Research Article

A Multiplier Theorem for Herz-Type Hardy Spaces Associated with the Dunkl Transform

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The main purpose of this paper is to establish a Hörmander multiplier theorem for Herz-type Hardy spaces associated with the Dunkl transform.

1. Introduction

Let $T_m(f)$ be a multiplier operator defined in terms of Fourier transforms by $T_m(f) = \mathcal{F}^{-1}(m\mathcal{F}(f))$ for suitable functions f . The multiplier theorem of Hörmander [1] gives a sufficient condition on m for the operator T_m to be bounded on $L^p(\mathbb{R}^n)$ whenever $1 < p < \infty$, namely, that m is a bounded C^ℓ -function on $\mathbb{R}^n \setminus \{0\}$ satisfying the Hörmander condition $M(2, \ell)$ as follows:

$$\left(\int_R^{2R} |m^{(s)}(\xi)|^2 d\xi \right)^{1/2} \leq CR^{(n+1)/2-s}, \quad \forall R > 0, \quad (1)$$

where ℓ is the least integer greater than $n/2$ and $s = 0, 1, \dots, \ell$. In [2], the authors proved that if m satisfies the Hörmander condition with $\ell > n(1/p - 1/2)$, then T_m is bounded on the Hardy spaces $H^p(\mathbb{R}^n)$ with $0 < p \leq 1$.

In [3], the authors considered the following multiplier operator which is associated with the Dunkl transform:

$$T_m^\alpha(f) = \mathcal{F}_\alpha^{-1}(m\mathcal{F}_\alpha(f)), \quad (2)$$

where $\mathcal{F}_\alpha$ designs the Dunkl transform and using Hörmander's technique proved the following theorem.

Theorem 1. Let ℓ be the least integer greater than $\alpha + 1$ and let m be a bounded C^ℓ -function on $\mathbb{R} \setminus \{0\}$ which satisfies the Hörmander condition $M_\alpha(2, \ell)$ as follows:

$$\left(\int_R^{2R} |m^{(s)}(\xi)|^2 d\mu_\alpha(\xi) \right)^{1/2} \leq CR^{\alpha+1-s}, \quad \forall R > 0, \quad (3)$$

where C is a constant independent of R and $s = 0, 1, \dots, \ell$. Then, the multiplier operator associated with the Dunkl transform can be extended to a bounded operator from $L^p(\mu_\alpha)$ into itself for $1 < p < \infty$, where $L^p(\mu_\alpha)$ is the Lebesgue space on $\mathbb{R}$ with respect to the following measure:

$$\mu_\alpha(x) = (2^{\alpha+1} \Gamma(\alpha+1))^{-1} |x|^{2\alpha+1}, \quad \left(\alpha > -\frac{1}{2}\right). \quad (4)$$

The Hardy spaces associated with Herz spaces can be regarded as the local version at the origin of the classical Hardy spaces H^p and they are good substitutes for H^p when we study the boundedness of nontranslation invariant operators. To establish the boundedness of operators in hardy-type spaces on $\mathbb{R}^n$, one usually appeals to the atomic decomposition characterization of these spaces. In [4, 5], the authors studied the Herz-type Hardy spaces $H\dot{K}_{\alpha,2}^{\beta,p}$ for the Dunkl operator in one-dimension and gave an atomic decomposition characterization of these spaces. The aim of this work is to prove the following Hörmander multiplier theorem on the spaces $H\dot{K}_{\alpha,2}^{\beta,p}$.

Theorem 2. Let $0 < p \leq 1$, $\beta = (1/p) - (1/2)$, and ℓ be an integer greater than $2(\alpha+1)\beta$. If m satisfies the Hörmander condition $M_\alpha(2, \ell)$, then the operator T_m^α is bounded on $H\dot{K}_{\alpha,2}^{\beta,p}$.

The paper is organized as follows. In Section 2, we recall some results about harmonic analysis and Herz-type Hardy spaces associated with the Dunkl operator on $\mathbb{R}$. In Section 3, we give the proof of the main result of this work. Then, as

an application, we obtain the boundedness of the generalized Hilbert transform on $H\dot{K}_{\alpha,2}^{\beta,p}$.

Throughout this paper, let $\mathcal{S}(\mathbb{R})$ be the usual Schwartz space and let $\mathcal{E}(\mathbb{R})$ be the space of C^∞ -functions on $\mathbb{R}$. We always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the shorter notation $\|f\|_{p,\alpha}$ instead of $\|f\|_{L^p(\mu_\alpha)}$.

2. Preliminaries

In this section, we recapitulate some results about harmonic analysis on Dunkl hypergroups and the Herz-type Hardy space and its atomic decomposition which will be used later. For details, the reader is referred to [6–8].

Let $\alpha > -1/2$. We consider the differential-difference operator introduced in [9] as follows:

$$\Lambda_\alpha(f)(x) = \frac{df}{dx}(x) + \frac{2\alpha+1}{x} \cdot \frac{f(x) - f(-x)}{2}, \quad f \in \mathcal{E}(\mathbb{R}), \quad (5)$$

and call it the *Dunkl operator*.

For $\lambda \in \mathbb{C}$, the following initial value problem:

$$\Lambda_\alpha(f)(x) = \lambda f(x), \quad f(0) = 1, \quad x \in \mathbb{R}, \quad (6)$$

has a unique solution $E_\alpha(\lambda \cdot)$ (called the *Dunkl kernel*) given by

$$E_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(iz), \quad z \in \mathbb{C}, \quad (7)$$

where j_α is the normalized Bessel function of the first kind (with order α) defined on $\mathbb{C}$ by

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} (-1)^n \frac{(z/2)^{2n}}{n! \Gamma(n+\alpha+1)}. \quad (8)$$

The integral representation of E_α is given by

$$E_\alpha(i\lambda x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi} \Gamma(\alpha+1/2)} \int_{-1}^1 (1-t)(1-t^2)^{\alpha-(1/2)} e^{-i\lambda xt} dt. \quad (9)$$

From which, we get

$$|\partial_x^n E_\alpha(i\lambda x)| \leq |\lambda|^n, \quad \lambda, x \in \mathbb{R}, \quad n \in \mathbb{N}. \quad (10)$$

The *Dunkl transform* $\mathcal{F}_\alpha$, which was introduced by [10] and studied in [11], is defined for $f \in L^1(\mu_\alpha)$ by

$$\mathcal{F}_\alpha(f)(x) = \int_{\mathbb{R}} E_\alpha(-ixy) f(y) d\mu_\alpha(y), \quad x \in \mathbb{R}. \quad (11)$$

This transform satisfies the following properties.

(i) For all $f \in L^1(\mu_\alpha)$, we have

$$\|\mathcal{F}_\alpha(f)\|_{\infty,\alpha} \leq \|f\|_{1,\alpha}. \quad (12)$$

(ii) For all $f \in L^1(\mu_\alpha)$ such that $\mathcal{F}_\alpha(f) \in L^1(\mu_\alpha)$, we have the following inversion formula:

$$\mathcal{F}_\alpha^{-1}(f)(x) = \mathcal{F}_\alpha(f)(-x), \quad \text{a.e. } x \in \mathbb{R}. \quad (13)$$

(iii) For all $f \in \mathcal{S}(\mathbb{R})$,

$$\mathcal{F}_\alpha(\Lambda_\alpha f)(x) = ix \mathcal{F}_\alpha(f)(x). \quad (14)$$

(iv) $\mathcal{F}_\alpha$ is a topological isomorphism from $\mathcal{S}(\mathbb{R})$ into itself.

(v) $\mathcal{F}_\alpha$ is an isometric isomorphism of $L^2(\mu_\alpha)$, and we have the following Parseval formula:

$$\int_{\mathbb{R}} f(x) \overline{g(x)} d\mu_\alpha(x) = \int_{\mathbb{R}} \mathcal{F}_\alpha(f)(x) \overline{\mathcal{F}_\alpha(g)(x)} d\mu_\alpha(x),$$

$$\|\mathcal{F}_\alpha(f)\|_{2,\alpha} = \|f\|_{2,\alpha}. \quad (15)$$

The following lemma can be proved, similar to Lemma 7.25, page 343, in [12].

Lemma 3. *Let ℓ be the least integer greater than $\alpha + 1$. If m satisfies the Hörmander condition $M_\alpha(2, \ell)$, then there is a constant C independent of m , such that if $q = 1$ or $s - \ell + \alpha + 1 < (\alpha + 1)/q \leq \alpha + 1$, the following inequality holds:*

$$\int_{\mathbb{R}} |m^{(s)}(\xi)|^{2q} d\mu_\alpha(\xi) \leq CR^{2(\alpha+1)-2qs}, \quad \forall R > 0. \quad (16)$$

Furthermore, in case $s - \ell + \alpha + 1 < 0$, then $|x|^s |m^{(s)}(x)| \leq C$ and $m^{(s)}$ is continuous on $\mathbb{R} \setminus \{0\}$.

Notation. For all $x, y, z \in \mathbb{R}$, we put

$$W_\alpha(x, y, z) = [1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x}] \Delta_\alpha(|x|, |y|, |z|), \quad (17)$$

where

$$\sigma_{x,y,z} = \begin{cases} \frac{x^2 + y^2 - z^2}{2xy}, & \text{if } x, y \in \mathbb{R} \setminus \{0\}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\Delta_\alpha(|x|, |y|, |z|)$$

$$= \begin{cases} d_\alpha \frac{[(|x| + |y|)^2 - z^2] (z^2 - (|x| - |y|)^2)^{\alpha-1/2}}{|xyz|^{2\alpha}}, & \text{if } |z| \in A_{x,y}, \\ 0, & \text{otherwise,} \end{cases}$$

$$d_\alpha = \frac{2^{1-\alpha} \Gamma(\alpha+1)^2}{\sqrt{\pi} \Gamma(\alpha+1/2)},$$

$$A_{x,y} = [||x| - |y||, |x| + |y|]. \quad (18)$$

The Dunkl translation operator τ_x , $x \in \mathbb{R}$ is defined for a continuous function f on $\mathbb{R}$ by

$$\tau_x f(y) = \int_{\mathbb{R}} f(z) d\gamma_{x,y}(z), \quad y \in \mathbb{R}, \quad (19)$$

where $\gamma_{x,y}$ is the signed measures given by

$$d\gamma_{x,y}(z) = \begin{cases} W_\alpha(x, y, z) d\mu_\alpha(z), & \text{if } x, y \in \mathbb{R} \setminus \{0\}, \\ d\delta_x(z), & \text{if } y = 0, \\ d\delta_y(z), & \text{if } x = 0. \end{cases} \quad (20)$$

The operator τ_x has the following properties.

- (i) For $x, y \in \mathbb{R}$ and a continuous function f on $\mathbb{R}$, we have

$$\tau_x(f)(y) = \tau_y(f)(x). \quad (21)$$

- (ii) For all $x \in \mathbb{R}$, the operator τ_x can be extended to $L^p(\mu_\alpha)$ ($p \geq 1$), and for $f \in L^p(\mu_\alpha)$, we have

$$\|\tau_x(f)\|_{p,\alpha} \leq 3\|f\|_{p,\alpha}. \quad (22)$$

- (iii) For all $x, \lambda \in \mathbb{R}$ and $f \in L^1(\mu_\alpha)$, we have

$$\mathcal{F}_\alpha(\tau_x(f))(\lambda) = E_\alpha(i\lambda x) \mathcal{F}_\alpha(f)(\lambda). \quad (23)$$

Let $p, q, r \in [1, \infty]$ such that $1/p + 1/q = 1/r + 1$. The convolution product of $f \in L^p(\mu_\alpha)$ and $g \in L^q(\mu_\alpha)$ is defined by

$$f *_\alpha g(x) = \int_{\mathbb{R}} \tau_x(f)(-y) g(y) d\mu_\alpha(y), \quad \text{a.e. } x, \quad (24)$$

and we have

$$\|f *_\alpha g\|_{r,\alpha} \leq 3\|f\|_{p,\alpha} \|g\|_{q,\alpha}. \quad (25)$$

If $f, g \in L^1(\mu_\alpha)$, then

$$\mathcal{F}_\alpha(f *_\alpha g) = \mathcal{F}_\alpha(f) \mathcal{F}_\alpha(g). \quad (26)$$

Now, let us recall the definition of the Herz-type Hardy space and its atomic decomposition. For $N \in \mathbb{N}$ being sufficiently large, we denote by F_N the subset of $S(\mathbb{R})$ constituted by all those $\phi \in S(\mathbb{R})$ such that $\text{supp}(\phi) \subset [-1, 1]$ and for all $m, n \in \mathbb{N}$ such that $m, n \leq N$, we have

$$\rho_{m,n}(\phi) = \sup_{x \in \mathbb{R}} (1 + |x|)^m |\Lambda_\alpha^n \phi(x)| \leq 1. \quad (27)$$

Moreover, the system of seminorms $\{\rho_{m,n}\}_{m,n \in \mathbb{N}}$ generates the topology of $S(\mathbb{R})$.

Let $f \in S'(\mathbb{R})$. We define the α -grand maximal function $G_\alpha(f)$ of f by

$$G_\alpha(f)(x) = \sup_{t>0, \phi \in F_N} |\phi_t *_\alpha f(x)|, \quad x \in \mathbb{R}, \quad (28)$$

where ϕ_t is the dilation of ϕ given by

$$\phi_t(x) = t^{-2(\alpha+1)} \phi\left(\frac{x}{t}\right), \quad x \in \mathbb{R}. \quad (29)$$

Definition 4. Let $\beta \in \mathbb{R}$, $p \in]0, \infty[$, and $q \in [1, \infty]$.

- (i) The homogeneous weighted Herz space $\dot{K}_{\alpha,q}^{\beta,p}$ is the space constituted by all functions $f \in L_{\text{loc}}^q(\mu_\alpha)$, such that

$$\|f\|_{\dot{K}_{\alpha,q}^{\beta,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)\beta kp} \|f \chi_k\|_{q,\alpha}^p \right]^{1/p} < \infty, \quad (30)$$

where χ_k is the characteristic function of $A_k = \{x \in \mathbb{R} / 2^{k-1} \leq |x| \leq 2^k\}$.

- (ii) The nonhomogeneous weighted Herz space $K_{\alpha,q}^{\beta,p}$ is defined, as usual, by $K_{\alpha,q}^{\beta,p} = L^q(\mu_\alpha) \cap \dot{K}_{\alpha,q}^{\beta,p}$. Moreover, $\|f\|_{K_{\alpha,q}^{\beta,p}} = \|f\|_{q,\alpha} + \|f\|_{\dot{K}_{\alpha,q}^{\beta,p}}$.

Note that $\dot{K}_{\alpha,q}^{0,q} = K_{\alpha,q}^{0,q} = L^q(\mu_\alpha)$.

Definition 5. Let $\beta \in \mathbb{R}$, $p \in]0, \infty]$, and $q \in [1, \infty]$. The Herz-type Hardy space $H\dot{K}_{\alpha,q}^{\beta,p}$ is the space of distributions $f \in S'(\mathbb{R})$ such that $G_\alpha(f) \in \dot{K}_{\alpha,q}^{\beta,p}$. Moreover, we define

$$\|f\|_{H\dot{K}_{\alpha,q}^{\beta,p}} = \|G_\alpha(f)\|_{\dot{K}_{\alpha,q}^{\beta,p}}. \quad (31)$$

In the same way, we define the space $HK_{\alpha,q}^{\beta,p}$ for the non-homogeneous case.

Definition 6. Let $q \in [1, \infty]$ and $\beta \geq 1 - 1/q$. A measurable function a on $\mathbb{R}$ is called a (central) (β, q, s) -atom if it satisfies the following:

- (i) $\text{supp}(a) \subset [-r, r]$, for some $r > 0$,
(ii) $\|a\|_{q,\alpha} \leq r^{-2(\alpha+1)\beta}$,
(iii) $\int_{\mathbb{R}} a(x) x^k d\mu_\alpha(x) = 0$, $k = 0, 1, \dots, s$, where $s = [2(\alpha+1)(\beta-1+1/q)]$ and $[\cdot]$ denotes the integer part function.

The following theorem is shown in [4].

Theorem 7. Let $0 < p \leq 1 < q \leq \infty$ and $\beta \geq 1 - 1/q$. Then, $f \in H\dot{K}_{\alpha,q}^{\beta,p}$ if and only if, for all $j \in \mathbb{N} \setminus \{0\}$, there exist a (β, q, s) -atom a_j and $\lambda_j \in \mathbb{C}$, such that $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$ and $f = \sum_{j=1}^{\infty} \lambda_j a_j$. Moreover,

$$\|f\|_{H\dot{K}_{\alpha,q}^{\beta,p}} \sim \inf \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p}, \quad (32)$$

where the infimum is taking over all atomic decompositions of f .

In the sequel, fix $q = 2$ and $\beta = 1/p - 1/2$.

Definition 8. For $0 < p \leq 1$. Set $s \geq [2(\alpha + 1)(1/p - 1)]$, $\varepsilon > s/2(\alpha + 1)$, $a = 1 - \frac{1}{p} + \varepsilon$, and $b = 1/2 + \varepsilon$. A central (p, s, ε) -molecule is a function $M \in L^2(\mu_\alpha)$ satisfying the following:

- (i) $M(x)|x|^{2(\alpha+1)b} \in L^2(\mu_\alpha)$,
- (ii) $\|M\|_{2,\alpha}^{a/b} \|M(x)|x|^{2(\alpha+1)b}\|_{2,\alpha}^{1-a/b} \equiv N(M) < \infty$,
- (iii) $\int_{\mathbb{R}} M(x)x^k d\mu_\alpha(x) = 0, k = 0, 1, \dots, s$.

Proposition 9. Let (p, s, ε) be the triple cited in the previous definition. Every central (p, s, ε) -molecule M belongs to $HK_{\alpha,2}^{\beta,p}$ and $\|M\|_{HK_{\alpha,2}^{\beta,p}} \leq CN(M)$, where the constant C is independent of M .

Proof. Let M be a central (p, s, ε) -molecule and suppose that $\|M\|_{HK_{\alpha,2}^{\beta,p}} = 1$. In the general case, letting $\tilde{M} = M/\|M\|_{HK_{\alpha,2}^{\beta,p}}$, we have $\|\tilde{M}\|_{HK_{\alpha,2}^{\beta,p}} = 1$.

Let $E_0 = \{|x| \leq 1\}$, $E_k = \{2^{k-1} < |x| \leq 2^k\}$, and $M_k = M\chi_k$, $k = 1, 2, 3, \dots$, where χ_k is the characteristic function of E_k . For each k , there exists a unique polynomial Q_k , of degree at most s , such that if $P_k = Q_k\chi_k$, then

$$\int_{\mathbb{R}} (M_k - P_k) x^j d\mu_\alpha(x) = 0, \quad j = 0, 1, \dots, s. \quad (33)$$

Using some ideas in [2], we can show that each $(M_k - P_k)$ is a multiple of a central $(\beta, 2, s)$ -atom with a sequence of coefficients in l^p . We also show that the sum $\sum_{k=0}^{+\infty} P_k$ can be written as an infinite linear combination of central (β, ∞, s) -atom with a sequence of coefficients in l^p . Since a (β, ∞, s) -atom is also $(\beta, 2, s)$ -atom, hence,

$$M = \sum_{k=0}^{+\infty} M_k = \sum_{k=0}^{+\infty} (M_k - P_k) + \sum_{k=0}^{+\infty} P_k = \sum_{i=0}^{+\infty} \lambda_i a_i, \quad (34)$$

where a_i is a central $(\beta, 2, s)$ -atom and $\sum_{k=0}^{+\infty} |\lambda_i|^p < \infty$. It follows from Theorem 7 that $M \in HK_{\alpha,2}^{\beta,p}$ and $\|M\|_{HK_{\alpha,2}^{\beta,p}} \leq CN(M)$. $\square$

The following Lemma plays an important role in the proof of the main result of this work.

Lemma 10. Let a be a $(\beta, 2, s)$ -atom. For all integer $0 \leq k \leq s$ and every $1 \leq u \leq \infty$, there exists a constant C independent of a , such that

$$\begin{aligned} (i) \quad & \left| (\mathcal{F}_\alpha(a))^{(k)}(y) \right| \leq C|y|^{s+1-k} \|a\|_{2,\alpha}^A, \\ A = 1 - \frac{1}{\beta} \left(\frac{1}{2} + \frac{s+1}{2(\alpha+1)} \right) \quad & y \in \mathbb{R}, \\ (ii) \quad & \left\| \left((\mathcal{F}_\alpha(a))^{(k)}(y) \right)^2 \right\|_{u',\alpha} \leq C \|a\|_{2,\alpha}^{2-(1/\beta)((k/\alpha+1)+(1/u))}, \\ \frac{1}{u} + \frac{1}{u'} = 1 \quad & y \in \mathbb{R}. \end{aligned} \quad (35)$$

Proof. (i) Let a be a $(\beta, 2, s)$ -atom. Consider that $r > 0$ such that $\text{supp}(a) \subset [-r, r]$ and that $\|a\|_{2,\alpha} \leq r^{-2(\alpha+1)\beta}$. From (9), (iii) of Definition (19), and the estimate for the remainder in Taylors' formula, it follows that

$$\begin{aligned} & (\mathcal{F}_\alpha(a))^{(k)}(y) \\ &= C_\alpha \int_{-1}^1 \left((1-t) (1-t^2)^{\alpha-1/2} t^k \right. \\ & \quad \times \int_{-r}^r (ix)^k \left[\exp(ixyt) - \sum_{n=0}^{s-k} \frac{(ixyt)^n}{n!} \right] \\ & \quad \times a(x) d\mu_\alpha(x) \Big) dt \\ &\leq C|y|^{s+1-k} \int_{-r}^r |x|^{s+1} |a(x)| d\mu_\alpha(x) \\ &\leq C|y|^{s+1-k} \|a\|_{2,\alpha} \left[\int_{-r}^r |x|^{2(s+1)} d\mu_\alpha(x) \right]^{1/2} \\ &\leq C|y|^{s+1-k} \|a\|_{2,\alpha} r^{s+\alpha+2}. \end{aligned} \quad (36)$$

From (ii) of Definition (19), we obtain

$$\begin{aligned} & \left| (\mathcal{F}_\alpha(a))^{(k)}(y) \right| \leq C|y|^{s+1-k} \|a\|_{2,\alpha}^A, \\ A = 1 - \frac{1}{\beta} \left(\frac{1}{2} + \frac{s+1}{2(\alpha+1)} \right). \end{aligned} \quad (37)$$

(ii) For $u = 1$,

$$(\mathcal{F}_\alpha(a))^{(k)}(y) = \int_{-r}^r \partial_y^k E_\alpha(-iyx) a(x) d\mu_\alpha(x). \quad (38)$$

Using (10), we get the following for all $y \in \mathbb{R}$:

$$\begin{aligned} & \left| (\mathcal{F}_\alpha(a))^{(k)}(y) \right| \leq C \int_{-r}^r |x|^k |a(x)| d\mu_\alpha(x) \\ &\leq C \left(\int_{-r}^r |a(x)|^2 d\mu_\alpha(x) \right)^{1/2} \\ &\quad \times \left(\int_{-r}^r |x|^{2k} d\mu_\alpha(x) \right)^{1/2} \\ &\leq C \|a\|_{2,\alpha} r^{k+\alpha+1}. \end{aligned} \quad (39)$$

From (ii) of Definition (19), we obtain the following for all $y \in \mathbb{R}$:

$$\left| (\mathcal{F}_\alpha(a))^{(k)}(y) \right|^2 \leq C \|a\|_{2,\alpha}^{2-(1/\beta)((k/\alpha+1)+(1/u))}. \quad (40)$$

For $u = \infty$,

$$\begin{aligned} & \int_{\mathbb{R}} \left| (\mathcal{F}_\alpha(a))^{(k)}(x) \right|^2 d\mu_\alpha(x) \leq C \int_{-r}^r |x|^{2k} |a(x)|^2 d\mu_\alpha(x) \\ &\leq C r^{2k} \|a\|_{2,\alpha}^2 \\ &\leq C \|a\|_{2,\alpha}^{2-(1/\beta)((k/\alpha+1)+(1/u))}. \end{aligned} \quad (41)$$

For $1 < u < \infty$,

$$\begin{aligned} & \int_{\mathbb{R}} \left| (\mathcal{F}_\alpha(a))^{(k)}(y) \right|^{2u'} d\mu_\alpha(x) \\ &= \int_{\mathbb{R}} \left| (\mathcal{F}_\alpha(a))^{(k)}(y) \right|^2 \left| (\mathcal{F}_\alpha(a))^{(k)}(y) \right|^{2u'-2} d\mu_\alpha(x) \\ &\leq C \|a\|_{2,\alpha}^{(u'-1)(2-(1/\beta)(k/(\alpha+1)))} \int_{\mathbb{R}} \left| (\mathcal{F}_\alpha(a))^{(k)}(x) \right|^2 d\mu_\alpha(x) \\ &\leq C \|a\|_{2,\alpha}^{u'(2-(1/\beta)(k/(\alpha+1)+(1/u'))}. \end{aligned} \quad (42)$$

Finally, we get the following for all $y \in \mathbb{R}$:

$$\left\| ((\mathcal{F}_\alpha(a))^{(k)}(y))^2 \right\|_{u',\alpha} \leq C \|a\|_{2,\alpha}^{2-(1/\beta)(k/(\alpha+1)+(1/u))}. \quad (43)$$

3. Proof of Theorem 2

Let $0 < p \leq 1$ and ℓ be an integer greater than $2(\alpha+1)\beta$. Set $s = [2(\alpha+1)(1/p-1)]$, $\epsilon = \ell/2(\alpha+1) - (1/2)$, $a = 1 - (1/p) + \epsilon$, and $b = \epsilon + (1/2)$.

We have $\ell - 1 \geq s$; then, according to Proposition 9 to prove Theorem 2 it suffices to prove that, for any $(\beta, 2, \ell)$ -atom f , $T_m^\alpha f$ is a central (p, s, ϵ) -molecule with $N(T_m^\alpha f) < C$ for some constant C independent of f . In other words, we need to check that

$$\begin{aligned} & \text{(i) } T_m^\alpha f(\xi), T_m^\alpha f(\xi) |\xi|^\ell \in L^2(\mu_\alpha), \\ & \text{(ii) } \|T_m^\alpha f\|_{2,\alpha}^{a/b} \|T_m^\alpha f(\xi) |\xi|^\ell\|_{2,\alpha}^{1-a/b} \equiv N(T_m^\alpha f) < C, \\ & \text{(iii) } \int_{\mathbb{R}} T_m^\alpha f(\xi) \xi^j d\mu_\alpha(\xi) = 0 \quad \forall j = 0, 1, \dots, s. \end{aligned} \quad (44)$$

Firstly, we prove (i) and (ii).

m satisfies the Hörmander condition $M_\alpha(2, \ell)$; then, by Theorem 1, there exists a constant C independent of f , such that

$$\|T_m^\alpha f\|_{2,\alpha} \leq C \|f\|_{2,\alpha}. \quad (45)$$

From (14) and (13), we have

$$\Lambda_\alpha^\ell(\mathcal{F}_\alpha(T_m^\alpha f))(\xi) = \mathcal{F}_\alpha((-ix)^\ell T_m^\alpha f(x))(\xi). \quad (46)$$

Then, by Plancherel theorem to estimate $\|T_m^\alpha f(\xi) |\xi|^\ell\|_{2,\alpha}$, it suffices to estimate $\|\Lambda_\alpha^\ell(m\mathcal{F}_\alpha(f))\|_{2,\alpha}$, which turns out to prove that

$$\|\Lambda_\alpha^\ell(m\mathcal{F}_\alpha(f))\|_{2,\alpha} \leq C \|f\|_{2,\alpha}^{1-(\ell/2(\alpha+1)\beta)}. \quad (47)$$

By induction, we have

$$\begin{aligned} \Lambda_\alpha^\ell(m\mathcal{F}_\alpha(f))(\xi) &= \sum_{r=0}^l a_r \xi^{r-\ell} (m\mathcal{F}_\alpha(f))^{(r)}(\xi) \\ &\quad + \sum_{r=0}^l b_r \xi^{r-\ell} (m\mathcal{F}_\alpha(f))^{(r)}(-\xi), \end{aligned} \quad (48)$$

where a_r and b_r are constants.

But, using Leibniz formula, we have the following for $r \in \{0, 1, \dots, \ell\}$:

$$(m\mathcal{F}_\alpha(f))^{(r)}(\xi) = \sum_{k=0}^r C_r^k (\mathcal{F}_\alpha(f))^{(k)}(\xi) (m)^{(r-k)}(\xi). \quad (49)$$

So, to establish (47), it suffices to claim that

$$\begin{aligned} & \|\xi^{r-\ell} (\mathcal{F}_\alpha(f))^{(k)}(\xi) (m)^{(r-k)}(\xi)\|_{2,\alpha} \\ &\leq C \|f\|_{2,\alpha}^{1-(\ell/2(\alpha+1)\beta)} \quad \text{for all integers } 0 \leq k \leq r \leq \ell. \end{aligned} \quad (50)$$

For the case $k = \ell$, we use Lemma 10 (ii) with $u = \infty$ and Lemma 3 to get the following:

$$\begin{aligned} \|\xi^{r-\ell} (\mathcal{F}_\alpha(f))^{(k)}(\xi) (m)^{(r-k)}(\xi)\|_{2,\alpha} &\leq C \|(\mathcal{F}_\alpha(f))^{(k)}\|_{2,\alpha} \\ &\leq C \|f\|_{2,\alpha}^{1-(\ell/2(\alpha+1)\beta)}. \end{aligned} \quad (51)$$

For $0 \leq k < \ell$, we have

$$\begin{aligned} & \|\xi^{r-\ell} (\mathcal{F}_\alpha(f))^{(k)}(\xi) (m)^{(r-k)}(\xi)\|_{2,\alpha}^2 \\ &= \sum_{j \in \mathbb{Z}} \int_{2^j < |\xi| < 2^{j+1}} |\xi|^{2(r-\ell)} \left| (\mathcal{F}_\alpha(f))^{(k)}(\xi) \right|^2 \\ &\quad \times \left| (m)^{(r-k)}(\xi) \right|^2 d\mu_\alpha(\xi) \\ &= S_1 + S_2, \end{aligned} \quad (52)$$

where

$$\begin{aligned} S_1 &= \sum_{j=-\infty}^{j_0} \int_{2^j < |\xi| < 2^{j+1}} |\xi|^{2(r-\ell)} \left| (\mathcal{F}_\alpha(f))^{(k)}(\xi) \right|^2 \\ &\quad \times \left| (m)^{(r-k)}(\xi) \right|^2 d\mu_\alpha(\xi), \\ S_2 &= \sum_{j=j_0+1}^{+\infty} \int_{2^j < |\xi| < 2^{j+1}} |\xi|^{2(r-\ell)} \left| (\mathcal{F}_\alpha(f))^{(k)}(\xi) \right|^2 \\ &\quad \times \left| (m)^{(r-k)}(\xi) \right|^2 d\mu_\alpha(\xi), \end{aligned} \quad (53)$$

and j_0 is the integer, such that

$$2^{2(\alpha+1)j_0} < \|f\|_{2,\alpha}^{1/\beta} \leq 2^{2(\alpha+1)(j_0+1)}. \quad (54)$$

Firstly, we estimate S_1 .

Using (i) of Lemma 10 and the fact that m satisfies the Hörmander condition $M_\alpha(2, \ell)$, we get

$$\begin{aligned} & \int_{2^j < |\xi| < 2^{j+1}} |\xi|^{2(r-\ell)} \left| (\mathcal{F}_\alpha(f))^{(k)}(\xi) \right|^2 \left| (m)^{(r-k)}(\xi) \right|^2 d\mu_\alpha(\xi) \\ &\leq C \|f\|_{2,\alpha}^{2-(1/\beta)(\ell/(\alpha+1)+1)} \\ &\quad \times \int_{2^j < |\xi| < 2^{j+1}} |\xi|^{2(r+1-k)} \left| (m)^{(r-k)}(\xi) \right|^2 d\mu_\alpha(\xi) \\ &\leq C \|f\|_{2,\alpha}^{2-(1/\beta)(\ell/(\alpha+1)+1)} 2^{2(\alpha+1)j}. \end{aligned} \quad (55)$$

By (54), we obtain

$$S_1 \leq C \|f\|_{2,\alpha}^{2-(1/\beta)(\ell/(\alpha+1)+1)} 2^{2(\alpha+1)j_0} \leq C \|f\|_{2,\alpha}^{2-(\ell/(\alpha+1)\beta)}. \quad (56)$$

Now, we estimate S_2 . By Holder's inequality, we have

$$\begin{aligned} & \int_{2^j < |\xi| < 2^{j+1}} |\xi|^{2(r-\ell)} \left| (\mathcal{F}_\alpha(f))^{(k)}(\xi) \right|^2 \left| (m)^{(r-k)}(\xi) \right|^2 d\mu_\alpha(\xi) \\ & \leq 2^{2j(r-\ell)} \left(\int_{2^j < |\xi| < 2^{j+1}} \left| (\mathcal{F}_\alpha(f))^{(k)}(\xi) \right|^{2u'} d\mu_\alpha(\xi) \right)^{1/u'} \\ & \quad \times \left(\int_{2^j < |\xi| < 2^{j+1}} \left| (m)^{(r-k)}(\xi) \right|^{2u} d\mu_\alpha(\xi) \right)^{1/u}. \end{aligned} \quad (57)$$

Using (ii) of Lemmas 10 and 3, we get

$$S_2 \leq C \|f\|_{2,\alpha}^{2-(1/\beta)(k/(\alpha+1)+(1/u))} \sum_{j=j_0+1}^{+\infty} \left(2^{(2(\alpha+1)/u)-2(\ell-k)} \right)^j. \quad (58)$$

To guarantee the convergence of this summation, we choose the pair (k, u) as follows:

- (a) if $l - k > \alpha + 1$, we choose $u = 1$;
- (b) if $0 < l - k \leq \alpha + 1$ and $k > \alpha + 1$, we choose $u = \infty$;
- (c) if $0 < l - k \leq \alpha + 1$ and $k \leq \alpha + 1$, we choose $0 < u < \infty$

such that $k > (\alpha + 1)(1 - (1/u))$.

Furthermore, by (54), we get

$$\begin{aligned} S_2 & \leq C \|f\|_{2,\alpha}^{2-(1/\beta)(k/(\alpha+1)+(1/u))} \left(2^{(2(\alpha+1)/u)-2(\ell-k)} \right)^{j_0+1} \\ & \leq C \|f\|_{2,\alpha}^{2-(\ell/(\alpha+1)\beta)}. \end{aligned} \quad (59)$$

Finally, combining (56) and (59), we obtain (47). (i) and (ii) are hence proved.

To prove (iii), it suffices to prove that $T_m^\alpha f(\xi) \xi^j \in L^1(\mu_\alpha)$ for all integer $0 \leq j \leq s$ and $\Lambda_\alpha^j(m\mathcal{F}_\alpha(f))(0) = 0$: indeed if $T_m^\alpha f(\xi) \xi^j \in L^1(\mu_\alpha)$ according to (14), which we have $\Lambda_\alpha^j(\mathcal{F}_\alpha(T_m^\alpha f))(x) = C\mathcal{F}_\alpha(T_m^\alpha f(\xi) \xi^j)(x)$ is continuous, and hence $\int_{\mathbb{R}} T_m^\alpha f(\xi) \xi^j d\mu_\alpha(\xi) = C\Lambda_\alpha^j(m\mathcal{F}_\alpha(f))(0)$.

Now, we check $T_m^\alpha f(\xi) \xi^j \in L^1(\mu_\alpha)$. We write $\int_{\mathbb{R}} |T_m^\alpha f(\xi) \xi^j| d\mu_\alpha(\xi) = I_1 + I_2$, where

$$\begin{aligned} I_1 &= \int_{|\xi| \leq 1} |T_m^\alpha f(\xi) \xi^j| d\mu_\alpha(\xi), \\ I_2 &= \int_{|\xi| > 1} |T_m^\alpha f(\xi) \xi^j| d\mu_\alpha(\xi). \end{aligned} \quad (60)$$

Using the fact that $T_m^\alpha f \in L^2(\mu_\alpha)$ and Schwarz's inequality, we get

$$\begin{aligned} I_1 & \leq \int_{|\xi| \leq 1} |T_m^\alpha f(\xi)| d\mu_\alpha(\xi) \\ & \leq \left(\int_{|\xi| \leq 1} |T_m^\alpha f(\xi)|^2 d\mu_\alpha(\xi) \right)^{1/2} \left(\int_{|\xi| \leq 1} d\mu_\alpha(\xi) \right)^{1/2} \\ & \leq C \|T_m^\alpha f\|_{2,\alpha} \leq \infty. \end{aligned} \quad (61)$$

For $0 \leq j \leq s$, we have

$$\begin{aligned} I_2 & \leq \int_{|\xi| > 1} |T_m^\alpha f(\xi) \xi^s| d\mu_\alpha(\xi) \\ & \leq \left(\int_{|\xi| > 1} |T_m^\alpha f(\xi)|^2 |\xi|^{2\ell} d\mu_\alpha(\xi) \right)^{1/2} \\ & \quad \times \left(\int_{|\xi| > 1} |\xi|^{2(s-\ell)} d\mu_\alpha(\xi) \right)^{1/2} \\ & = \|T_m^\alpha f(\xi) |\xi|^\ell\|_{2,\alpha} \left(\int_{|\xi| > 1} |\xi|^{2(s-\ell)} d\mu_\alpha(\xi) \right)^{1/2}. \end{aligned} \quad (62)$$

Using the fact that $s - \ell < \alpha + 1$, we get $I_2 \leq C$.

Finally, we check

$$\Lambda_\alpha^j(m\mathcal{F}_\alpha(f))(0) = 0, \quad 0 \leq j \leq s. \quad (63)$$

We have

$$\begin{aligned} \Lambda_\alpha^j(m\mathcal{F}_\alpha(f))(h) &= \sum_{r=0}^j a_r h^{r-j} (m\mathcal{F}_\alpha(f))^{(r)}(h) \\ & \quad + \sum_{r=0}^j b_r h^{r-j} (m\mathcal{F}_\alpha(f))^{(r)}(-h), \end{aligned} \quad (64)$$

where a_r and b_r are constants. Then, to prove (63), it suffices to prove that

$$\begin{aligned} \lim_{h \rightarrow 0} |h^{r-j} m^{(r-k)}(h) (\mathcal{F}_\alpha(f))^{(k)}(h)| &= 0, \\ \text{for all integers } 0 \leq k \leq r \leq j \leq s. \end{aligned} \quad (65)$$

By (i) of Lemma 10, we have

$$\begin{aligned} & |h^{r-j} m^{(r-k)}(h) (\mathcal{F}_\alpha(f))^{(k)}(h)| \\ & \leq C |h|^{s+1-j} |h|^{r-k} |m^{(r-k)}(h)| \|f\|_{2,\alpha}^A. \end{aligned} \quad (66)$$

According to Lemma 3, we have $|h|^{r-k} |m^{(r-k)}(h)| \leq C$; indeed $2(r-k-\ell) + \alpha + 1 < 0$; then, we obtain

$$\lim_{h \rightarrow 0} |h^{r-j} m^{(r-k)}(h) (\mathcal{F}_\alpha(f))^{(k)}(h)| \leq C \lim_{h \rightarrow 0} |h|^{s+1-j} = 0, \quad (67)$$

where (63) is hence proved. This finishes the proof of Theorem 2.

Corollary 11. Let $0 < p \leq 1$. Then, the generalized Hilbert transform H_α defined by

$$H_\alpha(f) = \frac{\Gamma(\alpha + (3/2))}{\sqrt{\pi}\Gamma(\alpha + 1)} \lim_{\varepsilon \rightarrow 0} \int_{|y| > \varepsilon} \frac{\tau_x(f)(-y)}{y} dy, \quad (68)$$

where τ_x is given by (19), is bounded on $H\dot{K}_{\alpha,2}^{\beta,p}$.

Proof. From Proposition 3.6 in [3], the generalized Hilbert transform H_α is a multiplier operator T_m^α with $m(\xi) = -\text{sign}(\xi)$; then the proof of the corollary follows from Theorem 2. $\square$

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