Research Article

On Solution of Fredholm Integrodifferential Equations Using Composite Chebyshev Finite Difference Method

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1. Introduction

Linear and nonlinear Fredholm integrodifferential equations can be used to model many problems of science and theoretical physics such as engineering, biological models, electrostatics, control theory of industrial mathematics, economics, fluid dynamics, heat and mass transfer, oscillation theory, and queuing theory [1].

In recent years, many authors have considered different numerical methods to solve these kinds of problems. In 2012, Dehghan and Salehi employed [2] the meshless moving least square method for solving nonlinear Fredholm integro-differential equations. A sequential method for the solution of Fredholm integro-differential equations was presented by Berenguer et al. [3] in 2012. The formulation of the Fredholm integro-differential equation in terms of an operator and the use of Schauder bases are the main tools of this method.

In [4], the operational Adomian-Tau method with Padé approximation was used for solving nonlinear Fredholm integro-differential equations. This approach is based on two matrices, and Padé approximation was used to improve the accuracy of the method. Chebyshev finite difference method was proposed in [5] in order to solve Fredholm integro-differential equations. In this scheme the problem is reduced to a set of algebraic equations. In [6], Legendre collocation matrix method was introduced for solving high-order linear Fredholm integro-differential equations. In this way, the equation and its conditions are converted to matrix equations using collocation points on the interval [−1, 1]. Atabakan et al. [7, 8] proposed a modification of homotopy analysis method (HAM) known as spectral homotopy analysis method (SHAM) to solve linear Volterra and Fredholm integro-differential equations. In this procedure, the Chebyshev pseudospectral method was used to obtain an approximation of solutions to higher-order equations. The semiorthogonal spline method was discussed in [9]. This approach is used to solve Fredholm integral and integro-differential equations.

In this paper, we applied a composite Chebyshev finite difference (ChFD) method for solving Fredholm integro-differential equations. Fredholm integro-differential equation is given by

\[
F \left( x, y(x), y'(x), y''(x), \ldots, y^{(n)}(x) \right) = f(x) + \lambda \int_{0}^{T} k(x, t) y(t) dt,
\]

\[
G_{r} \left( y(\tau_{0}), \ldots, y^{(n-1)}(\tau_{0}), \ldots, y(\tau_{n}), \ldots, y^{(n-1)}(\tau_{n}) \right) = 0,
\]

\[
r = 0, \ldots, n - 1,
\]

(1)
where \(k(x,t), f(x),\) and \(y(x)\) are analytic functions, \(\lambda\) is a constant value, \(G_r, r = 0, \ldots, n - 1,\) are linear functions and the points \(\tau_0, \tau_1, \ldots, \tau_n\) lie in \([0, T]\). It will always be assumed that \((I)\) possesses a unique solution \(y \in C^1[0,T].\)

The base of the proposed method is a hybrid of block-pulse functions and Chebyshev polynomials using Chebyshev-Gauss-Lobatto points. This method was introduced and applied for solving the optimal control of delay systems with a quadratic performance index in [10, 11].

Chebyshev polynomials which are the eigenfunctions of a singular Sturm-Liouville problem have many advantages. They can be considered as a good representation of smooth functions by finite Chebyshev expansions provided that the function is infinitely differentiable. The Chebyshev expansion coefficients converge faster than any finite power of \(1/m\) as \(m\) goes to infinity for problems with smooth solutions. The numerical differentiation and integration can be performed. Moreover, they have been applied to solve different kinds of boundary value problems [12–14].

The paper is organized in the following way. Section 2 includes some necessary preliminaries and notations. Chebyshev finite difference method and composite Chebyshev finite difference method for solving Fredholm integrodifferential equations are described in Sections 3 and 4, respectively. Convergence analysis of the proposed method is presented in Section 5. In Section 6 discretization of the method is introduced, and some numerical examples are presented in Section 7. In Section 8, concluding remarks are given.

2. Preliminaries and Notations

In this section, we present some notations, definitions, and preliminary facts that will be used further in this work.

2.1. Block-Pulse Functions (BPF). In order to introduce block-pulse functions, we first suppose the interval \([0, T)\) to be divided into \(K\) equidistant subintervals \([(k - 1)/K, k/K)\), \(k = 1, 2, \ldots, K\). A set of block-pulse functions \(B_{(K)}(t)\) composed of \(K\) orthogonal functions with piecewise constant values is defined on the semiopen interval \([0, T)\) as follows:

\[
B_{(K)}(t) = \left[ b_1(t), b_2(t), \ldots, b_k(t), \ldots, b_K(t) \right],
\]

where the \(k\)th component is given by

\[
b_k(t) = \begin{cases} 1, & \frac{(k - 1)}{K} T \leq t < \frac{k}{K} T, \\ 0, & \text{otherwise}. \end{cases}
\]

Block-pulse functions have some nice characteristics. They are disjoint and orthogonal; that is,

\[
b_k(t) b_l(t) = \begin{cases} b_k(t), & k = l, \\ 0, & k \neq l, \end{cases}
\]

\[
\int_0^T b_k(t) b_l(t) dt = \| b_k(t) \|^2 \delta_{k,l},
\]

where \(\delta_{k,l}\) is the Kronecker delta function. In addition, the set of block-pulse functionies satisfy Parseval’s identity when \(K\) tends to infinity. In other words, for any function \(v \in \mathcal{F}[0,T),\)

\[
\| v \|^2 = \int_0^T v^2(t) \, dt = \sum_{k=1}^\infty \left( \int_0^T v(t) b_k(t) \, dt \right)^2 = \sum_{k=1}^\infty \| b_k(t) \|^2,
\]

where

\[
c_k = \frac{1}{\| b_k(t) \|} \int_0^T v(t) b_k(t) \, dt, \quad k = 1, 2, 3, \ldots,
\]

so they are complete. For more information about block-pulse functions, interested reader is referred to [20–30].

2.2. Chebyshev Polynomials. Chebyshev polynomials of the first kind of degree \(m\) can be defined as follows [12]:

\[
T_m(t) = \cos m\beta, \quad \beta = \arccos t, \quad m = 0, 1, 2, \ldots,
\]

which are orthogonal with respect to the weight function \(w(t) = 1/\sqrt{1-t^2}\), where

\[
\langle T_m, T_n \rangle = \begin{cases} 0, & m \neq n, \\ \pi, & m = n = 0, \\ \frac{\pi}{2}, & m = n \geq 1. \end{cases}
\]

Chebyshev polynomials also satisfy the following recursive formula:

\[
T_0(t) = 1, \quad T_1(t) = t, \quad T_{m+1}(t) = 2t T_m(t) - T_{m-1}(t).
\]

The set of Chebyshev polynomials is a complete orthogonal set in the Hilbert space \(\mathcal{F}_2[-1,1]\). The Chebyshev expansion of a function \(f \in \mathcal{F}_2[-1,1]\) is

\[
f(t) = \sum_{m=0}^\infty \tilde{f}_m T_m(t),
\]

\[
\tilde{f}_m = \frac{2}{\pi c_m} \int_{-1}^1 f(t) T_m(t) w(t) \, dx,
\]

where

\[
c_m = \begin{cases} 2, & m = 0, \\ 1, & m \geq 1. \end{cases}
\]

They have also another useful characteristic; see [14]. If

\[
h(t) = \frac{1}{2} h_0 T_0(t) + \sum_{m=1}^\infty h_m T_m(t),
\]

then

\[
\int_{-1}^1 h(t) \, dt = h_0 - \sum_{m=2}^\infty \frac{1 + (-1)^m}{m^2 - 1} h_m.
\]
Table 1: A comparison of absolute errors between Wc, WG, Cfd, and present method.

<table>
<thead>
<tr>
<th></th>
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<tbody>
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<td>1.16 × 10^{-15}</td>
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<tr>
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<td>2.28 × 10^{-15}</td>
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<tr>
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</tr>
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<td>2.79 × 10^{-17}</td>
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<td>1.27 × 10^{-16}</td>
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</tr>
</tbody>
</table>

Table 2: The maximum errors of $E_{KM}$ for different values of $K$ and $M$.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$M$</th>
<th>$E_{KM}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>8</td>
<td>6.66 × 10^{-10}</td>
</tr>
<tr>
<td>10</td>
<td>8</td>
<td>3.01 × 10^{-13}</td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td>2.28 × 10^{-15}</td>
</tr>
</tbody>
</table>

3. Chebyshev Finite Difference Method

We can approximate a function $f(t)$ in terms of Chebyshev polynomials as follows [31]:

$$ (P_M) f(t) = \sum_{k=0}^{M} f_k T_k(t), \quad f_k = \frac{2}{M} \sum_{k=0}^{M} f(t_k) T_m(t_k), $$

with double primes meaning that the first and last terms should be halved. Moreover, $t_k$ are the extrema of the $M$th-order Chebyshev polynomial $T_M(t)$ and defined as

$$ t_k = \cos \left( \frac{k\pi}{M} \right), \quad k = 0, 1, 2, \ldots, M. $$

In view of (7), we have

$$ T_m(t_k) = \cos \left( \frac{mk\pi}{M} \right), $$

so $f_m$ can be rewritten as

$$ f_m = \frac{2}{M} \sum_{k=0}^{M} f(t_k) \cos \left( \frac{mk\pi}{M} \right). $$

The first three derivatives of the function $f(t)$ at the points $t_m, m = 0, 1, \ldots, M$, are given in [32, 33] as

$$ f^{(i)}(t_m) = \sum_{j=0}^{M} t_{m,j}^{(i)} f(t_j), \quad i = 1, 2, 3, $$
4. Abstract and Applied Analysis

Table 3: A comparison of absolute errors between Wc, WG, Cfd, and present method.

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.125</td>
<td>2.6 × 10⁻³</td>
<td>2.7 × 10⁻⁴</td>
<td>1.8 × 10⁻¹</td>
<td>1.45 × 10⁻¹</td>
</tr>
<tr>
<td>0.250</td>
<td>1.5 × 10⁻³</td>
<td>3.0 × 10⁻⁵</td>
<td>4.4 × 10⁻¹</td>
<td>1.78 × 10⁻¹</td>
</tr>
<tr>
<td>0.375</td>
<td>9.3 × 10⁻³</td>
<td>2.6 × 10⁻⁴</td>
<td>1.4 × 10⁻⁹</td>
<td>1.71 × 10⁻¹</td>
</tr>
<tr>
<td>0.500</td>
<td>5.1 × 10⁻³</td>
<td>4.3 × 10⁻⁴</td>
<td>2.4 × 10⁻¹</td>
<td>5.00 × 10⁻¹</td>
</tr>
<tr>
<td>0.625</td>
<td>2.5 × 10⁻³</td>
<td>5.6 × 10⁻⁴</td>
<td>1.7 × 10⁻⁹</td>
<td>1.22 × 10⁻¹</td>
</tr>
<tr>
<td>0.750</td>
<td>1.0 × 10⁻³</td>
<td>6.5 × 10⁻⁴</td>
<td>7.7 × 10⁻¹</td>
<td>8.66 × 10⁻⁶</td>
</tr>
<tr>
<td>0.875</td>
<td>2.3 × 10⁻⁴</td>
<td>7.2 × 10⁻⁴</td>
<td>1.3 × 10⁻⁹</td>
<td>4.52 × 10⁻⁶</td>
</tr>
</tbody>
</table>

Table 4: The maximum errors of $E_{KM}$ for different values of $K$ and $M$.

<table>
<thead>
<tr>
<th>K</th>
<th>8</th>
<th>10</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>6</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>$E_{KM}$</td>
<td>6.01 × 10⁻¹³</td>
<td>8.72 × 10⁻¹⁴</td>
<td>1.22 × 10⁻¹⁵</td>
</tr>
</tbody>
</table>

where

$$d_{m,j}^{(1)} = \frac{4\theta_j}{M} \sum_{n=1}^{M} \sum_{l=0}^{n-1} \frac{n\theta_n T_n(t_j) T_l(t_m)}{q_{l}}$$

$$= \frac{4\theta_j}{M} \sum_{n=1}^{M} \sum_{l=0}^{n-1} \frac{n\theta_n}{q_{l}} \cos \left( \frac{nj\pi}{M} \right) \cos \left( \frac{lm\pi}{M} \right)$$

As can be seen from (23), the first three derivatives of the function $f(t)$ at any point of the Chebyshev-Gauss-Lobatto points is expanded as a linear combination of the values of the function at these points.

In view of (13) and (19), we have

$$\int_{-1}^{1} f(t) \, dt \approx f_0 - \sum_{m=2}^{M-1} \frac{1 + (-1)^m}{m^2 - 1} f_m - \frac{1 + (-1)^M}{2(M^2 - 1)} f_M.$$  \hspace{1cm} (26)

4. Composite Chebyshev Finite Difference Method

In this Section, we present the composite Chebyshev finite difference (ChFD) method. Consider $t_{km}, k = 1,2,\ldots, K, m = 0,1,\ldots, M$, as the corresponding Chebyshev-Gauss-Lobatto collocation points at the $k$th subinterval $[(k-1)/K, k/K]$ such that

$$t_{km} = \frac{T}{2K} (m + 2k - 1).$$  \hspace{1cm} (27)

A function $f(t)$ can be written in terms of hybrid basis functions as follows:

$$(P_M) f(t) = \sum_{k=1}^{K} \sum_{m=0}^{M} f_{km} b_{km}(t),$$  \hspace{1cm} (28)

where $f_{km}, n = 1,2,\ldots, K, m = 0,1,\ldots, M$, are the expansion coefficients of the function $f(t)$ at the subinterval $[(k-1)/K, k/K]$ and $b_{km}(t), k = 1,2,\ldots, K, m = 0,1,\ldots, M$, are defined in (14).

In view of (14) and (19), we can obtain the coefficients $f_{km}$ as

$$f_{km} = \frac{2}{M} \sum_{p=0}^{M} f(t_{kp}) b_{km}(t_{kp})$$  \hspace{1cm} (29)
Table 5: A comparison of absolute errors between $T_m$, $C_f m$, and present method.

<table>
<thead>
<tr>
<th>$x$</th>
<th>Exact solution</th>
<th>Tau method [16]</th>
<th>Chebyshev finite difference [5]</th>
<th>Present method</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.0</td>
<td>0.367879441</td>
<td>$1.52 \times 10^{-6}$</td>
<td>$1.19 \times 10^{-8}$</td>
<td>$1.32 \times 10^{-16}$</td>
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<tr>
<td>-0.8</td>
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<td>$1.74 \times 10^{-6}$</td>
<td>$1.33 \times 10^{-8}$</td>
<td>$1.36 \times 10^{-16}$</td>
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<tr>
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<td>$1.29 \times 10^{-8}$</td>
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<tr>
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<td>$1.40 \times 10^{-16}$</td>
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<tr>
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<td>$1.97 \times 10^{-6}$</td>
<td>$1.27 \times 10^{-8}$</td>
<td>$1.20 \times 10^{-16}$</td>
</tr>
<tr>
<td>0.0</td>
<td>1.000000000</td>
<td>$1.83 \times 10^{-6}$</td>
<td>$1.02 \times 10^{-8}$</td>
<td>$9.99 \times 10^{-16}$</td>
</tr>
<tr>
<td>0.2</td>
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<td>$1.63 \times 10^{-6}$</td>
<td>$1.04 \times 10^{-8}$</td>
<td>$7.91 \times 10^{-17}$</td>
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<td>0.4</td>
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<td>$2.92 \times 10^{-9}$</td>
<td>$3.08 \times 10^{-17}$</td>
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<td>$1.52 \times 10^{-6}$</td>
<td>$1.19 \times 10^{-8}$</td>
<td>$1.32 \times 10^{-16}$</td>
</tr>
</tbody>
</table>

Table 6: The maximum errors of $E_{KM}$ for different values of $K$ and $M$.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$M$</th>
<th>$E_{KM}$</th>
<th>$d_{m,j}^{(3)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>10</td>
<td>$4.07 \times 10^{-12}$</td>
<td>$rac{32K^3}{T^3M} \sum_{n=2}^{M} \sum_{l=1}^{n-2} \sum_{r=0}^{(l+r) \text{ odd}} \frac{C_r}{\cos \left( \frac{nl(2T^2-1)}{T^2M} \right) \cos \left( \frac{lnx}{M} \right) \cos \left( \frac{lnM}{M} \right)}$</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>$3.93 \times 10^{-15}$</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td>$2.79 \times 10^{-17}$</td>
<td></td>
</tr>
</tbody>
</table>

Using (23)–(25), the first three derivatives of the function $f(t)$ at the points $t_{km}$, $k = 1, 2, \ldots, K$, $m = 0, 1, \ldots, M$, can be obtained as

$$f^{(i)}(t_{km}) = \sum_{j=0}^{M} d_{km,j}^{(i)} f(t_{kj}), \quad i = 1, 2, 3,$$ (30)

where

$$d_{km,j}^{(1)} = \frac{8N\Theta}{TM} \sum_{n=1}^{M} \sum_{l=0}^{n-1} \frac{n\Theta}{\cos \left( \frac{nl\pi}{M} \right) \cos \left( \frac{lnM}{M} \right)},$$

$$d_{km,j}^{(2)} = \frac{8K^2\Theta}{T^2M} \sum_{n=2}^{M} \sum_{l=0}^{n-2} \frac{n(2T^2-1)\Theta}{\cos \left( \frac{ln\pi}{M} \right) \cos \left( \frac{lnM}{M} \right)},$$

$$d_{m,j}^{(3)} = \frac{32K^3\Theta}{T^3M} \sum_{n=2}^{M} \sum_{l=1}^{n-2} \sum_{r=0}^{(l+r) \text{ odd}} \frac{C_r}{\cos \left( \frac{nl(2T^2-1)}{T^2M} \right) \cos \left( \frac{lnx}{M} \right) \cos \left( \frac{lnM}{M} \right)},$$

In view of (26) and (28), we get

$$\int_0^T f(t) dt = \frac{T}{2N} \sum_{k=1}^{K} f(k\Theta) - \sum_{m=2}^{M-1} \frac{1+(-1)^m}{m^2-1} f_{km} - \frac{1+(-1)^M}{2(M^2-1)} f_{KM}.$$ (32)

5. Convergence Analysis

A detailed proof of the following results can be found in [11].

**Lemma 1.** If the hybrid expansion of a continuous function $h(t)$ converges uniformly, then it converges to the function $h(t)$.

**Theorem 2.** A function $h(t) \in L_2^2[0, T]$ with bounded second derivative, say $|h''(t)| \leq B$, can be expanded as an infinite sum...
Table 7: A comparison of absolute errors between DTM, IHPM, Sa, and present method.

<table>
<thead>
<tr>
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</tr>
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<td>0.14 × 10^{-3}</td>
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<td>0.9</td>
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<td>1.00 × 10^{-2}</td>
<td>0.18 × 10^{-3}</td>
<td>6.13 × 10^{-6}</td>
<td>1.72 × 10^{-17}</td>
</tr>
</tbody>
</table>

Table 8: A comparison of absolute errors between Lps and ChFd. of hybrid functions and the series converges uniformly to \( h(t) \),

\[
h(t) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} h_{km} b_{km} \phi_k(t).
\]  

Theorem 3. Suppose that \( h(t) \in L^2_w[0, T) \) with bounded second derivative, say \( |h''(t)| \leq B \), and then its hybrid expansion converges uniformly to \( h(t) \), that is,

\[
\sum_{k=1}^{K} \sum_{m=0}^{\infty} h_{km} b_{km} \phi_k(t) = h(t),
\]  

where the summation symbol with prime denotes a sum with the first term halved.

Theorem 4 (accuracy estimation). Suppose that \( h(t) \in L^2_w[0, T) \) with bounded second derivative, say \( |h''(t)| \leq B \), and then one has the following accuracy estimation:

\[
|\sigma_{K,M}| \leq \left( S + \sum_{k=K+1}^{\infty} \sum_{m=M+1}^{\infty} \frac{C^2}{(m^2 - 1)^2} \times \frac{\pi T \omega_m^2}{4K} \right)^{1/2},
\]  

where

\[
\sigma_{K,M} = \left( \int_0^T \left( h(t) - \sum_{k=1}^{K} \sum_{m=0}^{M} f_{km} b_{km} \phi_k(t) \right)^2 \omega_k(t) dt \right)^{1/2},
\]

\[
\begin{align*}
C &= \frac{RN^2}{12} + \frac{BT^2}{K^2 \omega_m}, \\
R &= \max \left\{ \frac{d^2}{d\beta^2} \left( h \left( \frac{\cos(\beta) + 2k - 1}{2K} T \right) \cos(m\beta) \right), 0 \leq \beta \leq \pi \right\}, \\
S &= \frac{1}{4} h_{10}^2 \frac{\pi T}{2K} + \frac{1}{4} h_{KM}^2 \frac{\pi T}{4K}.
\end{align*}
\]  

Proof. Consider

\[
\begin{align*}
\sigma_{K,M}^2 &= \frac{1}{4} h_{10}^2 \int_0^T b_{10}^2(t) w_k(t) dt \\
&+ \frac{1}{4} h_{KM}^2 \int_0^T b_{KM}^2(t) w_k(t) dt \\
&+ \left( \int_0^T h(t) - \sum_{k=1}^{K} \sum_{m=0}^{M} f_{km} b_{km} \phi_k(t) \right)^2 w_k(t) dt \\
&= \frac{1}{4} h_{10}^2 \frac{\pi T}{2K} + \frac{1}{4} h_{KM}^2 \frac{\pi T}{4K} \\
&+ \int_0^T \sum_{k=K+1}^{\infty} \sum_{m=M+1}^{\infty} h_{km}^2 b_{km}^2(t) w_k(t) dt \\
&= S + \sum_{k=K+1}^{\infty} \sum_{m=M+1}^{\infty} h_{km}^2 \int_0^T b_{km}^2(t) w_k(t) dt \\
&= S + \sum_{k=K+1}^{\infty} \sum_{m=M+1}^{\infty} h_{km}^2 \frac{\pi T \omega_m^2}{4K}.
\end{align*}
\]
Table 9: A comparison of absolute errors between Lps and present method.

<table>
<thead>
<tr>
<th>x</th>
<th>Exact solution</th>
<th>Legendre polynomial solutions [6]</th>
<th>Present method</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.0</td>
<td>-0.8414709848</td>
<td>4.39 × 10^{-7}</td>
<td>7.00 × 10^{-20}</td>
</tr>
<tr>
<td>-0.8</td>
<td>-0.7173560909</td>
<td>4.69 × 10^{-8}</td>
<td>1.50 × 10^{-15}</td>
</tr>
<tr>
<td>-0.6</td>
<td>-0.5646424734</td>
<td>1.19 × 10^{-9}</td>
<td>4.91 × 10^{-15}</td>
</tr>
<tr>
<td>-0.4</td>
<td>-0.3894183423</td>
<td>2.30 × 10^{-9}</td>
<td>8.38 × 10^{-15}</td>
</tr>
<tr>
<td>-0.2</td>
<td>-0.1986693308</td>
<td>9.50 × 10^{-11}</td>
<td>1.05 × 10^{-14}</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1.99 × 10^{-17}</td>
<td>1.07 × 10^{-14}</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1986693308</td>
<td>1.04 × 10^{-10}</td>
<td>8.79 × 10^{-15}</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3894183423</td>
<td>1.06 × 10^{-8}</td>
<td>5.34 × 10^{-15}</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5646424734</td>
<td>5.00 × 10^{-8}</td>
<td>1.48 × 10^{-15}</td>
</tr>
<tr>
<td>0.8</td>
<td>0.7173560909</td>
<td>1.35 × 10^{-6}</td>
<td>1.11 × 10^{-15}</td>
</tr>
<tr>
<td>1.0</td>
<td>0.8414709848</td>
<td>4.65 × 10^{-7}</td>
<td>7.00 × 10^{-20}</td>
</tr>
</tbody>
</table>

Table 10: The maximum errors of $E_{KM}$ for different values of $K$ and $M$.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$M$</th>
<th>$E_{KM}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
<td>4.13 × 10^{-6}</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>3.32 × 10^{-6}</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
<td>3.77 × 10^{-14}</td>
</tr>
<tr>
<td>7</td>
<td>9</td>
<td>1.07 × 10^{-14}</td>
</tr>
</tbody>
</table>

With the aid of (15) and the proof of Theorem 3, we will have

$$a_{kM}^2 \leq S + \sum_{k=K+1}^{\infty} \sum_{m=M+1}^{\infty} \frac{C^2}{(m^2 - 1)^2} \times \frac{\pi T c_m}{4K},$$

where

$$C = \frac{Rn^3}{12} + \frac{BT^2}{k^2 c_m},$$

$$R = \max \left\{ \frac{d^2}{d\beta^2} \left( h \left( \cos \left( \frac{\beta}{2} + \frac{k - 1}{2K} T \right) \right) \cos (\beta) \right), \quad 0 \leq \beta \leq \pi \right\},$$

$$S = \frac{1}{4} h_0^2 \frac{\pi T}{2K} + \frac{1}{4} h_{kM}^2 \frac{\pi T}{4K}.$$ 

$$f_{km} = \frac{2}{M} \sum_{p=0}^{M} \left( k(x, t_{kp}) y(t_{kp}) \right) b_{km}(t_{kp})$$

$$= \frac{2}{M} \sum_{p=0}^{M} \left( k(x, t_{kp}) y(t_{kp}) \right) \cos \left( \frac{mp\pi}{M} \right);$$

with aid of (32), we will have

$$\int_0^T k(x, t) y(t) dt \approx \frac{T}{2K} \sum_{k=1}^{K} \sum_{m=2}^{M-1} \frac{1 + (-1)^m (1 + \frac{m}{2}) f_{km}}{m^2 - 1}.$$

In order to obtain the solution $y(x)$ in (1), by applying the composite ChFD method, we first collocate (1) in Chebyshev-Gauss-Lobatto collocation points $t_{km}, k = 1, \ldots, K, m = 0, \ldots, M - n$. In addition, substituting (28) and (30) into boundary conditions (1), we get $n$ equations. Moreover, the approximate solution and its first $n$ derivatives should be continuous at the interface of subintervals; that is,

$$y^{(i)}(t_{k,0}) = y^{(i)}(t_{k+1,M}), \quad k = 1, 2, \ldots, K - 1,$$

$$i = 0, 1, \ldots, n - 1.$$ 

Therefore, we have a system of $K(M + 1)$ algebraic equations, which can be solved by using Newton’s iterative method for the unknowns $y(t_{km}), k = 0, 1, \ldots, K, m = 0, 1, \ldots, M$. Consequently, the approximate solution $y(x)$ of (1) can be calculated.

7. Numerical Examples

In this section, we apply the technique described in Section 6 to some illustrative examples of higher-order linear Fredholm integrodifferential equations.
Example 1. Consider the second-order Fredholm integrodifferential equation [5, 15]

\[
y''(x) + 4xy'(x) = -\frac{8x^4}{(x^2 + 1)^5} - 2 \int_0^1 \frac{t^2 + 1}{(x^2 + 1)^2} y(t) \, dt, \quad 0 \leq x, t \leq 1
\]

subject to the boundary conditions

\[
y(0) = 1, \quad y(1) = 1,
\]

with the exact solution \(y(x) = 1/(x^2 + 1)\).

We solve the problem with \(M = 10\), and \(K = 8\). A comparison between absolute errors in solutions obtained by composite Chebyshev finite difference method, wavelet collocation method, wavelet Galerkin and Chebyshev finite difference method is tabulated in Table 1. As can be seen in Table 1, our results are much more accurate than those obtained by other methods specially wavelet collocation method.

The maximum errors for approximate solution \(y_{KM}(x)\) can be defined as

\[
E_{KM} = \|y_{KM} - y_{exact}(x)\|_{\infty} = \max \{\|y_{KM}(x) - y_{exact}(x)\|, 0 \leq x \leq 1\},
\]

where the computed result with \(K\) is shown by \(y_{KM}\) and \(y_{exact}(x)\) is the exact solution. For different values of \(K\), the errors of \(E_{KM}\) are presented in Table 2.

Example 2. Consider the second-order Fredholm integrodifferential equation [5, 15]

\[
x^2y''(x) + 50xy'(x) - 35y(x)
\]

\[
= \frac{1 - e^{x+1}}{x+1} + \left(x^2 + 50x - 35\right)e^x + \int_0^1 e^{-(x+1)t} y(t) \, dt, \quad 0 \leq x, t \leq 1
\]

subject to the boundary condition

\[
y(0) = 1, \quad y(1) = e.
\]

The exact solution of this equation is \(y(x) = e^x\).

The problem is solved with \(M = 7\), and \(K = 8\). A comparison between absolute errors in solutions by composite Chebyshev finite difference method, wavelet collocation method, wavelet Galerkin and Chebyshev finite difference method is tabulated in Table 3. It is clear from Table 3 that our method is reliable and applicable to handle Fredholm integrodifferential equations. For different values of \(K\), the errors of \(E_{KM}\) are shown in Table 4.

Example 3. Consider the second-order Fredholm integrodifferential equation [5, 16]

\[
e^x y''(x) + \cos(x) y'(x) + \sin(x) y(x) + \int_{-1}^1 e^{(x+1)t} y(t) \, dt
\]

\[
= \left(\cos(x) + \sin(x) + e^x\right)e^x - 2 \frac{\sin h(x + 2)}{x + 2}, \quad -1 \leq x, t \leq 1
\]

subject to

\[
y(-1) + y'(1) = e + \frac{1}{e},
\]

\[
y(-1) + y'(-1) + y(1) = e,
\]

with the exact solution \(y(x) = e^x\).

In order to apply the presented method for solving the given problem, we should transform using an appropriate change of variables as

\[
x = 2\zeta - 1, \quad \zeta \in [0, 1].
\]

In this example, we set \(M = 9\), and \(K = 10\). In Table 5, absolute errors in solutions obtained by composite Chebyshev finite difference method are compared with Tau method and Chebyshev finite difference method. According to Table 5 using the proposed method, we can obtain approximate solution which is almost same as exact solution. For different values of \(K\) the errors of \(E_{KM}\) are shown in Table 6.
Example 4. Consider the first-order Fredholm integrodifferential equation [3, 17–19]

\[
y' (x) = (x + 1) e^x - x + \int_0^1 x y (t) \, dt, \quad 0 \leq x, t \leq 1, \tag{52}
\]

subject to

\[
y(0) = 0. \tag{53}
\]

M = 9, and K = 4 are considered to solve Example 4. In Table 7, absolute errors in solutions obtained by composite Chebyshev finite difference method are compared with CAS wavelet method, differential transfer method, Improved homotopy perturbation method, and a sequential method. It is illustrated in Table 7 that the results obtained using current method are very close to exact solution. The graph of absolute errors for M = 9, and K = 4 is shown in Figure 1.

Example 5. Consider the first-order Fredholm integrodifferential equation [6, 34, 35]

\[
y'' (x) + xy' (x) - xy = e^x - 2 \sin (x) \\
+ \int_{-1}^{1} \sin (x) e^{-t} y (t) \, dt, \quad -1 \leq x, t \leq 1, \tag{54}
\]

subject to

\[
y(0) = 1, \quad y' (0) = 1. \tag{55}
\]

This example is solved for M = 8 and K = 4. In order to apply the presented method for solving the given problem, we should transform using an appropriate change of variables as

\[
x = 2\zeta - 1, \quad \zeta \in [0, 1]. \tag{56}
\]

In Table 8, absolute errors in solutions obtained by composite Chebyshev finite difference method are compared with Legendre polynomial method. As can be shown in Table 8, the introduced method is more efficient than Legendre polynomial method, and the numerical results are in good agreement with exact solutions up to 13 decimal places. The graph of absolute errors for K = 4, and M = 8 is shown in Figure 2.

Example 6. Consider the third-order linear Fredholm integrodifferential equation [6]

\[
y''' (x) - y' (x) = 2x (\cos 1 - \sin 1) - 2 \cos x + \int_{-1}^{1} x t y (t) \, dt, \tag{57}
\]

subject to

\[
y(0) = 0, \quad y' (0) = 1, \quad y'' (0) - 2y' (0) = -2. \tag{58}
\]

The exact solution for this problem is \( y(x) = \sin x \). We solve the problem with \( m = 9, \) and \( n = 10 \). In Table 9, absolute errors in solutions obtained by composite Chebyshev finite difference method are compared with Legendre polynomial solutions. For different values of K the errors of \( E_{KM} \) are shown in Table 10. The graph of absolute errors for K = 10, and M = 9 is shown in Figure 3.
8. Conclusion

In this paper, we presented the composite Chebyshev finite difference method for solving Fredholm integrodifferential equations. The composite ChFD method is indeed an extension of the ChFD scheme with $K = 1$. This method is based on a hybrid of block-pulse functions and Chebyshev polynomials using Chebyshev–Gauss–Lobatto collocation points.

The useful properties of Chebyshev polynomials and block-pulse functions make it a computationally efficient method to approximate the solution of Fredholm integrodifferential equations. We converted the given problem to a system of algebraic equations using collocation points.

The main advantage of the present method is the ability to represent smooth and especially piecewise smooth functions properly. It was also shown that the accuracy can be improved either by increasing the number of subintervals or by increasing the number of collocation points in subintervals. Several examples have been provided to demonstrate the powerfulness of the proposed method. A comparison was made among the present method, some other well-known approaches, and exact solution which confirms that the introduced method is more accurate and efficient.

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References


