Research Article

On Exponential Stability for a Class of Uncertain Neutral Markovian Jump Systems with Mode-Dependent Delays

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The exponential stability of neutral Markovian jump systems with interval mode-dependent time-varying delays, nonlinear perturbations, and partially known transition rates is investigated. A novel augmented stochastic Lyapunov functional is constructed, which employs the improved bounding technique and contains triple-integral terms to reduce conservativeness; then the delay-range-dependent and rate-dependent exponential stability criteria are developed by Lyapunov stability theory, reciprocally convex lemma, and free-weighting matrices. The corresponding results are extended to the uncertain case. Finally, numerical examples are given to illustrate the effectiveness of the proposed methods.

1. Introduction

Delay differential equations or systems are assuming an increasingly important role in many disciplines like mathematics, science, and engineering. In particular, the stability and stabilization problem for neutral delay differential dynamic systems have received considerable attention during the decades and neutral time-delay systems have been the focus of the research community, which are often encountered in such practical situations as distributed networks, population ecology, processes including steam or heat exchanges [1], and robots in contact with rigid environments [2]. Existing results can be roughly classified into two categories, delay-independent criteria and delay-dependent criteria, where the latter is generally regarded as less conservative. Moreover, since the derivative of the delayed state is involved, it should be pointed out that the stability of neutral time-delay systems is more difficult to tackle, which is identical with singular systems [3, 4]. The stability problem of them is more complicated than that for regular systems because more factors need to be considered. In the past decades, considerable attention has been devoted to the robust delay-independent stability and delay-dependent stability of linear neutral systems, which are mainly obtained based on the Lyapunov-Krasovskii (L-K) method [5–11], and references therein. It should be noted that the delay-partitioning approach is used in [6–8]. Furthermore, when nonlinear perturbations or parameter uncertainties appear in neutral systems, some results on stability analysis have been also presented [12–18]. Various techniques have been proposed in these papers, for example, model transformation techniques, the improved bounding techniques, and matrix decomposition approaches. In particular, He et al. [18] propose a new method for dealing with time-delay systems, which employs free weighting matrices to express the relationships between the terms in the Newton-Leibniz formula and has brought novel results. However, these results have conservativeness to some extent, which exist room for further improvement.

In another line, Markovian jump systems (MJSs) have attracted much attention during the past few decades since its first introduction by Krasovskii and Lidskii in 1961, which can be regarded as a special class of hybrid systems with finite operation modes whose structures are subject to random abrupt changes. The system parameters usually jump among finite modes, and the mode switching is governed by a Markov process. MJSs have many applications, such as failure
prone manufacturing systems, power systems, solar thermal central receivers, robotic manipulator systems, aircraft control systems, and economic systems. A large number of results on estimation and control problems related to such systems have been reported in the literature; see, for example, [19–25] and references therein for more details. However, these lines of literature about the transition probabilities in the jumping process have been assumed to be completely accessible. The ideal assumption on the transition probabilities inevitably limits the application of the traditional Markovian jump system theory. Actually, the likelihood of obtaining such available knowledge is questionable, and the cost may be very expensive. Thus, it is really significant and meaningful, from control perspectives, to further study more general jump systems with partially known transition rates. Recently, many results on the Markovian jump systems with partially known transition rates are obtained [26–31]. Most of these improved results just require some free matrices or the knowledge of the known elements in transition rate matrix, such as the structures of uncertainties, and some else of the unknown elements need not be considered. It is a great progress on the analysis of Markovian jump systems. However, few of these results are concerned with neutral Markovian jump systems with mode-dependent time-varying delays and perturbations. To the best of the authors’ knowledge, neutral Markovian jump systems with mode-dependent time-varying delays and partially known transition rates have not been fully investigated, and it is very challenging, especially when nonlinear perturbations exist. Besides, seeking and proposing less conservative delay-range-dependent criterion for uncertain neutral MJJs with nonlinear perturbations and partially known transition rates to desired performance are still open problems. These facts thus motivate our study.

In this paper, the investigated neutral Markovian jump systems are more general than the neutral MJJs with completely known or completely unknown transition rates, which can be viewed as two special cases of the ones tackled here. Specifically, a new augmented stochastic Lyapunov functional containing triple-term is constructed by dividing the delay interval into two subintervals, and then the delay-range-dependent and rate-dependent exponential stability criteria are obtained by reciprocally convex lemma and free weighting matrices. We further extend the criteria to the uncertain case. All the obtained results are presented in terms of LMIs that can be solved numerically. The remainder of the paper is organized as follows. Section 2 presents the problem and preliminaries. Section 3 gives the main results, which are then verified by numerical examples in Section 4. Section 5 concludes the paper.

Notations. The following notations are used throughout the paper. \( \mathbb{R}^n \) denotes the \( n \) dimensional Euclidean space and \( \mathbb{R}^{m \times n} \) is the set of all \( m \times n \) matrices. \( X < Y \) (\( X > Y \)), where \( X \) and \( Y \) are both symmetric matrices, means that \( X - Y \) is negative (positive) definite. \( I \) is the identity matrix with proper dimensions. For a symmetric block matrix, we use \( \ast \) to denote the terms introduced by symmetry. \( \mathcal{E} \) stands for the mathematical expectation, \( \| v \| \) is the Euclidean norm of vector \( v \), \( \| v \| = (v^T v)^{1/2} \), while \( \| A \| \) is spectral norm of matrix \( A \), \( \| A \| = [\lambda_{\max}(A^T A)]^{1/2} \). \( \lambda_{\max}(A) \) is the eigenvalue of matrix \( A \) with maximum (minimum) real part. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

2. Problem Statement and Preliminaries

Given a probability space \( \{\Omega, \mathcal{F}, P\} \) where \( \Omega \) is the sample space, \( \mathcal{F} \) is the algebra of events and \( P \) is the probability measure defined on \( \mathcal{F} \). \( \{r_i, t \geq 0\} \) is a homogeneous, finite-state Markovian process with right continuous trajectories taking values in a finite set \( S = \{1, 2, 3, \ldots, N\} \), with the mode transition probability matrix being

\[
P(r_{i+\Delta t} = j \mid r_i = i) = \begin{cases} 
\pi_{ij}\Delta t + o(\Delta t), & i \neq j, \\
1 + \pi_{ii}\Delta t + o(\Delta t), & i = j, 
\end{cases}
\]

where \( \Delta t > 0 \), \( \lim_{\Delta t \to 0} o(\Delta t)/\Delta t = 0 \), \( \pi_{ij} \geq 0 \) (\( i, j \in S, i \neq j \)) is the transition rate from mode \( i \) to \( j \) and for any state or mode \( i \in S \); it satisfies

\[
\pi_{ii} = -\sum_{j=1, j \neq i}^{N} \pi_{ij}.
\]

Since the transition rates of the Markov chain are partially known in this paper, some elements in matrix \( \Pi = [\pi_{ij}]_{N \times N} \) are inaccessible. For instance, the system with five operation modes, the jump rates matrix \( \Pi \) may be viewed as

\[
\pi_1 = \begin{bmatrix}
? & \pi_{12} & ? & ? & \pi_{15} \\
\pi_{21} & ? & ? & \pi_{24} & ? \\
? & \pi_{32} & ? & \pi_{34} & ? \\
? & ? & \pi_{42} & ? & ? \\
? & ? & ? & ? & \pi_{55}
\end{bmatrix},
\]

where ? represents the unknown element. For notation clarity, we denote \( \delta_k^i = \delta_k^i \cup \delta_{uk}^i \) for all \( i \in S \) and

\[
\delta_k^i \triangleq \begin{cases} 
j : \pi_{ij} \text{ is known for } j \in S, 
\end{cases}
\]

\[
\delta_{uk}^i \triangleq \begin{cases} 
j : \pi_{ij} \text{ is unknown for } j \in S. 
\end{cases}
\]

If \( \delta_k^i \neq \emptyset \), it is further described as

\[
\delta_k^i = \begin{bmatrix} k_1, k_2, \ldots, k_m \end{bmatrix}, \quad 1 \leq m \leq N,
\]

where \( k_j, (j = 1, 2, \ldots, m) \) represent the \( j \)th known element of the set \( \delta_k^i \) in the \( i \)th row of the transition rate matrix \( \Pi \). Furthermore, let \( \pi_{ii}^L \) and \( \pi_{ii}^U \) be the lower and upper bound for the diagonal elements of the jump rates matrix \( \Pi \).

In this paper, the following uncertain neutral Markovian jump systems with mode-dependent interval time-varying
Abstract and Applied Analysis delays, nonlinear perturbations, and partially known transition rates over the space \( \Omega, \mathcal{F}, \mathcal{P} \) are considered:
\[
\dot{x}(t) - C(t, r_{i}) \dot{x}(t - \tau(t, r_{i})) = A(t, r_{i}) x(t) + B(t, r_{i}) x(t - d(t, r_{i})) + D(t, r_{i}) f_{1}(x(t), t) + E(t, r_{i}) f_{2}(x(t - d(t, r_{i})), t) + F(t, r_{i}) f_{3}(\dot{x}(t - \tau(t, r_{i})), t),
\]
\[
x(s) = \varphi(s), \quad r_{i} = 0, \quad s \in [-\varsigma, 0],
\]
where \( x(t) \in \mathbb{R}^{n} \) is the system state and \( \tau(t, r_{i}) \) is mode-dependent interval time-varying neutral delay which satisfies \( 0 \leq \tau_{i}(t) \leq \overline{\tau}_{i}(t) \leq \gamma_{i} \) when \( r_{i} = i \in S \). The mode-dependent interval time-varying retarded delay \( d(t, r_{i}) \) is assumed that
\[
0 \leq d_{i_{1}} \leq d_{i_{2}} \leq d_{i_{3}}, \quad \max_{i \in S} \{d_{i_{1}}\} \leq \min_{i \in S} \{d_{i_{2}}\},
\]
where \( \overline{\tau}_{i}, \gamma_{i} \geq 0, d_{i_{1}}, d_{i_{2}}, d_{i_{3}}, \mu_{i_{1}} \geq 0, \) and \( \varsigma = \max_{i \in S} \{\overline{\tau}_{i}, d_{i_{2}}\} \) are real constant scalars. The initial condition \( \varphi(s) \) is a continuously differentiable vector-valued function. \( f_{1}(x(t), t) \in \mathbb{R}^{n}, f_{2}(x(t - d(t, r_{i})), t) \in \mathbb{R}^{n}, \) and \( f_{3}(\dot{x}(t - \tau(t, r_{i})), t) \in \mathbb{R}^{n} \) are unknown nonlinear perturbations which are, with respect to the current state \( x(t) \), the delayed state \( x(t - d(t, r_{i})) \) and the neutral delay state \( \dot{x}(t - \tau(t, r_{i})) \), respectively. For all \( t \) and \( r_{i} = i \in S \), they are assumed to be bounded in magnitude as
\[
\begin{align*}
\| f_{1}(x(t), t) \| & \leq \alpha \| x(t) \|, \\
\| f_{2}(x(t - d_{i}(t)), t) \| & \leq \beta \| x(t - d_{i}(t)) \|, \\
\| f_{3}(\dot{x}(t - \tau_{i}(t)), t) \| & \leq \gamma \| \dot{x}(t - \tau_{i}(t)) \|,
\end{align*}
\]
where \( \alpha \geq 0, \beta \geq 0, \) and \( \gamma \geq 0 \) are given constants, for simplicity, \( f_{1} \equiv f_{1}(x(t), t), f_{2} \equiv f_{2}(x(t - d_{i}(t)), t), \) and \( f_{3} \equiv f_{3}(\dot{x}(t - \tau_{i}(t)), t) \).

For notational simplicity further, when \( r_{i} = i \in S \), the parametric matrices \( A(t, r_{i}) \in \mathbb{R}^{n \times n}, B(t, r_{i}) \in \mathbb{R}^{n \times n}, C(t, r_{i}) \in \mathbb{R}^{n \times n}, D(t, r_{i}) \in \mathbb{R}^{n \times n}, E(t, r_{i}) \in \mathbb{R}^{n \times n}, \) and \( F(t, r_{i}) \in \mathbb{R}^{n \times n} \) are denoted by \( A_{i}(t), B_{i}(t), C_{i}(t), D_{i}(t), E_{i}(t), \) and \( F_{i}(t) \), which can be described as
\[
\begin{align*}
A_{i}(t) = A_{i} + \Delta A_{i}(t), & \quad B_{i}(t) = B_{i} + \Delta B_{i}(t), \\
C_{i}(t) = C_{i}, & \quad D_{i}(t) = D_{i} + \Delta D_{i}(t), \\
E_{i}(t) = E_{i} + \Delta E_{i}(t), & \quad F_{i}(t) = F_{i} + \Delta F_{i}(t),
\end{align*}
\]
where \( A_{i}, B_{i}, C_{i}, D_{i}, E_{i}, \) and \( F_{i} \) are known constant matrices with appropriate dimensions. \( \Delta A_{i}(t) \in \mathbb{R}^{n \times n}, \Delta B_{i}(t) \in \mathbb{R}^{n \times n}, \Delta D_{i}(t) \in \mathbb{R}^{n \times n}, \Delta E_{i}(t) \in \mathbb{R}^{n \times n}, \) and \( \Delta F_{i}(t) \in \mathbb{R}^{n \times n} \) are uncertainties. The parametric matrix \( G_{i} \) is less than 1 and the admissible parametric uncertainties satisfy the following condition:
\[
\begin{align*}
\begin{bmatrix}
\Delta A_{i}(t) & \Delta B_{i}(t) & \Delta D_{i}(t) & \Delta E_{i}(t) & \Delta F_{i}(t)
\end{bmatrix} = L_{i} H_{i}(t) \begin{bmatrix}
N_{A_{i}} & N_{B_{i}} & N_{D_{i}} & N_{E_{i}} & N_{F_{i}}
\end{bmatrix},
\end{align*}
\]
where \( L_{i}, N_{A_{i}}, N_{B_{i}}, N_{D_{i}}, N_{E_{i}}, \) and \( N_{F_{i}} \) are known mode-dependent constant matrices with appropriate dimensions and \( H_{i}(t) \) is an unknown and time-varying matrix satisfying
\[
H_{i}^{T}(t) H_{i}(t) \leq I, \quad \forall t.
\]
Particularly, the following nominal systems can be obtained for \( H_{i}(t) = 0 \):
\[
\dot{x}(t) - C_{i} \dot{x}(t - \tau_{i}(t)) = A_{i} x(t) + B_{i} x(t - d_{i}(t)) + D_{i} f_{1} + E_{i} f_{2} + F_{i} f_{3}.
\]
Before proceeding with the main results, we present the following assumptions, definitions, and lemmas.

**Assumption 1.** System matrices \( A_{i} \) (for all \( i \in S \)), are Hurwitz and all the eigenvalues have negative real parts for each mode. \( L_{i} \) (for all \( i \in S \)), is full rank in row.

**Assumption 2.** The Markov process is irreducible and the system mode \( r_{i} \) is available at time \( t \).

**Definition 3** (see [32]). Define operator \( \mathcal{D} : C([-\varsigma, 0], \mathbb{R}^{n}) \to \mathbb{R}^{n} \) as \( \mathcal{D}(x_{t}) = x(t) - Cx(t - \tau) \). \( \mathcal{D} \) is said to be stable if the homogeneous difference equation
\[
\mathcal{D}(x_{t}) = 0, \quad t \geq 0,
\]
\[
x_{0} = \psi \in \mathcal{B} \in C([-\varsigma, 0], \mathbb{R}^{n}) : \mathcal{D}(x) = 0
\]
is uniformly asymptotically stable. In this paper, that is, \( \| C_{i} \| \geq 1 \).

**Definition 4** (see [33]). The system in (6) is exponentially stable with a decay rate \( \varepsilon \) for all \( r_{i} = i \in S \), if there exist scalars \( \varepsilon > 0 \) and \( \kappa \geq 1 \) such that for all \( x(t) \),
\[
\| x(t) \| \leq \kappa \exp \{-\varepsilon (t - t_{0})\} \| x_{0} \|,
\]
where \( \varepsilon \) is the exponential decay rate, \( \| * \| \) denotes the Euclidean norm, and
\[
\| x_{t} \|_{c} = \sup_{\theta \in [\varsigma, 0]} \left\{ \| x(t_{0} + \theta) \|, \| \dot{x}(t_{0} + \theta) \| \right\}
\]
\[
= \sup_{\theta \in [\varsigma, \varsigma]} \left\{ \| x(t) \|, \| \dot{x}(t) \| \right\}.
\]

**Definition 5** (see [34]). Define the stochastic Lyapunov-Krasovskii function of system (6) as \( V(x(t), r_{i} = i, t > 0) = V(x_{i}, t, t) \), where its infinitesimal generator is defined as
\[
\Gamma V(x(t), i, t)
\]
\[
= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{E} \left[ V(x(t + \Delta t), r_{i} + \Delta r_{i}, t + \Delta t) \mid x(t) = x, \right]
\]
\[
= \frac{\partial}{\partial t} V(x(t), i, t) + \frac{\partial}{\partial x} V(x(t), i, t) \dot{x}(t)
\]
\[
+ \sum_{j=1}^{N} \pi_{ij} V(x(t), j, t).
\]
Lemma 6 (see [35]). Given constant matrices $\Omega_1, \Omega_2$, and $\Omega_3$, where $\Omega_1 = \Omega_1^T$ and $\Omega_2 = \Omega_2^T > 0$, $\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0$ if and only if
\[
\begin{bmatrix}
\Omega_1 & \Omega_3^T \\
\Omega_3 & -\Omega_2
\end{bmatrix} < 0, \quad \text{or} \quad 
\begin{bmatrix}
-\Omega_2 & \Omega_3^T \\
\Omega_3 & \Omega_1
\end{bmatrix} < 0.
\] (18)

Lemma 7. For any constant matrix $Q = Q^T > 0$, continuous functions $0 \leq h_1(t) \leq h_2(t)$, constant scalars $0 \leq \tau_1 < \tau_2$, and constant $\epsilon > 0$ such that the following integrals are well defined,
\[
\begin{aligned}
(a) \quad & \frac{\exp \left[2\epsilon h_2(t)\right] - \exp \left[2\epsilon h_1(t)\right]}{2\epsilon} \\
& \times \int_{-\tau_2}^{t-\tau_1} \exp \left[2\epsilon (s-t)\right] x^T(s) Q x(s) \, ds \\
& \geq \int_{-\tau_2}^{t-\tau_1} x^T(s) Q \left[\int_{0}^{\tau} \eta(s) \, ds\right] \\
\int_{-\tau_2}^{t-\tau_1} \exp \left[-2\epsilon \tau_2\right] \\
& \times Q \left[\int_{0}^{\tau} \eta(s) \, ds\right].
\end{aligned}
\] (19)
\[
\begin{aligned}
(b) \quad & \frac{\tau_2^2 - \tau_1^2}{2} \\
& \int_{-\tau_2}^{-\tau_1} \int_{-\tau_2}^{\tau_2} \exp \left[2\epsilon (s-t)\right] x^T(s) Q x(s) \, ds \, d\theta \\
& \geq \exp \left[-2\epsilon \tau_2\right] \\
& \times Q \left[\int_{0}^{\tau} \eta(s) \, ds\right].
\end{aligned}
\] (20)

Proof. (a) Is directly obtained from [36]. In addition, from $-\tau_2 \leq \theta \leq -\tau_1$ and $t + \theta \leq s \leq t$, it is held that $-\tau_2 \leq \theta \leq s - t \leq 0$. Then
\[
\int_{-\tau_2}^{-\tau_1} \int_{-\tau_2}^{\tau_2} \exp \left[2\epsilon (s-t)\right] x^T(s) Q x(s) \, ds \, d\theta
\] (21)
\[
\begin{aligned}
\int_{-\tau_2}^{-\tau_1} \int_{-\tau_2}^{\tau_2} \exp \left[-2\epsilon \tau_2\right] \\
& \times Q \left[\int_{0}^{\tau} \eta(s) \, ds\right].
\end{aligned}
\] (b) Is thus true by [37].

Lemma 8 (see [38]). For functions $\lambda_1(t), \lambda_3(t) \in [0, 1]$, $\lambda_1(t) + \lambda_3(t) = 1$, and $\eta_1 = 0$ with $\lambda_1(t) = 0$ and $\eta_2 = 0$ with $\lambda_3(t) = 0$, matrices $P > 0$, $Q > 0$, then there exists matrix $T$ such that
\[
\begin{bmatrix}
P & T \\
T^T & Q
\end{bmatrix} > 0
\] (22)

and the following inequality holds:
\[
\frac{1}{\lambda_1(t)} \eta_1 P \eta_1^T + \frac{1}{\lambda_3(t)} \eta_2 Q \eta_2^T \\
\geq \eta_1 \eta_2 \begin{bmatrix}
P & T \\
T^T & Q
\end{bmatrix} \begin{bmatrix}
\eta_1^T \\
\eta_2^T
\end{bmatrix}.
\] (23)

Lemma 9 (see [39]). For given matrices $Q = Q^T$, $M$, and $N$ with appropriate dimensions,
\[
Q + MF(t) N + N^T F^T(t) M^T < 0
\] (24)

for all $F(t)$ satisfying $F^T(t) F(t) \leq I$, if and only if there exists a scalar $\delta > 0$, such that
\[
Q + \delta^{-1} MM^T + \delta NN^T < 0.
\] (25)

3. Main Results

This section will state the exponential stability analysis for neutral Markovian jump systems with mode-dependent interval time-varying delays, nonlinear perturbations, and partially known transition rates. With creative Lyapunov functional and novel matrix inequalities analysis, delay-range-dependent and rate-dependent exponential stability conditions are presented.

3.1. Exponential Stability for the Nominal Systems

Theorem 10. For given scalars $\pi_i^m, \pi_i^M, \alpha, \beta, \gamma, \epsilon, \tau_i, \nu_i, d_{ij}, d_{2j}, \mu$, and constant scalar $d_{mi}$ satisfying $d_{ij} < d_{mi} < d_{2j}$, the systems described by (13) with partially known transition rates are exponentially stable with decay rate $\epsilon \in (0, 1]$ and there exist symmetric positive matrices $P_i > 0$, $Q_{ii} > 0$, $Q_{2i} > 0$, $R_{ii} > 0$, $T_{ii} > 0$, $(i \in S)$, $Q_{j} > 0$, $(j = 3, 4)$, $R_{k} > 0$, $T_{j} > 0$, $(k, l = 2, 3, 4)$, $U_{m} > 0$, $V_{n} > 0$, $W_{s} > 0$, $(m, n, s = 1, 2, 3, 4)$ and matrices $M_{1}, M_{2}, N_{1}, N_{2}, N_{3}, N_{4}$ for any scalars $\alpha_i, \beta_i, \gamma_i$, any symmetric matrices $X_i, Y_{1i}, Y_{2i}, Z_{1i}, Z_{2i}, (i \in S)$ and any matrices $I_k, (k = 1, 2, \ldots, 24)$ with appropriate dimensions, such that the following linear matrix inequalities hold.

When $i \in \delta^i_k$
\[
\sum_{j \in \delta^i_k \setminus \{i\}} \pi_{ij} (Q_{ij} - Y_{ij}) - \pi_{ij} Y_{ij} \leq 0,
\]
\[
\sum_{j \in \delta^i_k \setminus \{i\}} \pi_{ij} (Q_{2j} - Y_{2ij}) - \pi_{ij} Y_{2ij} \leq 0,
\]
\[
\sum_{j \in \delta^i_k \setminus \{i\}} \pi_{ij} (R_{ij} - Z_{ij}) - \pi_{ij} Z_{ij} \leq 0,
\]
\[
\sum_{j \in \delta^i_k \setminus \{i\}} \pi_{ij} (T_{ij} - Z_{2ij}) - \pi_{ij} Z_{2ij} \leq 0,
\] (26)

and the following inequalities hold:
\[
\begin{aligned}
P_j - X_i \leq 0, & \quad j \in \delta^i_k, \\
Q_{ij} - Y_{ij} \leq 0, & \quad j \in \delta^i_k, \\
Q_{2j} - Y_{2ij} \leq 0, & \quad j \in \delta^i_k, \\
R_{ij} - Z_{ij} \leq 0, & \quad j \in \delta^i_k, \\
T_{ij} - Z_{2ij} \leq 0, & \quad j \in \delta^i_k.
\end{aligned}
\]
When \( i \in \mathcal{S}_{uk} \)

\[
\sum_{j \in \mathcal{S}_{uk}} \pi_{ij} (Q_{ij} - Y_{ii}) \leq 0; \sum_{j \in \mathcal{S}_{uk}} \pi_{ij} (Q_{2j} - Y_{2i}) \leq 0, (27)
\]

\[
\sum_{j \in \mathcal{S}_{uk}} \pi_{ij} (R_{ij} - Z_{ii}) \leq 0; \sum_{j \in \mathcal{S}_{uk}} \pi_{ij} (T_{ij} - Z_{2i}) \leq 0, (28)
\]

\[
P_j - X_i \leq 0, \quad j \in \mathcal{S}_{uk}, \quad j \neq i,
\]

\[
P_j - X_i \geq 0, \quad j \in \mathcal{S}_{uk}, \quad j = i,
\]

\[
- \sum_{j \in \mathcal{S}_{uk}, j \neq i} \pi_{ij}^M (Q_{1j} - Y_{1j}) - \pi_{ij}^M Y_{ii} \leq 0,
\]

\[
- \sum_{j \in \mathcal{S}_{uk}, j \neq i} \pi_{ij}^M (Q_{2j} - Y_{2i}) - \pi_{ij}^M Y_{ii} \leq 0,
\]

\[
- \sum_{j \in \mathcal{S}_{uk}, j \neq i} \pi_{ij}^M (R_{ij} - Z_{ii}) - \pi_{ij}^M Z_{ii} \leq 0,
\]

\[
- \sum_{j \in \mathcal{S}_{uk}, j \neq i} \pi_{ij}^M (T_{ij} - Z_{2i}) - \pi_{ij}^M Z_{2i} \leq 0,
\]

\[
\begin{bmatrix}
U_1 & M_1 \\
M_1^T & U_1
\end{bmatrix}
> 0, \quad \begin{bmatrix}
V_1 & M_2 \\
M_2^T & V_1
\end{bmatrix}
> 0,
\]

\[
\text{(i)} \quad \begin{bmatrix}
U_3 & N_1 \\
N_1^T & U_3
\end{bmatrix}
> 0, \quad \begin{bmatrix}
V_3 & N_2 \\
N_2^T & V_3
\end{bmatrix}
> 0,
\]

\[
\text{(ii)} \quad \Omega_{i0} + \Omega_{i1} < 0,
\]

\[
\begin{bmatrix}
U_4 & N_3 \\
N_3^T & U_4
\end{bmatrix}
> 0, \quad \begin{bmatrix}
V_4 & N_4 \\
N_4^T & V_4
\end{bmatrix}
> 0,
\]

\[
\text{(iv)} \quad \Omega_{i0}^e + \Omega_{i0} < 0,
\]

where

\[
\Omega_{i0}^e = \sum_{m=1}^{24} \eta_m X_m e_m^T + \mathcal{L} (\Xi)
\]

\[
\begin{align*}
&- \frac{2e_{d_{ij}}}{\exp (2e_{d_{ij}})} - 1 \begin{bmatrix}
e_{20} & e_{21} - e_{20}
\end{bmatrix} \\
&- \exp \left[ -2e_{d_{ij}} \right] (\tau e_{1j} - e_{1j}) W_1 \begin{bmatrix}
e_{1j} e_{1j} - e_{1j}
\end{bmatrix} \\
&- \exp \left[ -2e_{d_{ij}} \right] (d_{ij} e_{1j} - e_{1j}) W_2 \begin{bmatrix}
e_{1j} e_{1j} - e_{1j}
\end{bmatrix} \\
&- \exp \left[ -2e_{d_{ij}} \right] (e_{1j} e_{1j} - e_{1j}) W_3 \begin{bmatrix}
e_{1j} e_{1j} - e_{1j}
\end{bmatrix} \\
&- \exp \left[ -2e_{d_{ij}} \right] (e_{1j} e_{1j} - e_{1j}) W_4 \begin{bmatrix}
e_{1j} e_{1j} - e_{1j}
\end{bmatrix} \\
&- \frac{2e_{T_i}}{\exp (2e_{T_i})} - 1 \begin{bmatrix}
e_{20} & e_{21} - e_{20}
\end{bmatrix}
\end{align*}
\]

\[
\times \begin{bmatrix}
U_1 & M_1 \\
M_1^T & U_1
\end{bmatrix}
\begin{bmatrix}
e_{20} \\
e_{21} - e_{20}
\end{bmatrix}
\times \begin{bmatrix}
V_1 & M_2 \\
M_2^T & V_1
\end{bmatrix}
\begin{bmatrix}
e_{18} \\
e_{18} - e_{16}
\end{bmatrix},
\]

where \( \mathcal{L} \) is a linear operator on \( \mathbb{R}^{n \times n} \) by

\[
\mathcal{L} (G) = G + G^T, \quad \forall G \in \mathbb{R}^{n \times n},
\]

\[
\Xi = -e_1 e_1^T + e_1 (P_i B_i + J_i B_i) e_1^T
\]

\[
+ e_1 (P_i C_i) e_1^T + \sum_{m=2}^{24} e_1 (A_i^T J_m) e_m^T
\]

\[
+ e_1 P_i (D_i e_{23} + E_i e_{23} + F_i e_{24})
\]

\[
+ e_2 (J_i B_i) e_2^T - \sum_{m=3}^{24} e_2 (J_m^T) e_m^T
\]

\[
+ \sum_{m=4}^{24} e_3 (B_i^T J_m) e_m^T + \sum_{m=1}^{18} e_m (J_m C_i) e_1^T
\]

\[
+ \sum_{m=20}^{24} e_{19} (C_i^T J_m) e_m^T
\]

\[
+ \sum_{m=1}^{21} e_m J_m (D_i e_{23} + E_i e_{23} + F_i e_{24})
\]

\[
+ e_{21} J_{22} (E_i e_{23} + F_i e_{24}) + \sum_{m=23}^{24} e_{22} (D_i^T J_m) e_m^T
\]

\[
+ e_{23} (J_{23} F_i) e_{23}^T + e_{23} (F_i^T J_{24}) e_{24}^T
\]

\[
Y_1 = A_i^T P_i + P_i A_i + \sum_{j \in \mathcal{S}_{uk}} \pi_{ij} (P_j - X_j)
\]

\[
+ 2e P_i + Q_i + Q_i + R_i + V_i + d_{ii}^2 U_2
\]

\[
+ e_i^2 U_3 + e_i^2 U_4 + e_i \alpha^2 I + J_i A_i + A_i^T I
\]

\[
Y_2 = Q_{ii} + Q_i + T_i + \tau V_i + d_{ii}^2 U_2
\]

\[
+ e_i^2 V_3 + e_i^2 V_4 + \frac{\tau^2}{4} W_i + \frac{d_{ii}^2}{4} W_2
\]

\[
+ e_i^2 W_3 + e_i^2 W_4 - I - J_i^2
\]

\[
Y_3 = \alpha (e, \mu_i) R_{ii} + e_i \beta^2 I + J_i B_i + B_i^T J_i
\]

\[
Y_4 = \alpha (e, \mu_i) T_{ii}
\]

\[
Y_5 = \exp \left[ -2e_{d_{ij}} \right] (R_{ij} + R_j - R_i)
\]

\[
Y_6 = \exp \left[ -2e_{d_{ij}} \right] (R_4 - R_3)
\]

\[
Y_7 = -\exp \left[ -2e_{d_{ij}} \right] R_4
\]
\[
Y_6 = \exp \{-2\epsilon d_{1i}\} (T_{1i} + T_3 - T_2),
\]
\[
Y_9 = \exp \{-2\epsilon d_{mi}\} (T_4 - T_3),
\]
\[
Y_{10} = -\exp \{-2\epsilon d_{2i}\} T_4,
\]
\[
Y_{11} = -\exp \{-2\epsilon d_{2i}\} T_4,
\]
\[
\lambda = \min_{i\in S} \{\lambda_{\max}(P_i)\}, \quad e_i = U_{ij} e_{ij}, \quad \epsilon_i = d_{mi} - d_{ui}, \quad e_{ij} = d_{2i} - d_{mi}, \quad e_{ij} = \frac{1}{2} (d_{mi} - d_{ui}), \quad e_i = \frac{1}{2} (d_{2i} - d_{mi}).
\]

where \(e_i \mid i = 1, 2, \ldots, 24\) are block entry matrices; that is,
Proof. Construct the following stochastic Lyapunov functional:

\[
V(x(t), i, t) = \sum_{k=1}^{7} V_k(x_i, i),
\]

where

\[
V_1(x_i, i) = x^T(t) P_i x(t),
\]

\[
V_2(x_i, i) = \int_{t-	au_i(t)}^{t} \exp{\{2\epsilon (s-t)\}} x^T(s) Q_i x(s) \, ds + \int_{t-	au_i(t)}^{t} \exp{\{2\epsilon (s-t)\}} x^T(s) Q_{i2} \dot{x}(s) \, ds + \int_{t-	au_i(t)}^{t} \exp{\{2\epsilon (s-t)\}} x^T(s) Q_{i3} \dot{x}(s) \, ds + \int_{t-	au_i(t)}^{t} \exp{\{2\epsilon (s-t)\}} x^T(s) Q_{i4} \dot{x}(s) \, ds,
\]

\[
V_3(x_i, i) = \int_{t-d_{i2}(t)}^{t-d_{i3}(t)} \exp{\{2\epsilon (s-t)\}} x^T(s) R_{i1} x(s) \, ds + \int_{t-d_{i2}(t)}^{t-d_{i3}(t)} \exp{\{2\epsilon (s-t)\}} x^T(s) R_{i2} \dot{x}(s) \, ds + \int_{t-d_{i2}(t)}^{t-d_{i3}(t)} \exp{\{2\epsilon (s-t)\}} x^T(s) R_{i3} \dot{x}(s) \, ds + \int_{t-d_{i2}(t)}^{t-d_{i3}(t)} \exp{\{2\epsilon (s-t)\}} x^T(s) R_{i4} \dot{x}(s) \, ds,
\]

\[
V_4(x_i, i) = \int_{t-d_{i2}(t)}^{t-d_{i3}(t)} \exp{\{2\epsilon (s-t)\}} \dot{x}^T(s) T_{i1} \dot{x}(s) \, ds + \int_{t-d_{i2}(t)}^{t-d_{i3}(t)} \exp{\{2\epsilon (s-t)\}} \dot{x}^T(s) T_{i2} \dot{x}(s) \, ds + \int_{t-d_{i2}(t)}^{t-d_{i3}(t)} \exp{\{2\epsilon (s-t)\}} \dot{x}^T(s) T_{i3} \dot{x}(s) \, ds + \int_{t-d_{i2}(t)}^{t-d_{i3}(t)} \exp{\{2\epsilon (s-t)\}} \dot{x}^T(s) T_{i4} \dot{x}(s) \, ds,
\]

\[
V_5(x_i, i) = \int_{t-d_{i2}(t)}^{t-d_{i3}(t)} \int_{t+\theta}^{t} \exp{\{2\epsilon (s-t)\}} \dot{x}^T(s) U_{i1} x(s) \, ds \, d\theta + \int_{t-d_{i2}(t)}^{t-d_{i3}(t)} \int_{t+\theta}^{t} \exp{\{2\epsilon (s-t)\}} \dot{x}^T(s) U_{i2} x(s) \, ds \, d\theta + \int_{t-d_{i2}(t)}^{t-d_{i3}(t)} \int_{t+\theta}^{t} \exp{\{2\epsilon (s-t)\}} \dot{x}^T(s) U_{i3} x(s) \, ds \, d\theta + \int_{t-d_{i2}(t)}^{t-d_{i3}(t)} \int_{t+\theta}^{t} \exp{\{2\epsilon (s-t)\}} \dot{x}^T(s) U_{i4} x(s) \, ds \, d\theta,
\]

\[
V_6(x_i, i) = \int_{t-d_{i2}(t)}^{t-d_{i3}(t)} \int_{t+\theta}^{t} \exp{\{2\epsilon (s-t)\}} \dot{x}^T(s) V_1 x(s) \, ds \, d\theta + \int_{t-d_{i2}(t)}^{t-d_{i3}(t)} \int_{t+\theta}^{t} \exp{\{2\epsilon (s-t)\}} \dot{x}^T(s) V_2 x(s) \, ds \, d\theta + \int_{t-d_{i2}(t)}^{t-d_{i3}(t)} \int_{t+\theta}^{t} \exp{\{2\epsilon (s-t)\}} \dot{x}^T(s) V_3 x(s) \, ds \, d\theta + \int_{t-d_{i2}(t)}^{t-d_{i3}(t)} \int_{t+\theta}^{t} \exp{\{2\epsilon (s-t)\}} \dot{x}^T(s) V_4 x(s) \, ds \, d\theta,
\]

\[
V_7(x_i, i) = \int_{t-d_{i2}(t)}^{t-d_{i3}(t)} \int_{t+\theta}^{t} \exp{\{2\epsilon (s-t)\}} \dot{x}^T(s) W_1 x(s) \, ds \, d\theta + \int_{t-d_{i2}(t)}^{t-d_{i3}(t)} \int_{t+\theta}^{t} \exp{\{2\epsilon (s-t)\}} \dot{x}^T(s) W_2 x(s) \, ds \, d\theta + \int_{t-d_{i2}(t)}^{t-d_{i3}(t)} \int_{t+\theta}^{t} \exp{\{2\epsilon (s-t)\}} \dot{x}^T(s) W_3 x(s) \, ds \, d\theta
\]

Remark 11. It should be pointed out that the proposed stochastic augmented Lyapunov functional (38) contains some triple-integral terms, which has not been used in of the existing literature in the same context before. Compared with the existing ones, [37] has shown that such triple-integral terms are very effective in the reduction of conservativeness.

Taking \( \Gamma \) as its infinitesimal generator along the trajectory of system (13), we obtain the following from Definition 5 and (38)-(45):

\[
\Gamma V(x(t), i, t) = \sum_{k=1}^{7} \Gamma V_k(x_i, i),
\]

\[
\Gamma V_1(x_i, i) \text{ can be easily obtained by the following equation:}
\]

\[
\Gamma V_1(x_i, i) = 2 \left[ x^T(t) A_i^T + x^T(t - d_i(t)) B_i^T \right] + x^T(t - \tau_i(t)) C_i + f_i^T P_i + 2 \epsilon x^T(t) P_i x(t) - 2 \epsilon V_1(x_i, i).
\]
With regard to $\Gamma V_2(x_i, i)$, the detailed procedures are given as follows.

Define

$$V_2(x_i, i) = V_{21}(x_i, i) + V_{22}(x_i, i) + V_{23}(x_i, i) + V_{24}(x_i, i),$$

where

$$V_{21}(x_i, i) = \int_{t-\tau_i}^t \exp\{2\varepsilon(s-t)\} x^T(s) Q_{4i} x(s) \, ds,$$

$$V_{22}(x_i, i) = \int_{t-\tau_i}^t \exp\{2\varepsilon(s-t)\} \dot{x}^T(s) Q_{2i} \dot{x}(s) \, ds,$$

$$V_{23}(x_i, i) = \int_{t-\tau_i}^t \exp\{2\varepsilon(s-t)\} x^T(s) Q_2 x(s) \, ds,$$

$$V_{24}(x_i, i) = \int_{t-\tau_i}^t \exp\{2\varepsilon(s-t)\} \dot{x}^T(s) Q_2 \dot{x}(s) \, ds. \tag{49}$$

By the infinitesimal generator $\Gamma$, we obtain

$$\Gamma V_{21}(x_i, i) = \exp\{-2\varepsilon t\} \times \Gamma \left( \int_{t-\tau_i}^t \exp\{2\varepsilon s\} x^T(s) Q_{4i} x(s) \, ds \right) \tag{50}$$

$$- 2eV_{21}(x_i, i),$$

where

$$\Gamma \left( \int_{t-\tau_i}^t \exp\{2\varepsilon s\} x^T(s) Q_{4i} x(s) \, ds \right) = \lim_{\Delta \to 0^+} \frac{1}{\Delta} \left\{ \mathcal{G} \left[ \int_{t+\Delta-(t+\Delta)}^{t+\Delta} \exp\{2\varepsilon s\} x^T(s) Q_1 (r_{i+\Delta}) \times x(s) \, ds \mid x(t), r_i = i \right] + \int_{t-\tau_i}^t \exp\{2\varepsilon s\} x^T(s) Q_{4i} x(s) \, ds \right\}$$

$$= \lim_{\Delta \to 0^+} \frac{1}{\Delta} \left\{ \sum_{j=1}^N \left( \sum_{\tau_j \leq t \leq \tau_{j+1}} \cdot \exp\{2\varepsilon s\} x^T(s) Q_{4j} x(s) \, ds \right) \right\}$$

Following the same procedure, $\Gamma V_{22}(x_i, i)$ is also obtained:

$$\Gamma V_{22}(x_i, i) = \dot{x}^T(t) Q_{2i} \dot{x}(t) - (1 - \tau_i(t)) \exp\{2\varepsilon(t - \tau_i(t))\} \times \dot{x}^T(t - \tau_i(t)) Q_{2i} \dot{x}(t - \tau_i(t)) \tag{51}$$

Moreover, $\Gamma V_{23}(x_i, i), \Gamma V_{24}(x_i, i)$ are easily calculated as shown in the following:

$$\Gamma V_{23}(x_i, i) = x^T(t - \tau_i(t)) Q_3 x(t) - \exp\{-2\varepsilon \tau_i(t)\} \times x^T(t - \tau_i(t)) Q_3 x(t - \tau_i(t)) - 2eV_{23}(x_i, i) \tag{53}$$

According to (50), (51), (52), and (53), we can easily obtain the following (54):

$$\Gamma V_2(x_i, i)$$

$$= x^T(t) [Q_{1i} + Q_{4i}] x(t)$$

$$+ \dot{x}^T(t) [Q_{2i} + Q_{4i}] \dot{x}(t) - 2eV_2(x_i, i)$$

$$- (1 - \tau_i(t)) \exp\{-2\varepsilon \tau_i(t)\} x^T(t - \tau_i(t)) Q_3 x(t)$$

$$\times (t - \tau_i(t)) - (1 - \tau_i(t)) \exp\{-2\varepsilon \tau_i(t)\} \times \dot{x}^T(t - \tau_i(t)) Q_{2i} \dot{x}(t - \tau_i(t))$$

$$- \exp\{-2\varepsilon \tau_i(t)\} x^T(t - \tau_i(t)) Q_3 x(t - \tau_i(t)) - \exp\{-2\varepsilon \tau_i(t)\} \dot{x}^T(t - \tau_i(t)) Q_{4i} x(t - \tau_i(t))$$

$$+ \sum_{j \in S} \pi_{ij} \int_{t-\tau_i(t)}^t \exp\{2\varepsilon(s-t)\} x^T(s) x(s) + \dot{x}^T(s) Q_{2j} \dot{x} \, ds.$$
\[
\leq x^T(t) [Q_{1i} + Q_3] x(t) \\
+ x^T(t) [Q_{2i} + Q_4] \dot{x}(t) - 2eV_2(x_i,i) \\
- (1 - \tau_i(t)) \exp [-2e\tau_i(t)] x^T(t - \tau_i(t)) Q_{1i} x(t) \\
\times (t - \tau_i(t)) - (1 - \tau_i(t)) \exp [-2e\tau_i(t)] \dot{x}^T \\
\times (t - \tau_i(t)) Q_{2i} \dot{x}(t - \tau_i(t)) \\
- \exp [-2e\tau_i^T] x^T(t - \tau_i) Q_{3i} x(t - \tau_i) \\
- \exp [-2e\tau_i^T] \dot{x}^T(t - \tau_i) Q_{4i} \dot{x}(t - \tau_i) \\
+ \sum_{j\neq i} \int_{t-\tau_i}^{t-\tau_j} \exp[2e(s-t)] \\
\times \left[ x^T(s) Q_{1j} x(s) + \dot{x}^T(s) Q_{2j} \dot{x}(s) \right] ds.
\]

(54)

Then, in the same method, \( \Gamma V_3(x_i,i) \), \( \Gamma V_4(x_i,i) \) can be calculated and the results are given by the following, respectively:

\[
\Gamma V_3(x_i,i) = x^T(t) R_2 x(t) + \exp [-2ed_{1i}] x^T \\
\times (t - d_{1i}) [R_{1i} + R_3 - R_2] x(t - d_{1i}) \\
+ \exp [-2ed_{mi}] x^T(t - d_{mi}) [R_4 - R_3] x(t - d_{mi}) \\
- \exp [-2ed_{2i}] x^T(t - d_{2i}) R_4 x(t - d_{2i}) \\
- 2eV_3(x_i,i) - (1 - d_i(t)) \\
\times \exp [-2e\dot{d}_1(t)] x^T (t - d_i(t)) R_{1i} x(t - d_i(t)) \\
+ \sum_{j\neq i} \int_{t-d_{1i}}^{t-d_{1j}} \exp[2e(s-t)] x^T(s) R_{1j} x(s) ds
\]

\[
\leq x^T(t) R_2 x(t) + \exp [-2ed_{1i}] x^T(t - d_{1i}) \\
\times [R_{1i} + R_3 - R_2] x(t - d_{1i}) \\
+ \exp [-2ed_{mi}] x^T(t - d_{mi}) [R_4 - R_3] x(t - d_{mi}) \\
\times (t - d_{mi}) - \exp [-2ed_{2i}] x^T \\
\times (t - d_{2i}) R_4 x(t - d_{2i}) - 2eV_3(x_i,i) \\
- (1 - d_i(t)) \exp [-2e\dot{d}_1(t)] x^T \\
\times (t - d_i(t)) R_{1i} x(t - d_i(t)) \\
+ \sum_{j\neq i} \int_{t-d_{1i}}^{t-d_{1j}} \exp[2e(s-t)] x^T(s) R_{1j} x(s) ds.
\]

(55)

Moreover, \( \Gamma V_5(x_i,i) \), \( \Gamma V_6(x_i,i) \), and \( \Gamma V_7(x_i,i) \) can be directly obtained as follows:

\[
\Gamma V_5(x_i,i) = x^T(t) \left[ \tilde{r}_1 U_1 + d_{1j}^2 U_2 + \phi_1^2 U_3 + \phi_2^2 U_4 \right] x(t) \\
- 2eV_5(x_i,i) \\
- \int_{t-\tau_i}^{t} \tilde{r}_2 \exp [2e(s-t)] x^T(s) U_{1i} x(s) ds \\
- \int_{t-d_{1i}}^{t} d_{1i} \exp [2e(s-t)] x^T(s) U_{2i} x(s) ds \\
- \int_{t-d_{mi}}^{t} \phi_{1i} \exp [2e(s-t)] x^T(s) U_{3i} x(s) ds \\
- \int_{t-d_{mi}}^{t} \phi_{2i} \exp [2e(s-t)] x^T(s) U_{4i} x(s) ds.
\]
\[ \Gamma V_6 (x_i, i) = \dot{x}^T(t) \left[ \tau_i^2 V_1 + d_{1i}^2 V_2 + \epsilon_{1i}^2 V_3 + \epsilon_{2i}^2 V_4 \right] \dot{x}(t) - 2\epsilon V_6 (x_i, i) - \int_{t-t_i}^t \tau_i \exp \left[ 2\epsilon (s-t) \right] \dot{x}^T(s) V_1 \dot{x}(s) \, ds \]

\[ - \int_{t-d_{1i}}^{t-d_{1i}} \epsilon_{1i} \exp \left[ 2\epsilon (s-t) \right] \dot{x}^T(s) V_2 \dot{x}(s) \, ds \]

\[ - \int_{t-d_{mi}}^{t-d_{mi}} \epsilon_{mi} \exp \left[ 2\epsilon (s-t) \right] \dot{x}^T(s) V_3 \dot{x}(s) \, ds \]

\[ - \int_{t-d_{ni}}^{t-d_{ni}} \epsilon_{ni} \exp \left[ 2\epsilon (s-t) \right] \dot{x}^T(s) V_4 \dot{x}(s) \, ds, \]

\[ \Gamma V_7 (x_i, i) = \dot{x}^T(t) \left[ \frac{\tau_i^4}{4} W_1 + \frac{d_{1i}^4}{4} W_2 + \epsilon_{1i}^4 W_3 + \epsilon_{2i}^4 W_4 \right] \dot{x}(t) - 2\epsilon V_7 (x_i, i) - \int_{t-t_i}^t \frac{\tau_i^2}{2} \exp \left[ 2\epsilon (s-t) \right] \dot{x}^T(s) W_1 \dot{x}(s) \, ds \, d\theta \]

\[ - \int_{t-d_{1i}}^{t-d_{1i}} \frac{d_{1i}^2}{2} \exp \left[ 2\epsilon (s-t) \right] \dot{x}^T(s) W_2 \dot{x}(s) \, ds \, d\theta \]

\[ - \int_{t-d_{mi}}^{t-d_{mi}} \frac{d_{mi}^2}{2} \exp \left[ 2\epsilon (s-t) \right] \dot{x}^T(s) W_3 \dot{x}(s) \, ds \, d\theta \]

\[ - \int_{t-d_{ni}}^{t-d_{ni}} \epsilon_{ni} \exp \left[ 2\epsilon (s-t) \right] \dot{x}^T(s) W_4 \dot{x}(s) \, ds \, d\theta. \]

(57)

Define

\[ \xi(t) = \text{col} \left\{ x(t), \dot{x}(t), x(t-d_1(t)), \dot{x}(t-d_1(t)), f(t), f_1(t), f_2(t), f_3(t) \right\}. \]

(58)

Then, there exist matrices \( J = \text{col} \{ f_k, (k = 1, 2, \ldots, 24) \} \) with appropriate dimensions, such that the following equality holds according to (13):

\[ 2\xi^T(t) J \left[ -\dot{x}(t) + A \dot{x}(t) + B \dot{x}(t-d_1(t)) + C \dot{x}(t-t_1(t)) + D f_1(t) + E f_2(t) + F f_3(t) \right] = 0. \]

(59)

Due to \( \sum_{j=1}^N \pi_j = 0 \), the following zero equations hold for arbitrary matrices \( X_i = X_i^T, \ Y_{j1} = Y_{j1}^T, \ Y_{2i} = Y_{2i}^T, \ Z_{j1} = Z_{j1}^T, \ Z_{2i} = Z_{2i}^T, \ i \in S \); that is,

\[ -\dot{x}^T(t) \left[ \sum_{j \in S_{i}^k} \pi_{ij} X_j + \sum_{j \in S_{ak}} \pi_{ij} X_j \right] x(t) = 0, \]

\[ -\int_{t-t_i}^t \exp \left[ 2\epsilon (s-t) \right] x^T(s) \, ds = 0, \]

\[ -\int_{t-d_{1i}}^{t-d_{1i}} \exp \left[ 2\epsilon (s-t) \right] x^T(s) \, ds = 0, \]

(60)

In view of (9), the following inequalities hold for any scalars \( e_1 > 0, e_2 > 0, \) and \( e_3 > 0 \):

\[ e_1 \left[ \alpha' x^T(t) x(t) - f_1^T(t) x(t), f_2(t) x(t), f_3(t) \right] \geq 0, \]
\( \epsilon_2 \left[ \beta_1^2 x^T(t - d_i(t)) x(t - d_i(t)) - f_1^T x(t - d_i(t), t) f_1 x(t - d_i(t), t) \right] \geq 0, \)

\( \epsilon_3 \left[ \gamma_1^2 x^T(t - \tau_i(t)) \dot{x}(t - \tau_i(t)) - f_3^T x(t - \tau_i(t), t) f_3 x(t - \tau_i(t), t) \right] \geq 0. \)

(61)

From (46) and (61), we have

\[
\Gamma V(x(t), i, t) \leq \sum_{k=1}^{g} \Gamma V_k(x_t, i) + e_1 \left[ \alpha^2 x^T(t) x(t) - f_1^T f_1 \right] + e_2 \left[ \beta^2 x^T(t - d_i(t)) x(t - d_i(t)) - f_2^T f_2 \right] + e_3 \left[ \gamma^2 x^T(t - \tau_i(t)) \dot{x}(t - \tau_i(t)) - f_3^T f_3 \right].
\]

(62)

Since it is easy to see \(-1 - \eta(t)\) \(\exp \{-2\epsilon \eta(t)\} \leq -1 - \gamma(t)\), then from (8) we also obtain that

\[
-
\left(1 - d_i(t)\right) \exp \{-2\epsilon d_i(t)\} \leq \omega(\epsilon, \eta_i),
\]

(63)

where \(\omega(\epsilon, \eta_i)\) is defined in Theorem 10. Notice (a) of Lemma 7, then

\[
- \int_{t-d_{d_i}}^{t} d_i \exp \{2\epsilon (s - t)\} x^T(s) U_2 x(s) \, ds
\leq - \frac{2\epsilon d_i}{\exp \{2\epsilon d_i\} - 1} \chi(t) e_1 U_2 \chi_{e_{11}}^T \xi(t),
\]

(64)

Notice (b) of Lemma 7, then

\[
- \int_{t-d_{d_i}}^{t} d_i \exp \{2\epsilon (s - t)\} x^T(s) V_2 \dot{x}(s) \, ds
\leq - \frac{2\epsilon d_i}{\exp \{2\epsilon d_i\} - 1} \chi(t) \left( e_1 - e_5 \right) V_2 \left( e_1^T - e_5^T \right) \xi(t).
\]

(65)

For \(d_i(t) \in [d_{d_{11}}, d_{d_{11}}]\), the following is held from (a) of Lemma 7:

\[
- \int_{t-d_{d_{11}}}^{t} d_{11} \exp \{2\epsilon (s - t)\} \dot{x}(s) V_3 \dot{x}(s) \, ds
= - \left( \int_{t-d_{d_{11}}}^{t} \frac{d_{11}}{\exp \{2\epsilon d_{11}\} - 1} \chi(t) \exp \{2\epsilon d_{11}\} \right) \exp \{2\epsilon d_{11}\} \chi(t)
\]

\[
- \int_{t-d_{d_{11}}}^{t} d_{d_{11}} \exp \{2\epsilon (s - t)\} x^T(s) V_3 \dot{x}(s) \, ds
\leq - \frac{2\epsilon d_{11}}{\exp \{2\epsilon d_{11}\} - 1} \chi(t) \left( e_1 - e_5 \right) V_2 \left( e_1^T - e_5^T \right) \xi(t).
\]

(66)

where

\[
\lambda_{11}(t) = \frac{\exp \{2\epsilon d_{11}(t)\} - \exp \{2\epsilon d_{11}\}}{\exp \{2\epsilon d_{11}\} - \exp \{2\epsilon d_{11}\}},
\]

\[
\lambda_{21}(t) = \frac{\exp \{2\epsilon d_{11}(t)\} - \exp \{2\epsilon d_{11}\}}{\exp \{2\epsilon d_{11}\} - \exp \{2\epsilon d_{11}\}}.
\]

(67)
By Lemma 8, there exists matrix $N_2$ with appropriate dimensions such that

$$-\int_{t-d_{12}}^{t} Q_{ij} \exp \{2e(s - t)\} x^T(s) V_j x(s) \; ds \leq - \frac{2e Q_{ij}}{\exp \{2ed_{mi} \} - \exp \{2ed_{ii} \}} \xi^T(t) \times [e_{12} \; e_{14} - e_{12}] \left[ \begin{array}{c}
U_3 \\ N_1^T \\ N_2^T \\ U_3 \end{array} \right] > 0.$$  \hspace{1cm} (68)

Similarly, considering $-\int_{t-d_{12}}^{t} Q_{ij} \exp \{2e(s - t)\} x^T(s) U_j x(s) \; ds$ and following the same procedure, there exists matrix $N_j$ with appropriate dimensions such that

$$-\int_{t-d_{12}}^{t} Q_{ij} \exp \{2e(s - t)\} U_j^T x(s) \; ds \leq - \frac{2e Q_{ij}}{\exp \{2ed_{mi} \} - \exp \{2ed_{ii} \}} \xi^T(t) \times [U_3 \\ N_1 \\ U_3] > 0.$$  \hspace{1cm} (69)

For $\tau(t) \in [0, \overline{\tau}]$, with the same matrix inequalities technique, we obtain the following:

$$-\int_{t-\tau}^{t} \tau \exp \{2e(s - t)\} x^T(s) U_j x(s) \; ds \leq - \frac{2e \tau}{\exp \{2e\tau \} - 1} \xi^T(t) \left[ \begin{array}{c}
e_{20} \\
e_{21} - e_{20} \end{array} \right] \left[ \begin{array}{c}
U_1 \\ M_1^T \\ U_1 \end{array} \right] > 0,$$

$$-\int_{t-\tau}^{t} \tau \exp \{2e(s - t)\} x^T(s) V_j x(s) \; ds \leq - \frac{2e \tau}{\exp \{2e\tau \} - 1} \xi^T(t) \left[ e_{1} - e_{18} \; e_{18} - e_{16} \right] \left[ V_1 \\ M_2^T \\ V_1 \right] > 0.$$  \hspace{1cm} (70)

Consider $-\int_{t-d_{12}}^{t} Q_{ij} \exp \{2e(s - t)\} x^T(s) U_j x(s) \; ds$ and $-\int_{t-d_{12}}^{t} Q_{ij} \exp \{2e(s - t)\} x^T(s) V_j x(s) \; ds$, which are directly estimated by (a) of Lemma 7; that is,

$$-\int_{t-d_{12}}^{t} Q_{ij} \exp \{2e(s - t)\} x^T(s) U_j x(s) \; ds$$

$$\leq - \frac{2e Q_{ij}}{\exp \{2ed_{mi} \} - \exp \{2ed_{ii} \}} \xi^T(t) \left[ e_{1} - e_{18} \; e_{18} - e_{16} \right] \left[ V_1 \\ M_2^T \\ V_1 \right] > 0.$$  \hspace{1cm} (71)

Substituting (47), (54)–(60), and (63)–(71) into (62) we obtain

$$\Gamma V(x(t), i, t) + 2e V(x(t), i, t)$$

$$\leq \xi^T(t) (\Gamma V + \Omega_i) \xi(t) + \mathcal{U},$$  \hspace{1cm} (72)

where

$$\mathcal{U} = \int_{t-\tau_j}^{t} \exp \{2e(s - t)\} x^T(s) \left[ \sum_{i \neq j} \pi_{ij} Q_{ij} - \sum_{j \neq i} \pi_{ij} Y_{ij} - \sum_{j \neq i} \pi_{ij} Y_{ij} \right] x(s) \; ds$$

$$+ \int_{t-\tau_j}^{t} \exp \{2e(s - t)\} x^T(s) \left[ \sum_{j \neq i} \pi_{ij} Q_{ij} - \sum_{j \neq i} \pi_{ij} Y_{ij} \right] x(s) \; ds$$

$$+ \int_{t-d_{ij}}^{t} \exp \{2e(s - t)\} x^T(s) \left[ \sum_{j \neq i} \pi_{ij} R_{ij} - \sum_{j \neq i} \pi_{ij} Z_{ij} \right] x(s) \; ds$$

$$+ \int_{t-d_{ij}}^{t} \exp \{2e(s - t)\} x^T(s) \left[ \sum_{j \neq i} \pi_{ij} T_{ij} - \sum_{j \neq i} \pi_{ij} Z_{ij} \right] x(s) \; ds$$

$$+ x^T(t) \left[ \sum_{j \neq i} \pi_{ij} (P_j - X_j) \right] x(t).$$  \hspace{1cm} (73)
On the other hand, for $d_i(t) \in [d_{m_i}, d_{2i}]$, the integral terms

$$-\int_{t-d_m}^{t-d_u} \varrho_2 \exp \{2\varepsilon (s-t)\} x^T (s) U_i x (s) \, ds,$$

$$-\int_{t-d_u}^{t-d_m} \varrho_2 \exp \{2\varepsilon (s-t)\} x^T (s) V_i x (s) \, ds$$

are disposed and estimated by Lemma 8, and

$$-\int_{t-d_m}^{t-d_u} \varrho_2 \exp \{2\varepsilon (s-t)\} x^T (s) U_i x (s) \, ds,$$

$$-\int_{t-d_u}^{t-d_m} \varrho_2 \exp \{2\varepsilon (s-t)\} x^T (s) V_i x (s) \, ds$$

are directly estimated by (a) of Lemma 7. Therefore,

$$\Gamma V (x(t), i, t, t) + 2\varepsilon V (x(t), i, t)$$

$$\leq \xi^T (t) (\Omega_{ij}^c + \Omega_{ij}^s) \xi (t) + U.$$  \hspace{1cm} (76)

With (72) and (76), the following inequality (77) is held for $d_i(t) \in [d_{1i}, d_{2i}]$ if (26)–(34) are satisfied

$$\Gamma V (x(t), i, t, t) + 2\varepsilon V (x(t), i, t, t) < 0.$$ \hspace{1cm} (77)

From the stochastic Lyapunov functional (38) and (77), it is held that

$$V (x(t), i, t) \geq \min_{\lambda \in \Lambda} \{\lambda_{\min} (P_i)\} \|x(t)\|^2$$

$$= \bar{\lambda} \|x(t)\|^2,$$ \hspace{1cm} (78)

$$V (x(t), i, t) < \exp \{-2\varepsilon (t-t_0)\}$$

$$\times V (x_{i_0}, r_{i_0}, t_0).$$

Moreover, we have

$$V (x_{i_0}, r_{i_0}, t_0) = \sum_{k=1}^{6} V_k (x_{i_0}, r_{i_0}) \leq \bar{\lambda} \|x_{i_0}\|^2.$$ \hspace{1cm} (79)

Then from (78) and (79), it is readily seen that

$$\|x(t)\| \leq \sqrt{\frac{\bar{\lambda}}{\lambda}} \exp \{-\varepsilon (t-t_0)\} \|x_{i_0}\|,$$ \hspace{1cm} (80)

where $\kappa = \sqrt{\bar{\lambda}/\lambda} \geq 1$.

Therefore, by Definition 4, the system (13) is exponentially stable with a decay rate $\varepsilon$. This completes the proof. \hspace{1cm} \Box

\textbf{Remark 12.} It is noted that the integral intervals in (79) are enlarged as follows:

$$\sum_{k=1}^{6} V_k (x_{i_0}, r_{i_0}) \leq \int_{t_0}^{t_\theta} \exp \{2\varepsilon (s-t_0)\} x^T (s)$$

$$\times \left\{Q_{1i} + Q_3 + R_{1i} + \sum_{k=2}^{4} R_k \right\} x (s) \, ds,$$

$$+ \int_{t_0}^{T_\theta} \exp \{2\varepsilon (s-t_0)\} x^T (s)$$

$$\times \left\{Q_{2i} + Q_4 + T_{1i} + \sum_{k=2}^{4} T_k \right\} x (s) \, ds,$$

$$+ \int_{t_0}^{0} \int_{t_0+\tau}^{T_\theta} \varsigma \exp \{2\varepsilon (s-t_0)\} x^T (s)$$

$$\times \left(\sum_{n=1}^{4} U_n \right) x (s) \, ds \, d\theta,$$

$$+ \int_{t_0}^{0} \int_{t_0+\tau}^{T_\theta} \sum_{n=1}^{4} W_n x (s) \, ds \, d\theta$$

$$+ \int_{t_0}^{0} \int_{t_0+\tau}^{T_\theta} \frac{\varsigma}{2} \exp \{2\varepsilon (s-t_0)\} x^T (s)$$

$$\times \left(\sum_{n=1}^{4} W_n \right) x (s) \, ds \, d\lambda \, d\theta$$

$$+ x^T (t_0) P_i x (t_0).$$ \hspace{1cm} (81)

(79) can be obtained by letting $\bar{\lambda}$ be defined as previously mentioned.

\textbf{Remark 13.} In Theorem 10, the factors

$$\frac{\exp \{2\varepsilon d_{1i}\} - \exp \{2\varepsilon d_i (t)\}}{2\varepsilon},$$

$$\frac{\exp \{2\varepsilon d_i (t)\} - \exp \{2\varepsilon d_{2i}\}}{2\varepsilon}$$

may be enlarged as $(\exp \{2\varepsilon d_{1i}\} - \exp \{2\varepsilon d_{2i}\})/2\varepsilon$. This will lead conservative results due to the fact that $d_i(t)$ cannot achieve $d_{1i}$ and $d_{2i}$ at the same time. While we apply Lemma 7 to these terms, the method by using reciprocally convex lemma [38] can achieve less conservative results. Moreover, for $\varepsilon > 0$, the factor $(d_i(t) - 1) \exp \{-2\varepsilon d_i (t)\}$ that appeared in the derivative of Lyapunov functional may be directly enlarged as $\mu_i \exp \{-2\varepsilon d_{1i}\} - \exp \{-2\varepsilon d_{2i}\}$. In this paper, we
enlarge it as \( \omega(\epsilon, \mu_l) \) to reduce the conservativeness of the obtained criteria. In the literature \([36, 45, 46]\), this factor is enlarged as \((\mu_l - 1) \exp \{ -2\epsilon d_{2i} \}, \) which only holds for \( \mu_l < 1 \).

**Remark 14.** The information on the lower bound of the delay is sufficiently used in the Lyapunov functional by introducing the terms such as \( \int_{t-d_{2i}}^{t-d_{2i}} \exp \{ 2\epsilon(s-t) \} x_i^T(s) R_i x_i(s) ds \), \( \int_{t-d_{2i}}^{t-d_{1i}} \exp \{ 2\epsilon(s-t) \} x_i^T(s) R_i x_i(s) ds \), and \( \int_{t-d_{1i}}^{t-d_{0i}} \exp \{ 2\epsilon(s-t) \} x_i^T(s) R_i x_i(s) ds \), which is equivalent to the improved bounding technique.

**Remark 15.** It should be also mentioned that the result obtained in Theorem 10 is delay-range-dependent and decay rate-dependent stability condition for (13), which is less conservative than the previous ones and will be verified in Section 4. Although the large number of introduced free weighting matrices may increase the complexity of computation, utilizing the technique of free weighting matrices would reduce the conservativeness. In addition, the given results can be extended to more general systems with neutral delay \( \tau_i(t) \). That is, \( \tau_{ij} \leq \tau_i(t) \leq \tau_{2i} \). The corresponding results can be obtained by using the similar methods.

In Theorem 10, it is assumed that \( \epsilon \neq 0 \). For \( \epsilon = 0 \), by l’Hôpital rule, the following asymptotic stability criterion can be obtained.

**Corollary 16.** For given scalars \( \pi^m_i, \pi^M_i, \alpha, \beta, \gamma, \tau_i, \eta_i, d_{1i}, d_{2i}, \mu_i \), and constant scalar \( d_{mi} \) satisfying \( d_{1i} < d_{mi} < d_{2i} \), the systems described by (13) with partially known transition rates are asymptotically stable if \( \| C_i \| + \gamma < 1 \) and there exist symmetric positive matrices \( P_i > 0, Q_{1i} > 0, Q_{2i} > 0, R_{1i} > 0, T_{1i} > 0 \), \( (i \in S) \), \( Q_i > 0 \), \( (j = 3, 4) \), \( R_k > 0 \), \( T_k > 0 \), \( (k, l = 2, 3, 4) \), \( U^T_m > 0 \), \( V_k > 0 \), \( W_l > 0 \), \( (m, n, s = 1, 2, 3, 4) \) and matrices \( M_{1i}, M_{2i}, N_{1i}, N_{2i}, N_{3i}, N_{4i} \) for any scalars \( \epsilon_i, \eta_i, \zeta_i \), any symmetric matrices \( X_i, Y_{1i}, Y_{2i}, Z_{1i}, Z_{2i} \), \( (i \in S) \) and any matrices \( f_{ij} \), \( (k = 1, 2, \ldots, 24) \) with appropriate dimensions, such that (26)–(32) and the following linear matrix inequalities hold:

\[
\begin{align*}
(i) & \quad \begin{bmatrix} U_3 & N_1 \\ N_1^T & U_3 \end{bmatrix} > 0, \quad \begin{bmatrix} V_3 & N_2 \\ N_2^T & V_3 \end{bmatrix} > 0, \\
(ii) & \quad \Omega_{10} + \Omega_{11} < 0, \\
(iii) & \quad \begin{bmatrix} U_4 & N_3 \\ N_3^T & U_4 \end{bmatrix} > 0, \quad \begin{bmatrix} V_4 & N_4 \\ N_4^T & V_4 \end{bmatrix} > 0, \\
(iv) & \quad \Omega_{10} + \Omega_{12} < 0,
\end{align*}
\]

where

\[
\begin{align*}
\Omega_{10} = \sum_{m=1}^{24} e_{m} \tilde{\gamma}_{m} e_{m}^T + \mathcal{L}(\Xi) \\
- (e_1 - e_3) V_2 (e_1^T - e_3^T) \\
- (\tilde{\tau}_i e_1 - e_{21}) W_1 (\tilde{\tau}_i e_1^T - e_{21}^T) \\
- (d_{1i} e_1 - e_{11}) W_2 (d_{1i} e_1^T - e_{11}^T)
\end{align*}
\]

and the remaining notations are the same as Theorem 10.

Further, the information on the transition rates may be completely unknown in some circumstances, which viewed the systems as switched systems with arbitrary switching. The following corollary is therefore given to guarantee the exponential stability for this case.

**Corollary 17.** For given scalars \( \pi^m_i, \pi^M_i, \alpha, \beta, \gamma, \tau_i, \eta_i, d_{1i}, d_{2i}, \mu_i \), and constant scalar \( d_{mi} \) satisfying \( d_{1i} < d_{mi} < d_{2i} \), the
systems described by \((13)\) with completely unknown transition rates are exponentially stable with decay rate \(\varepsilon\) and \(\kappa = \sqrt{\frac{\lambda}{\lambda}}\). if \(\|C\| + \gamma < 1\) and there exist symmetric positive matrices \(P > 0\), \(Q_j > 0\), \(R_k > 0\), \(T_i > 0\), \(U_m > 0\), \(V_n > 0\), \(W_s > 0\), \((j, k, l, m, n, s = 1, 2, 3, 4)\) and matrices \(M_1, M_2, N_1, N_2, N_3, N_4\) for any scalars \(\varepsilon_1, \varepsilon_2, \varepsilon_3, \) any symmetric matrices \(Y_1, Y_2, Z_1, Z_2,\) and any matrices \(J_k, (k = 1, 2, \ldots, 24)\) with appropriate dimensions, such that the following linear inequality holds:

\[
- \sum_{j \in D_{\text{out}, j} / i } \pi^M_i (Q_1 - Y_1) - \pi^M_i Y_1 \leq 0; \\
- \sum_{j \in D_{\text{out}, j} / i } \pi^M_i (Q_2 - Y_2) - \pi^M_i Y_2 \leq 0, \tag{86}
\]

\[
- \sum_{j \in D_{\text{out}, j} / i } \pi^M_i (R_1 - Z_1) - \pi^M_i Z_1 \leq 0; \\
- \sum_{j \in D_{\text{out}, j} / i } \pi^M_i (T_1 - Z_2) - \pi^M_i Z_2 \leq 0, \tag{87}
\]

\[
\left[ \begin{array}{c}
U_3 \\
N_1^T \\
U_1
\end{array} \right] > 0, \\
\begin{bmatrix}
V_3 & N_2 \\
N_2^T & V_5
\end{bmatrix} > 0, \tag{88}
\]

\[
\Omega_{10} + \Omega_{21} < 0,
\]

\[
\begin{bmatrix}
U_4 \\
N_3^T \\
U_4
\end{bmatrix} > 0, \\
\begin{bmatrix}
V_4 & N_4 \\
N_4^T & V_4
\end{bmatrix} > 0, \tag{89}
\]

\[
\Omega_{40} + \Omega_{25} < 0,
\]

where

\[
\Omega_{90}^e = \sum_{m=1}^{24} \omega_m e_m e_m^T + \mathcal{Z}(\Xi)
\]

\[
- \frac{2e_{d_{ij}}}{\exp[2e_{d_{ij}}] - 1} \left( e_{i1} - e_{ij} \right) V_2 \left( e_{i1} - e_{ij} \right) \\
- \exp[-2e_{T_1}] \left( \bar{c}_{e1} - e_{11} \right) W_1 \left( e_{i1} - e_{11} \right) \\
- \exp[-2e_{d_{ij}}] \left( d_{i1} e_{i1} - e_{11} \right) W_2 \left( d_{i1} e_{i1} - e_{11} \right) \\
- \exp[-2e_{d_{m1}}] \left( e_{i1} e_{i1} - e_{11} \right) W_3 \left( e_{i1} e_{i1} - e_{11} \right) \\
- \exp[-2e_{d_{j1}}] \left( e_{i1} e_{i1} - e_{11} \right) W_4 \left( e_{i1} e_{i1} - e_{11} \right) \\
- \frac{2e_{T_1}}{\exp[2e_{T_1}] - 1} \left[ e_{20} e_{21} - e_{20} \right] \\
\times \begin{bmatrix}
U_1 \\
M_1^T \\
U_1
\end{bmatrix} \begin{bmatrix}
0 \\
e_{i1} - e_{11}
\end{bmatrix} \\
- \frac{2e_{T_1}}{\exp[2e_{T_1}] - 1} \left[ e_{i1} - e_{11} \right] e_{18} - e_{16} \right]
\]

\[
\Omega_{90}^e = \Omega_{11}^e + \Omega_{12}^e = \Omega_{90},
\]

\[
\bar{\Omega}_{11} = \Omega_{11}^e, \\
\bar{\Omega}_{12} = \Omega_{12}^e,
\]

\[
\bar{X} = \lambda_{\max}(P) + \frac{1 - \exp[-2\varepsilon]}{2\varepsilon} \times \left\{ \frac{4}{j=1} \lambda_{\max}(Q_j) + \frac{4}{k=1} \lambda_{\max}(R_k) + \frac{4}{i=1} \lambda_{\max}(T_i) \right\} \\
+ \frac{\exp[-2\varepsilon] + 2\varepsilon - 1}{4\varepsilon^2} \times \left\{ \frac{4}{m=1} \lambda_{\max}(U_m) + \frac{4}{n=1} \lambda_{\max}(V_n) \right\} \\
+ \frac{1 - \exp[-2\varepsilon] + 2\varepsilon^2 - 2\varepsilon}{8\varepsilon^3} \left\{ \frac{4}{j=1} \lambda_{\max}(W_j) \right\}, \tag{91}
\]

and the remaining notations are the same as Theorem 10.
Proof. Since the elements of transition rates are entirely unknown, we choose the Lyapunov functional as follows:

\[ V(x(t), i, t) = x^T(t)Px(t) + \sum_{k=2}^{2} V_k(x_i, i), \]  

(92)

where

\[ V_2(x_i, i) = \int_{t-\tau_i(t)}^{t} \exp [2\epsilon (s-t)] x^T(s) Q_1 x(s) ds \]

\[ + \int_{t-\tau_i(t)}^{t} \exp [2\epsilon (s-t)] x^T(s) Q_2 x(s) ds \]

\[ + \int_{t-\tau_i(t)}^{t} \exp [2\epsilon (s-t)] x^T(s) Q_3 x(s) ds \]

\[ + \int_{t-\tau_i(t)}^{t} \exp [2\epsilon (s-t)] x^T(s) Q_4 x(s) ds, \]

(93)

\[ V_3(x_i, i) = \int_{t-d_i(t)}^{t} \exp [2\epsilon (s-t)] x^T(s) R_1 x(s) ds \]

\[ + \int_{t-d_i(t)}^{t-d_i(t)} \exp [2\epsilon (s-t)] x^T(s) R_2 x(s) ds \]

\[ + \int_{t-d_i(t)}^{t-d_i(t)} \exp [2\epsilon (s-t)] x^T(s) R_3 x(s) ds \]

\[ + \int_{t-d_i(t)}^{t-d_i(t)} \exp [2\epsilon (s-t)] x^T(s) R_4 x(s) ds, \]

(94)

\[ V_4(x_i, i) = \int_{t-d_i(t)}^{t-d_i(t)} \exp [2\epsilon (s-t)] x^T(s) T_1 x(s) ds \]

\[ + \int_{t-d_i(t)}^{t-d_i(t)} \exp [2\epsilon (s-t)] x^T(s) T_2 x(s) ds \]

\[ + \int_{t-d_i(t)}^{t-d_i(t)} \exp [2\epsilon (s-t)] x^T(s) T_3 x(s) ds \]

\[ + \int_{t-d_i(t)}^{t-d_i(t)} \exp [2\epsilon (s-t)] x^T(s) T_4 x(s) ds, \]

(95)

and \( V_2(x_i, i), V_3(x_i, i), V_4(x_i, i) \) are the same as (43), (44), and (45).

Then, we follow a similar line as in proof of Theorem 10 and obtain the result. \( \square \)

3.2. Extension to the Uncertain Case. In this subsection, the uncertain neutral Markovian jump systems described by (6) with partially unknown transition rates are considered. The delay-range-dependent and rate-dependent exponential stability conditions are presented in the following theorems and corollaries.

Theorem 18. For given scalars \( \pi^m, \pi^M, \alpha, \beta, \gamma, \epsilon, \mu, d_{1i}, d_{2i}, \mu_i \) and constant scalar \( d_{m} \) satisfying \( d_{1i} < d_{m} < d_{2i} \), the systems described by (6) with partially known transition rates are exponentially stable with decay rate \( \epsilon \) and \( \kappa = \sqrt{\lambda/\lambda} \) if \( \|C_i\| + \gamma < 1 \) and there exist scalars \( \delta_i \geq 0, \delta_i \geq 0, \) symmetric positive matrices \( P_i, Q_i, R_i, T_i, \) \( i \in S \), and matrices \( A_i, B_i, C_i, D_i, E_i \) \( i \in S \), \( k = 1, 2, \ldots, 24 \) with appropriate dimensions, such that (26)-(32) and the following inequalities hold:

(i) \( U_3 N_1 > 0, \quad V_3 N_2 > 0, \)

(96)

(ii) \( \delta_0 \Omega^c + \delta_1 \Omega^c \geq \delta_0 \Omega^c + \delta_1 \Omega^c > 0, \)

(97)

where

\[ \Omega = \begin{bmatrix} e_1 N_1 & e_3 N_3 + e_2 N_{13} + e_5 N_{31} & e_4 N_{23} + e_6 N_{32} \end{bmatrix}, \]

(98)

and other notations are the same as Theorem 10.

Proof. Defining \( E = \text{col} \{e_m, (m = 1, 2, \ldots, 24) \} \) and \( J = \text{col} \{P_j + J_j, (m = 2, \ldots, 24) \} \), we replace \( A_i, B_i, D_i, E_i \), and \( F_i \) with \( A_i + \Delta A_i(t), B_i + \Delta B_i(t), D_i + \Delta D_i(t), E_i + \Delta E_i(t), \) and \( F_i + \Delta F_i(t) \) on the basis of Theorem 10. That is,

(i) \( U_3 N_1 > 0, \quad V_3 N_2 > 0, \)

(99)

(ii) \( \Omega^c + \Omega^c t > 0, \quad e_1 A_i^c (t) + e_3 B_i^c (t) E_i^c \epsilon < 0, \)

(100)
Considering (ii) of (99) and combining the uncertainties condition (II), we have
\[ \Omega_i^T + \Omega_j^T + \mathcal{L} \]

\[
\times \begin{bmatrix}
   e_1 & e_3 & e_{22} & e_{23} & e_{24} \\
   N_{A1}^T & N_{B1}^T & N_{D1}^T & N_{E1}^T & N_{F1}^T \\
   H_i^T(t) L_i^T y_i(t) E \\
\end{bmatrix} < 0.
\]

(102)

By the definition of \( \mathcal{L} \), we obtain
\[ \Omega_i^T + \Omega_j^T + \mathcal{M} H_i^T(t) N + N_i^T H_j(t) M^T < 0. \]

(102)

According to (12), by Lemmas 9 and 6, with (102) we obtain (ii) of (96). Following the same procedure, (iv) of (100) is considered and (iv) of (97) can be obtained. Finally, following the proof of Theorem 10, the systems described by (6) are exponentially stable with a decay rate \( \varepsilon \). This completes the proof.

Considering the uncertain case, the following corollaries are given, for \( \varepsilon = 0 \) and entirely unknown transition rates, respectively.

**Corollary 19.** For given scalars \( \pi_i^m, \pi_i^M, \alpha, \beta, \gamma, \overline{\alpha}, \overline{\beta}, \overline{\gamma}, d_{1i}, d_{2i}, \mu \) and constant scalar \( d_{mi} \) satisfying \( d_{1i} < d_{mi} < d_{2i} \), the systems described by (6) with partially known transition rates are asymptotically stable if \( \|C_i\| + \gamma < 1 \) and there exist \( \hat{\delta}_1 > 0, \hat{\delta}_2 > 0, \) symmetric positive matrices \( P_i > 0, Q_i > 0, R_i > 0, T_i > 0, (i \in \mathcal{S}) \), \( J > 0 \), (\( j = 3, 4 \)), \( R_k > 0, T_k > 0, (k, l = 2, 3, 4) \), \( U_m > 0, V_n > 0, \) and \( W_s > 0 \), \( (m, n, s = 1, 2, 3, 4) \) and matrices \( M_1, M_2, N_1, N_2, N_3, N_4, M \) for any scalars \( e_1, e_2, e_3 \), any symmetric matrices \( X_1, Y_1, Y_2, Z_1, Z_2, (i \in \mathcal{S}) \) and any matrices \( I_k, (k = 1, 2, \ldots, 24) \) with appropriate dimensions, such that (26)–(32), and the following inequalities hold:

\[
\begin{align*}
(i) & \quad U_3 N_1 > 0, & V_3 N_2 > 0, \\
(ii) & \quad \hat{\delta}_1 \hat{\Omega}_{i0} + \hat{\delta}_1 \hat{\Omega}_{i1} + \hat{\delta}_2 \hat{\Omega}_{i2} > 0, & \hat{\delta}_1 \hat{\Omega}_{i0} + \hat{\delta}_1 \hat{\Omega}_{i1} + \hat{\delta}_2 \hat{\Omega}_{i2} > 0, \\
(iii) & \quad U_3 N_1 > 0, & V_3 N_2 > 0, \\
(iv) & \quad \hat{\delta}_2 \hat{\Omega}_{i0} + \hat{\delta}_2 \hat{\Omega}_{i2} > 0, & \hat{\delta}_2 \hat{\Omega}_{i0} + \hat{\delta}_2 \hat{\Omega}_{i2} > 0,
\end{align*}
\]

(103)

where
\[
\mathcal{M} = e_1 N_{A1}^T + e_3 N_{D1}^T + e_{22} N_{E1}^T + e_{23} N_{F1}^T,
\]
\[
\mathcal{N} = L_i^T L_i^T e_1 + \sum_{m=1}^{24} L_i^T L_i^T e_m,
\]

(104)

and other notations are the same as Corollary 17.

**Corollary 20.** For given scalars \( \pi_i^m, \pi_i^M, \alpha, \beta, \gamma, \overline{\alpha}, \overline{\beta}, \overline{\gamma}, d_{1i}, d_{2i}, \mu \), and constant scalar \( d_{mi} \) satisfying \( d_{1i} < d_{mi} < d_{2i} \), the systems described by (6) with completely unknown transition rates are exponentially stable with decay rate \( \varepsilon \) and \( \kappa = \sqrt{\lambda/\lambda} \) if \( \|C_i\| + \gamma < 1 \) and there exist scalars \( \hat{\delta}_1 > 0, \hat{\delta}_2 > 0, \) symmetric positive matrices \( P_i > 0, Q_i > 0, R_i > 0, T_i > 0, U_m > 0, V_n > 0, W_s > 0 \), \( (j = 3, 4) \), \( R_k > 0, T_k > 0, (k, l = 2, 3, 4) \), \( U_m > 0, V_n > 0, \) and \( W_s > 0 \), \( (m, n, s = 1, 2, 3, 4) \) and matrices \( M_1, M_2, N_1, N_2, N_3, N_4, M \) for any scalars \( e_1, e_2, e_3 \), any symmetric matrices \( X_1, Y_1, Y_2, Z_1, Z_2, (i \in \mathcal{S}) \) and any matrices \( I_k, (k = 1, 2, \ldots, 24) \) with appropriate dimensions, such that (86), (87), and the following inequalities hold:

\[
\begin{align*}
(i) & \quad U_3 N_1 > 0, & V_3 N_2 > 0, \\
(ii) & \quad \hat{\delta}_1 \hat{\Omega}_{i0} + \hat{\delta}_1 \hat{\Omega}_{i1} + \hat{\delta}_2 \hat{\Omega}_{i2} > 0, & \hat{\delta}_1 \hat{\Omega}_{i0} + \hat{\delta}_1 \hat{\Omega}_{i1} + \hat{\delta}_2 \hat{\Omega}_{i2} > 0, \\
(iii) & \quad U_3 N_1 > 0, & V_3 N_2 > 0, \\
(iv) & \quad \hat{\delta}_2 \hat{\Omega}_{i0} + \hat{\delta}_2 \hat{\Omega}_{i2} > 0, & \hat{\delta}_2 \hat{\Omega}_{i0} + \hat{\delta}_2 \hat{\Omega}_{i2} > 0,
\end{align*}
\]

(105)

and other notations are the same as Corollary 17.

**4. Numerical Examples**

In this section, numerical examples are given to demonstrate that the proposed theoretical results in this paper are effectiveness. In particular, when \( i \in \mathcal{S} = \{1\}, d_1(t), d_1, d_m, \) and \( d_2 \) are expressed into \( d_1, d_1, d_m, \) and \( d_2 \).

**Example 1.** We consider an interval mode-dependent time-varying delay system in the form of (6) with three modes and nonlinear perturbations:

\[
\| f_1 (x(t), t) \| \leq 0.1 \| x(t) \|, \]

\[
\| f_2 (x(t) − d(t), t) \| \leq 0.2 \| x(t) − d(t) \|, \]

\[
\| f_3 (x(t − \tau(t), t) \| \leq 0.1 \| x(t − \tau(t) \|
\]

(107)

that is, \( \alpha = 0.1, \beta = 0.2, \gamma = 0.1 \). The parametric matrices of the system are given as follows:

\[
A_1 = \begin{bmatrix}
    -0.75 & -0.75 \\
    1.50 & -1.50
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
    -0.15 & -0.09 \\
    1.50 & -0.10
\end{bmatrix}, \quad A_3 = \begin{bmatrix}
    -0.30 & -0.15 \\
    0.50 & -0.50
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
    0.11 & 0.23 \\
    -0.52 & -0.37
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
    -0.59 & 0.02 \\
    -0.06 & -0.61
\end{bmatrix}, \quad B_3 = \begin{bmatrix}
    0.51 & 0.24 \\
    0.02 & -0.44
\end{bmatrix},
\]

(104)
The interval mode-dependent time-varying neutral delays are assumed to be
\[
\tau_1(t) = 0.6 \left(1 + \sin^2(3t)\right), \quad \tau_2(t) = 0.5 \left(1 + \cos^2(4t)\right),
\]
\[
\tau_3(t) = 0.6 \sin^2 t.
\]
(111)

They are governed by the Markov process \( \{r_t, t \geq 0\} \) and shown in Figures 1 and 2. It can be readily obtain that
\[
\tau_1 = 0.6, \quad \tau_2 = 1.6, \quad \tau_3 = 2\sqrt{2};
\]
\[
\nu_1 = 0.3, \quad \nu_2 = 0.8, \quad \nu_3 = \sqrt{2};
\]
\[
d_{11} = 0.4, \quad d_{21} = 0.8; \quad d_{12} = 0.5, \quad d_{22} = 1;
\]
\[
d_{13} = 0, \quad d_{23} = 0.6;
\]
\[
\mu_1 = 1.2, \quad \mu_2 = 2, \quad \mu_3 = \frac{9\sqrt{3}}{20}.
\]
(112)

Providing that decay rate \( \varepsilon = 0.5 \), we choose qualified values of \( d_{m1}, d_{m2}, d_{m3} \), for example, \( d_{m1} = 0.5, d_{m2} = 0.7, d_{m3} = 0.3 \), and then we solve (26)–(32), (96), and (97) in Theorem 18. Thus, it is facile to establish the exponential stability of this uncertain neutral system described by (6), which shows that the approach presented in this paper is effectiveness.

Example 2. Consider the nominal system (13) with four operation modes \( S = \{1, 2, 3, 4\} \), \( \alpha = \beta = \gamma = 0 \), and the following parameters:
\[
A_1 = \begin{bmatrix} -1.15 & -0.75 \\ 1.50 & -1.50 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1.20 & 0.12 \\ 0.24 & -0.25 \end{bmatrix},
\]
\[
C_1 = \begin{bmatrix} -0.15 & -0.06 \\ 0.50 & -0.50 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2.15 & -0.49 \\ 1.50 & -2.10 \end{bmatrix},
\]
\[
B_2 = \begin{bmatrix} -1.45 & -0.96 \\ 0.47 & -1.57 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.25 & -0.35 \\ 0.05 & -0.65 \end{bmatrix},
\]
\[
A_3 = \begin{bmatrix} -1.30 & -0.15 \\ 1.50 & -1.80 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.58 & -0.68 \\ -0.13 & 0.96 \end{bmatrix},
\]
(113)

The partially known transition rates matrix is described by
\[
\begin{bmatrix} -0.8 & ? & ? \\ ? & -0.9 & ? \\ 0.7 & 0.4 & -1.1 \end{bmatrix},
\]
(109)

where \( \sigma_1^M = -1.1, \sigma_2^M = -0.8 \).

The interval mode-dependent time-varying neutral delays are assumed to be
\[
\tau_1(t) = 0.3 \left(1 + \cos t\right), \quad \tau_2(t) = 0.8 \left(1 + \sin t\right),
\]
\[
\tau_3(t) = \sqrt{2} \sin t + \cos t.
\]
(110)
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0.3 0.4 0.5 0.6 0.7 0.8 0.9

Retarded time-varying delay

Time, t

0 1 2 3 4 5 6

Figure 2: Retarded time-varying delay $d_i(t)$ at Mode 1, Mode 2, and Mode 3.

### Table 1: Maximum upper bound of $d_2$ with $\Pi_1$ and different parameter $\mu$.

<table>
<thead>
<tr>
<th>Methods</th>
<th>$\mu$</th>
<th>0.18</th>
<th>0.19</th>
<th>0.20</th>
<th>0.21</th>
<th>0.22</th>
<th>0.23</th>
<th>0.24</th>
<th>0.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>Xiong et al. [40]</td>
<td>$d_2$</td>
<td>0.4489</td>
<td>0.3317</td>
<td>0.2242</td>
<td>0.1814</td>
<td>0.1486</td>
<td>0.1220</td>
<td>0.0983</td>
<td>0.0781</td>
</tr>
<tr>
<td>Corollary 16</td>
<td>$d_2$</td>
<td>0.4771</td>
<td>0.3664</td>
<td>0.2583</td>
<td>0.2165</td>
<td>0.1897</td>
<td>0.1473</td>
<td>0.1144</td>
<td>0.0956</td>
</tr>
</tbody>
</table>

\[
C_3 = \begin{bmatrix} 0.33 & 0.07 \\ 0.19 & -0.36 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -1.90 & -0.34 \\ 1.50 & -1.65 \end{bmatrix},
\]
\[
B_i = \begin{bmatrix} -0.67 & -1.50 \\ 1.39 & 1.23 \end{bmatrix}, \quad C_4 = \begin{bmatrix} -0.23 & 0.16 \\ 0.02 & -0.57 \end{bmatrix}.
\]

(113)

The partially known transition rate matrix is considered as the following two cases:

\[
\Pi_1 = \begin{bmatrix} -1.3 & 0.2 & ? & ? \\ ? & ? & 0.3 & 0.3 \\ 0.6 & ? & -1.5 & ? \\ 0.4 & ? & ? & ? \end{bmatrix},
\]
\[
\Pi_2 = \begin{bmatrix} -1.3 & 0.2 & 0.4 & 0.7 \\ ? & ? & 0.3 & 0.3 \\ 0.6 & ? & -1.5 & ? \\ 0.4 & ? & ? & ? \end{bmatrix},
\]

(114)

where $\pi_{i}^{m}=1.8$ and $\pi_{i}^{M}=-1.3$.

Providing that $\varepsilon = 0$ and $\tau(t) = d_i(t) = d(t), i \in S = \{1, 2, 3, 4\}$, we have $d_1 = 0, \tau = d_2, \gamma = \mu$. Set $d_m = d_3/2$ and employ Corollary 16; the maximum upper bound of the time delay $d_2$ which satisfies LMIs (26)–(32), and (83), can be calculated by solving a quasi-convex optimization problem. This neutral Markovian jump system with partially unknown transition rates was also considered in [40]. The results on the maximum upper bound of $d_2$ are compared in Tables 1 and 2.

From Tables 1 and 2, we consider the previous system with $\Pi_1, \Pi_2$ and obtain the maximum upper bound of delay $d_2 = 0.4771, d_2 = 12.4394$, respectively, in this paper by setting $\mu = 0.18$, while the maximum upper bound of delay $d_2 = 0.4489, d_2 = 12.0750$, respectively, for [40]. The results are also given by setting $\mu = 0.19, \mu = 0.20, \mu = 0.21, \mu = 0.22, \mu = 0.23, \mu = 0.24$, and $\mu = 0.25$, and it is found that the maximum upper bound of delay in this paper is larger than that of [40]. So it can be seen that our proposed method is less conservative than the result in [40]. Besides, according to the data in Tables 1 and 2, we know that the maximum upper bound of delay to guarantee stability is dependent on transition rates knowledge.

**Example 3.** To show the reduced conservativeness of the exponential stability condition in Theorem 10, consider the time-delay system in the form of (13) with $C_i = 0$ and $\alpha = \beta = \gamma = 0, d_i(t) = d$ and

\[
A_i = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} -0.5 & 0.1 \\ 0.3 & 0 \end{bmatrix}, \quad i \in S = \{1\}.
\]

(115)

For given $d$, the maximum exponential decay rate $\varepsilon$, which satisfies the LMIs (32), (33), and (34) in Theorem 10, can be calculated by solving a quasi-convex optimization problem. The results are presented in Table 3.

From Table 3, we know that the maximum exponential decay rate $\varepsilon = 0.9569$ in this paper by setting $d = 0.8, \mu$, while the maximum exponential decay rate $\varepsilon = 0.7344$ for [41], $\varepsilon = 0.9367$ for [42], and $\varepsilon = 0.9366$ for [43]. The results are also given by setting $d = 1.0, d = 1.2, d = 1.4, d = 1.6, d = 1.8, d = 2.0$, and it is found that the maximum exponential decay rate in this paper is larger than those in [41–43]. So it can be demonstrated that Theorem 10 in this paper yields less conservative results than [41–43].

Consider the previous system again, but with parametric matrices and parameter uncertainties as follows:

\[
A_i = \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0.1 & 0 \\ 4 & 0.1 \end{bmatrix}, \quad i \in S = \{1\}
\]

(116)

and the uncertain matrices $\Delta A_i(t)$ and $\Delta B_i(t)$ satisfy

\[
L_i = 0.2I, \quad N_{Ai} = N_{Bi} = I, \quad \|H_i(t)\| < 1,
\]

(117)

where $i \in S = \{1\}$.
Table 2: Maximum upper bound of \( d_2 \) with \( \Pi_2 \) and different parameter \( \mu \).

<table>
<thead>
<tr>
<th>Methods</th>
<th>( \mu )</th>
<th>0.18</th>
<th>0.19</th>
<th>0.20</th>
<th>0.21</th>
<th>0.22</th>
<th>0.23</th>
<th>0.24</th>
<th>0.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>Xiong et al. [40]</td>
<td>( d_2 )</td>
<td>12.0750</td>
<td>0.9924</td>
<td>0.5147</td>
<td>0.3749</td>
<td>0.2742</td>
<td>0.2139</td>
<td>0.1737</td>
<td>0.1433</td>
</tr>
<tr>
<td>Corollary 16</td>
<td>( d_2 )</td>
<td>12.4394</td>
<td>1.2587</td>
<td>0.6263</td>
<td>0.4037</td>
<td>0.3003</td>
<td>0.2485</td>
<td>0.2018</td>
<td>0.1796</td>
</tr>
</tbody>
</table>

Table 3: Maximum upper bound of \( \varepsilon \) with different parameter \( d \).

<table>
<thead>
<tr>
<th>Methods</th>
<th>( d )</th>
<th>0.8</th>
<th>1</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mondié and Kharitonov [41]</td>
<td>( \varepsilon )</td>
<td>0.7344</td>
<td>0.6715</td>
<td>0.6145</td>
<td>0.5642</td>
<td>0.5202</td>
<td>0.4818</td>
<td>0.4481</td>
</tr>
<tr>
<td>Liu [42]</td>
<td>( \varepsilon )</td>
<td>0.9367</td>
<td>0.5903</td>
<td>0.3400</td>
<td>0.1813</td>
<td>0.0752</td>
<td>0.0014</td>
<td>0</td>
</tr>
<tr>
<td>Xu et al. [43]</td>
<td>( \varepsilon )</td>
<td>0.9366</td>
<td>0.9192</td>
<td>0.8991</td>
<td>0.8115</td>
<td>0.6990</td>
<td>0.6148</td>
<td>0.5494</td>
</tr>
<tr>
<td>Theorem 10</td>
<td>( \varepsilon )</td>
<td>0.9569</td>
<td>0.9407</td>
<td>0.9218</td>
<td>0.9024</td>
<td>0.8154</td>
<td>0.7394</td>
<td>0.6933</td>
</tr>
</tbody>
</table>

Table 4: Maximum upper bound of \( \varepsilon \) with different parameter \( d \).

<table>
<thead>
<tr>
<th>Methods</th>
<th>( d )</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
<th>1.1</th>
<th>1.3</th>
<th>1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mondié and Kharitonov [41]</td>
<td>( \varepsilon )</td>
<td>0.6255</td>
<td>0.4760</td>
<td>0.3825</td>
<td>0.3191</td>
<td>0.2735</td>
<td>0.2392</td>
<td>0.2125</td>
</tr>
<tr>
<td>Xu et al. [43]</td>
<td>( \varepsilon )</td>
<td>1.0108</td>
<td>0.8366</td>
<td>0.7103</td>
<td>0.6156</td>
<td>0.5425</td>
<td>0.4845</td>
<td>0.4375</td>
</tr>
<tr>
<td>Theorem 18</td>
<td>( \varepsilon )</td>
<td>1.0419</td>
<td>0.9207</td>
<td>0.8035</td>
<td>0.7048</td>
<td>0.6234</td>
<td>0.5876</td>
<td>0.5043</td>
</tr>
</tbody>
</table>

For given \( d \), the maximum exponential decay rate \( \varepsilon \), which satisfies the LMIs in (32), (96), and (97) in Theorem 18, can be calculated by solving a quasi-convex optimization problem. The results are presented in Table 4 and it also can be seen that the delay-range-dependent and rate-dependent exponential stability conditions in this paper are less conservative than previous results in [41, 43].

Example 4. Consider the system (6) with \( \tau(t) = d_1(t) = d(t) \) and the parameters are listed in the following:

\[
A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},
\]

\[
D_1 = E_2 = F_1 = I, \quad \alpha = 0.05, \quad \beta = 0.1, \quad \gamma = 0,
\]

\[
d_1 = 0, \quad d_2 = \bar{\tau}, \quad \mu = \nu.
\]

(118)

For given \( \varepsilon \) and \( \mu \), choose \( d_m = d_2/2 \) and utilize Theorem 10, the maximum upper bound of \( d_2 \), which satisfies the LMIs in (32), (33), and (34), can be obtained by solving a quasi-convex optimization problem. The results are presented in Tables 5, 6, and 7.

From Tables 5, 6, and 7, we consider \( \mu = 0, \mu = 0.5 \), and \( \mu = 0.9 \) and obtain the maximum upper bound of delay \( d_2 = 1.5167, d_2 = 1.0643, \) and \( d_2 = 0.7136 \), respectively, in this paper by setting \( \varepsilon = 0.1 \), while the maximum upper bound of delay \( d_2 = 1.2999, d_2 = 0.9442, \) and \( d_2 = 0.5471 \), respectively, for [44], the maximum upper bound of delay \( d_2 = 1.4008, d_2 = 1.0120, \) and \( d_2 = 0.6438 \), respectively, for [13]. The results are also given by setting \( \varepsilon = 0.3, \varepsilon = 0.5, \varepsilon = 0.7, \) and \( \varepsilon = 0.9 \), and it is found that the maximum upper bound of delay is larger than those in [13, 44].

According to the comparative result, it can be seen that our proposed method is less conservative than those in [13, 44].

Example 5. Partial element equivalent circuit (PEEC) model can be represented as a stochastic jump system as in (6) with the abrupt variation in structures and parameters [47]. Consider the practical PEEC system described by (6) with \( \alpha = \beta = \gamma = 0 \) and completely unknown transition rates \( \Pi \),

\[
\Pi = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}, \quad i \in S = \{1, 2\},
\]

(119)

where \( \pi_i^m = -1.0 \) and \( \pi_i^M = -0.6 \).

The parametric matrices of the system are given as follows:

\[
A_1 = \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix},
\]

\[
B_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 & 0 \\ -0.9 & -1.2 \end{bmatrix},
\]

\[
C_1 = 0.5I, \quad C_2 = 0.3I,
\]

\[
L_1 = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 \\ -0.3 \end{bmatrix},
\]

\[
N_{A1} = \begin{bmatrix} 0.2 & 0 \end{bmatrix}, \quad N_{A2} = \begin{bmatrix} 0 & 0.2 \end{bmatrix},
\]

\[
N_{B1} = \begin{bmatrix} -0.3 & 0.3 \end{bmatrix}, \quad N_{B2} = \begin{bmatrix} 0.2 & 0.2 \end{bmatrix}.
\]

Given the decay rate \( \varepsilon = 0.4 \), the bound of mode-dependent time-varying neutral delay \( 0 \leq \tau_1(t) \leq 0.5, 0 \leq \tau_2(t) \leq 0.6 \), and the bound of mode-dependent time-varying retarded delay \( 0.1 \leq d_1(t) \leq 0.2, 0.3 \leq d_2(t) \leq 0.4 \). Without loss of generality, we choose \( d_{m1} = 0.12 \) and \( d_{m2} = 0.33 \). Since the information on the delay derivative is not available, by setting \( Q_1 = Q_2 = 0, R_1 = 0, T_1 = 0 \) in the Lyapunov
Table 5: Maximum upper bound of $d_2$ with different $\epsilon$ and parameter $\mu = 0$.

<table>
<thead>
<tr>
<th>Methods</th>
<th>$\epsilon$</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chen et al. [44]</td>
<td>$d_2$</td>
<td>1.2999</td>
<td>0.8781</td>
<td>0.6917</td>
<td>0.5792</td>
<td>0.5015</td>
</tr>
<tr>
<td>Qiu and Cui [13]</td>
<td>$d_2$</td>
<td>1.4008</td>
<td>1.0199</td>
<td>0.8457</td>
<td>0.7395</td>
<td>0.6667</td>
</tr>
<tr>
<td>Theorem 10</td>
<td>$d_2$</td>
<td>1.5167</td>
<td>1.1365</td>
<td>1.0674</td>
<td>0.9677</td>
<td>0.7558</td>
</tr>
</tbody>
</table>

Table 6: Maximum upper bound of $d_2$ with different $\epsilon$ and parameter $\mu = 0.5$.

<table>
<thead>
<tr>
<th>Methods</th>
<th>$\epsilon$</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chen et al. [44]</td>
<td>$d_2$</td>
<td>0.9442</td>
<td>0.7275</td>
<td>0.6096</td>
<td>0.5321</td>
<td>0.4761</td>
</tr>
<tr>
<td>Qiu and Cui [13]</td>
<td>$d_2$</td>
<td>1.0120</td>
<td>0.8324</td>
<td>0.7311</td>
<td>0.6941</td>
<td>0.6063</td>
</tr>
<tr>
<td>Theorem 10</td>
<td>$d_2$</td>
<td>1.0643</td>
<td>0.9156</td>
<td>0.7985</td>
<td>0.7368</td>
<td>0.6849</td>
</tr>
</tbody>
</table>

Table 7: Maximum upper bound of $d_2$ with different $\epsilon$ and parameter $\mu = 0.9$.

<table>
<thead>
<tr>
<th>Methods</th>
<th>$\epsilon$</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chen et al. [44]</td>
<td>$d_2$</td>
<td>0.5471</td>
<td>0.5015</td>
<td>0.4650</td>
<td>0.4350</td>
<td>0.4089</td>
</tr>
<tr>
<td>Qiu and Cui [13]</td>
<td>$d_2$</td>
<td>0.6438</td>
<td>0.5789</td>
<td>0.5214</td>
<td>0.4954</td>
<td>0.4231</td>
</tr>
<tr>
<td>Theorem 10</td>
<td>$d_2$</td>
<td>0.7136</td>
<td>0.6344</td>
<td>0.5987</td>
<td>0.5473</td>
<td>0.4963</td>
</tr>
</tbody>
</table>

5. Conclusions

The problem of exponential stability for neutral Markovian jumping systems with interval mode-dependent time-varying delays, nonlinear perturbations, and partially unknown transition rates is investigated in this paper. A novel augmented stochastic Lyapunov-Krasovskii functional is constructed, which contains some triple-integral terms and sufficiently takes advantage of the delay bound. Then, less conservative delay-range-dependent and rate-dependent exponential stability criteria are obtained by novel technique of matrix inequalities and free weighting matrices. These theoretical results are successfully verified through some numerical examples. Finally, the main contributions of this paper can be summarized as follows: (1) the constructed stochastic Lyapunov functional contains some triple-integral terms which are very effective in the reduction of conservativeness and has not appeared in the context of neutral Markovian jump systems with partially known transition rates and nonlinear perturbations before; (2) the bound of the delay is fully utilized in this paper; that is, improved bounding technique is used to reduce the conservativeness; (3) the reciprocally convex lemma is used to derive the delay-range-dependent and rate-dependent stability conditions, which can well reduce the conservativeness of the investigated systems; and (4) the proposed results are applicable to the partially known transition rates and expressed in a new representation, which are proved to be less conservative than some existing ones.

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References


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[40] L. Xiong, J. Tian, and X. Liu, “Stability analysis for neutral Markovian jump systems with partially unknown transition
Abstract and Applied Analysis


