(\frac{1}{2}\right) \approx 0.6355, \quad \rho_2(1) = \sqrt{3 - \sqrt{5}}. \quad (89)$$

The radii are sharp.

Proof. Let

$$h(z) = \frac{2}{1-z} + (1-z)^2 - 1 \quad (z \in \mathbb{D}). \quad (90)$$

(a) We claim that $\operatorname{Re} h(z) > \alpha$ in $|z| < \rho_1$. By (22) and (23), it follows that

$$\begin{aligned} \operatorname{Re} h(z) &= 2R \cos \theta - 1 + \frac{\cos 2\theta}{R^2} \\ &= \frac{1 + t^2 + r^4 t^2 + 2t^3 - 2r^2(t + t^2 + t^3)}{2t^2} \quad (t := R^2) \\ &:= \phi(t). \end{aligned} \quad (91)$$

A calculation shows that $\phi'(t) = 0$ if $t = t_0 = 1 \in (1/(1+r)^2, 1/(1-r)^2)$, $\phi''(t_0) = 3 - 2r^2 > 0$, and that

$$\min \phi(t) = \phi(1) = \frac{1}{2}(4 - 6r^2 + r^4) > \alpha \quad (92)$$

over all $t \in (1/(1+r)^2, 1/(1-r)^2)$ provided

$$r^4 - 6r^2 + 4 - 2\alpha > 0. \quad (93)$$

This inequality reduces to $r \leq \rho_1$. Thus, $\operatorname{Re} h_1(z) > \alpha$ in $\mathbb{D}$. Figure 10 shows that, for $\alpha = 0.5$, the radius $\rho_1 = \sqrt{3 - \sqrt{6}}$ is sharp.

(b) Let $h(re^{it}) = u + iv$. Then

$$\begin{aligned} u &= (2(1 - r \cos t) + (r^2(\cos^2 t - \sin^2 t) - 2r \cos t)) \\ &\quad \times (1 + r^2 - 2r \cos t) \\ &\quad \times (1 + r^2 - 2r \cos t)^{-1}, \\ v &= \frac{2r \sin t (1 - (1 - r \cos t)(1 + r^2 - 2r \cos t))}{1 + r^2 - 2r \cos t}. \end{aligned} \quad (94)$$

By (94), it follows that

$$\begin{aligned} \arg h(re^{it}) &= \arctan \left((2r^2(-r + \cos t(3 + r^2 - 2r \cos t)) \sin t) \right. \\ &\quad \times (2 + 2r^2 - r(4 + 3r^2) \cos t \\ &\quad \left. + r^2(3 + r^2) \cos 2t - r^3 \cos 3t)^{-1} \right). \end{aligned} \quad (95)$$

Let

$$\begin{aligned} g(x) &= (2r^2(-r + x(3 + r^2 - 2rx)) \sqrt{1 - x^2}) \\ &\quad \times (2 + 2r^2 - r(4 + 3r^2)x \\ &\quad + r^2(3 + r^2)(2x^2 - 1) - r^3(4x^3 - 3x))^{-1}. \end{aligned} \quad (96)$$

A calculation shows that there exists $x_0 \in [0, 1]$ such that $g'(x_0) = 0$ and $g''(x_0) < 0$. Thus

$$g(x) \leq g(x_0), \quad x \in [-1, 1]. \quad (97)$$

By (95), (96), and (97), evidently

$$|\arg h(re^{it})| \leq \frac{\gamma\pi}{2} \quad (98)$$

$$\begin{aligned} &\frac{2r^2(-r + x_0(3 + r^2 - 2rx_0)) \sqrt{1 - x_0^2}}{2 - r^2 - r^4 - 4rx_0 + (6r^2 + 2r^4)x_0^2 - 4r^3x_0^3} \\ &\leq \tan\left(\frac{\gamma\pi}{2}\right); \end{aligned} \quad (99)$$

that is,

$$\begin{aligned} &2r^2 \sqrt{1 - x_0^2} (r - (3 + r^2)x_0 + 2rx_0^2) \\ &+ \tan\left(\frac{\gamma\pi}{2}\right) (2 - r^2 - r^4 - 4rx_0 + (6r^2 + 2r^4)x_0^2 - 4r^3x_0^3) \\ &\geq 0. \end{aligned} \quad (100)$$

Thus, $|\arg h_2(z)| < \gamma\pi/2$ in $\mathbb{D}$. Figure 11 shows that, for $\gamma = 0.5$, the radius $\rho_2 \approx 0.6335$ is sharp.

To conclude the proof, let $g(z) = z/(1 - \rho_i z)^2 \in \mathcal{ST}$. Then,

$$\begin{aligned} &\frac{f(z) * (z/(1 - \rho_i z)^2) h_i(z)}{f(z) * (z/(1 - \rho_i z)^2)} \\ &= \frac{f(z) * (z/(1 - \rho_i z)^2) (2/(1 - \rho_i z) - 1 + (1 - \rho_i z)^2)}{f(z) * (z/(1 - \rho_i z)^2)} \\ &= \frac{f(z) * (2z/(1 - \rho_i z)^3 + z - z/(1 - \rho_i z)^2)}{f(z) * (z/(1 - \rho_i z))} \\ &= \left(1 + \frac{\rho_i z f''(\rho_i z)}{f'(\rho_i z)}\right) + \frac{1}{f'(\rho_i z)} \\ &= \chi(\rho_i z) = \chi_i(z). \end{aligned} \quad (101)$$

As in the earlier proofs, Lemma 2 together with (101) and the corresponding inequality for the function h_i shows

FIGURE 10: Image of $|z| \leq \sqrt{3 - \sqrt{6}}$ touches $\operatorname{Re} w = 0.5$.

FIGURE 11: Image of $|z| \leq 0.6335$ touches $|\arg w| = \pi/4$.

that the function χ_i satisfies the required condition. For sharpness, consider $f_0(z) = z/(1-z) \in \mathcal{CV} \subset \mathcal{ST}(1/2)$. Then,

$$\left(1 + \frac{zf_0''(z)}{f_0'(z)}\right) + \frac{1}{f_0'(z)} = \frac{2}{1-z} - 1 + (1-z)^2 = h(z). \quad (102)$$

□

For $\alpha = 0$, Theorem 7(a) reduces to the following result.

Corollary 8 ([14, Theorem 4, page 723]). If $f \in \mathcal{CV}$, then

$$\operatorname{Re} \left(\left(1 + \frac{zf''(z)}{f'(z)}\right) + \frac{1}{f'(z)} \right) > 0 \quad (103)$$

in $|z| < \rho = \sqrt{3 - \sqrt{5}} = (\sqrt{5} - 1)/\sqrt{2} \approx 0.874032$. The result is sharp.

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References

- [1] B. A. Uralegaddi, M. D. Ganigi, and S. M. Sarangi, "Univalent functions with positive coefficients," *Tamkang Journal of Mathematics*, vol. 25, no. 3, pp. 225–230, 1994.
- [2] R. M. Ali, M. Haji Mohd, L. S. Keong, and V. Ravichandran, "Radii of starlikeness, parabolic starlikeness and strong starlikeness for Janowski starlike functions with complex parameters," *Tamsui Oxford Journal of Information and Mathematical Sciences*, vol. 27, no. 3, pp. 253–267, 2011.
- [3] R. M. Ali, N. E. Cho, N. K. Jain, and V. Ravichandran, "Radii of starlikeness and convexity for functions with fixed second coefficient defined by subordination," *Filomat*, vol. 26, no. 3, pp. 553–561, 2012.
- [4] S. Owa and H. M. Srivastava, "Some generalized convolution properties associated with certain subclasses of analytic functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 3, no. 3, article 42, 13 pages, 2002.
- [5] V. Ravichandran, M. Hussain Khan, H. Silverman, and K. G. Subramanian, "Radius problems for a class of analytic functions," *Demonstratio Mathematica*, vol. 39, no. 1, pp. 67–74, 2006.
- [6] J. Sokół and J. Stankiewicz, "Radius of convexity of some subclasses of strongly starlike functions," *Zeszyty Naukowe Politechniki Rzeszowskiej. Matematyka*, no. 19, pp. 101–105, 1996.
- [7] R. M. Ali, N. K. Jain, and V. Ravichandran, "Radii of starlikeness associated with the lemniscate of Bernoulli and the left-half plane," *Applied Mathematics and Computation*, vol. 218, no. 11, pp. 6557–6565, 2012.
- [8] J. Sokół, "Coefficient estimates in a class of strongly starlike functions," *Kyungpook Mathematical Journal*, vol. 49, no. 2, pp. 349–353, 2009.
- [9] J. Sokół, "Radius problems in the class $\mathcal{SL}^*$," *Applied Mathematics and Computation*, vol. 214, no. 2, pp. 569–573, 2009.
- [10] F. Rønning, "Uniformly convex functions and a corresponding class of starlike functions," *Proceedings of the American Mathematical Society*, vol. 118, no. 1, pp. 189–196, 1993.
- [11] R. M. Ali and V. Ravichandran, "Uniformly convex and uniformly starlike functions," *Mathematics Newsletter*, vol. 21, no. 1, pp. 16–30, 2011.
- [12] A. W. Goodman, *Univalent Functions*, vol. 1, Mariner, Tampa, Fla, USA, 1983.

- [13] P. L. Duren, *Univalent Functions*, vol. 259 of *Grundlehren der Mathematischen Wissenschaften*, Springer, New York, NY, USA, 1983.
- [14] R. Singh and S. Paul, "Linear sums of certain analytic functions," *Proceedings of the American Mathematical Society*, vol. 99, no. 4, pp. 719–725, 1987.
- [15] St. Ruscheweyh and T. Sheil-Small, "Hadamard products of Schlicht functions and the Pólya-Schoenberg conjecture," *Commentarii Mathematici Helvetici*, vol. 48, pp. 119–135, 1973.