Research Article

Existence of Solutions for a Periodic Boundary Value Problem via Generalized Weakly Contractions

Sirous Moradi, Erdal Karapınar, and Hassen Aydi

1 Department of Mathematics, Faculty of Science, Arak University, Arak 38156-8-8349, Iran
2 Department of Mathematics, Atilim University, İncek, 06836 Ankara, Turkey
3 Université de Sousse, Institut Supérieur d’Informatique et des Technologies de Communication de Hammam Sousse, Route GP1-4011 H., Sousse, Tunisia
4 Department of Mathematics, Jubail College of Education, Dammam University 31961, Saudi Arabia

Correspondence should be addressed to Erdal Karapınar; erdalkarapinar@yahoo.com

Received 21 December 2012; Accepted 19 February 2013

1. Introduction

Existence of solutions for a periodic boundary value problem by using upper and lower solution methods has attracted the attention of many authors (see, e.g., [1–5]).

We consider a special case of the following boundary value problem:

\[ u'(t) = f(t, u(t)) \quad \text{if} \quad t \in [0, T], \]
\[ u(0) = u(T) + \zeta_0, \]  \hspace{1cm} (1)

where \( T > 0, f : [0, T] \times \mathbb{R} \to \mathbb{R} \) is a continuous map and \( \zeta_0 \) is constant.

Obviously, if \( \zeta_0 = 0 \), then the problem (1) becomes the following periodic boundary value problem:

\[ u'(t) = f(t, u(t)) \quad \text{if} \quad t \in [0, T], \]
\[ u(0) = u(T). \]  \hspace{1cm} (2)

Definition 1. A lower solution for (1) is a function \( \alpha \in C^1([0, T]) \) such that

\[ \alpha'(t) \leq f(t, \alpha(t)) \quad \text{if} \quad t \in [0, T], \]
\[ \alpha(0) \leq \alpha(T) + \zeta_0. \]  \hspace{1cm} (3)

Let \( \mathcal{A} \) stand for the class of functions \( \phi : [0, +\infty) \to [0, +\infty) \), which satisfy the following conditions:

(i) \( \phi \) is nondecreasing,
(ii) \( \phi(x) < x \), for each \( x > 0 \),
(iii) \( \beta(x) = \phi(x)/x \in \mathcal{A} \).

Very recently, Amini-Harandi and Emami [1] proved the following existence theorem, which extended the main theorem of Harjani and Sadarangani [2].

Theorem 2. Consider problem (2), with \( f \) being continuous. Suppose that there exists \( \lambda > 0 \) such that for \( x, y \in \mathbb{R} \) with \( y \geq x \),

\[ 0 \leq f(t, y) + \lambda y - [f(t, x) + \lambda x] \leq \lambda \phi(y - x), \]  \hspace{1cm} (4)

where \( \phi \in \mathcal{A} \). Then, the existence of a lower solution for (2) provides the existence of a unique solution of (2).

In this paper, we solve (2) by extending a fixed point theorem in the context of partially ordered metric space. Our results improve/extend/generalize some results in the literature, in particular, the results of Amini-Harandi and
2. Preliminaries

In this section, we state a necessary background on the topic of fixed point theory, one of the core subjects of nonlinear analysis, for the sake of completeness of the paper. Fixed point theory has a wide potential application not only in the branches of mathematics, but also in several disciplines such as economics, computer science, and biology (see, e.g., [6, 7]). The most beautiful and elementary result in this direction is the Banach contraction mapping principle [8]. After this substantial result of Banach, several authors have extended this principle in many different ways (see, e.g., [1–7, 9–31]). In particular, the authors have introduced new types of contractions and researched the existence and uniqueness of the fixed point in various spaces. One of the important contraction types, a ϕ-contraction, was introduced by Boyd and Wong [14]. In 1997, Alber and Duerre-Delabriere [10] defined the concept of a weak-ϕ-contraction which is a generalization of the ϕ-contraction. A self-mapping f on a metric space (X, d) is said to be weak-ϕ-contractive if there exists a map ϕ : [0, +∞) → [0, +∞) with ϕ(0) = 0 and ϕ(t) > 0 for all t > 0 such that
\[ d(f(x), f(y)) \leq d(x, y) - \varphi(d(x, y)), \] for all x, y ∈ X.

Later, Zhang and Song [31] introduced the notion of a generalized weak-ϕ-contraction which is a natural extension of the weak-ϕ-contraction. A self-mapping f on a metric space (X, d) is said to be generalized weak-ϕ-contractive if there exists a map ϕ : [0, +∞) → [0, +∞) with ϕ(0) = 0 and ϕ(t) > 0 for all t > 0 such that
\[ d(f(x), f(y)) \leq N(x, y) - \varphi(N(x, y)), \] for all x, y ∈ X, where
\[ N(x, y) = \max \left\{ d(x, y), d(x, f(x)), d(y, f(y)), \frac{d(x, f(y)) + d(y, f(x))}{2} \right\}. \] For more details on weak-ϕ-contractions, we refer to, for example, [20, 21, 28].

On the other hand, the existence and uniqueness of a fixed point in the context of partially ordered metric spaces were first investigated in 1986 by Turinici [30]. After this pivotal paper, a number of results were reported in this direction with applications to matrix equations, ordinary differential equations, and integral equations (see, e.g., [1, 2, 4, 5, 7, 9, 11–13, 15–19, 22, 25–27]).

Recently, the main theorem of Geraghty [16, Theorem 2.1] is reproved by Amini-Harandi and Emami [1] in the context of partially ordered metric space. On the other hand, the main theorem of Amini-Harandi and Emami [1, Theorem 2.1] extends the theorem of Harjani and Sadarangani [2]. The authors in [1, 2] also proved the existence and uniqueness of a solution for a periodic boundary value problem.

Before stating the main theorem in [1], we recall the following class of functions introduced by Geraghty [16]. Let \( \mathcal{S} \) denote the set of all functions \( \psi : [0, +\infty) \to [0, 1) \) such that
\[ \psi(t_n) \to 1 \quad \text{implies} \quad t_n \to 0. \] Theorem 3. Let \((X, \preceq)\) be a partially ordered set and suppose that there exists a metric d in X such that \((X, d)\) is a complete metric space. Let \( f : X \to X \) be a nondecreasing mapping such that there exists an element \( x_0 \in X \) with \( x_0 \preceq f(x_0) \). Suppose that there exists \( \beta \in \mathcal{S} \) such that
\[ d(f(x), f(y)) \preceq \beta(d(x, y))d(x, y) \] for each \( x, y \in X \) with \( x \preceq y \).

Assume that either
(a) \( f \) is continuous or
(b) for every nondecreasing sequence \( \{x_n\} \) if \( x_n \to x \), then \( x_n \preceq x \) for all \( n \in \mathbb{N} \).

Moreover, if for each \( x, y \in X \) there exists \( z \in X \) which is comparable to \( x \) and \( y \), then \( f \) has a unique fixed point.

Let \( F(f) \) denote the set of fixed points of \( f \).

We give the following classes of functions. Let \( \Phi \) denote the set of all mappings \( \varphi : [0, +\infty) \to [0, +\infty) \) verifying that
\[ \varphi(t_n) \to 0 \quad \text{implies} \quad t_n \to 0. \] It is clear that if \( \varphi \in \Phi \), we have that
\[ \varphi(t) = 0 \quad \text{implies} \quad t = 0. \] 3. Some Auxiliary Fixed Point Theorems

In the following theorem, we prove the existence and uniqueness of a fixed point for generalized weak-ϕ-contractive mappings in partially ordered complete metric spaces.

Theorem 4. Let \((X, \preceq)\) be a partially ordered set and suppose that there exists a metric d in X such that \((X, d)\) is a complete metric space. Let \( f : X \to X \) be a nondecreasing mapping such that there exists an element \( x_0 \in X \) with \( x_0 \preceq f(x_0) \). Suppose that there exists \( \varphi \in \Phi \) such that
\[ d(f(x), f(y)) \leq N(x, y) - \varphi(N(x, y)), \] for each \( x, y \in X \) with \( x \preceq y \) (i.e., a generalized weak-ϕ-contraction).

Suppose also that either
(a) \( f \) is continuous or
(b) for every nondecreasing sequence \( \{x_n\} \) if \( x_n \to x \), then \( x_n \preceq x \) for all \( n \in \mathbb{N} \).
Then $f$ has a fixed point. Moreover, if every $x, y \in F(f)$ is comparable, then the fixed point of $f$ is unique.

**Proof.** First, we prove the existence of a fixed point of $f$. Since the self-mapping $f$ is nondecreasing and $x_0 \preceq f(x_0)$, we get that

$$x_0 \preceq f(x_0) \preceq f^2(x_0) \preceq \cdots \preceq f^n(x_0) \preceq \cdots$$

Define $x_n = f^n(x_0), n = 1, 2, 3, \ldots$. Then, expression (13) is equivalent to

$$x_n \preceq x_{n+1} \quad \forall n \in \mathbb{N}.$$  \hspace{1cm} (14)

Assume that $x_n \not\preceq x_{n+1}$ for each $n \in \mathbb{N}$. Otherwise, the proof is completed. From (12), we derive that

$$d(x_{n+1}, x_n) \preceq N(x_n, x_{n-1}) - \varphi(N(x_n, x_{n-1})),$$  \hspace{1cm} (15)

where

$$N(x_n, x_{n-1}) = \max \left\{ d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), \right\} \leq \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}.$$

If $N(x_n, x_{n-1}) = d(x_n, x_{n+1})$ for some $n$, then from (15) and (16), we have

$$0 < d(x_{n+1}, x_n) \preceq d(x_n, x_{n+1}) - \varphi(d(x_n, x_{n+1})).$$  \hspace{1cm} (17)

This is a contradiction. Hence, $N(x_n, x_{n-1}) = d(x_n, x_{n+1})$ for all $n \geq 1$. So by (15) and (16), we have for all $n \geq 1$,

$$d(x_{n+1}, x_n) \preceq d(x_n, x_{n+1}) - \varphi(d(x_n, x_{n+1})).$$  \hspace{1cm} (18)

Thus, we conclude that the nonnegative sequence $\{d(x_n, x_{n+1})\}$ is decreasing. Therefore, there exists $r \geq 0$ such that $\lim_{n \to \infty} d(x_n, x_{n+1}) = r$. By using (18), we find that

$$0 \leq \varphi(d(x_n, x_{n+1})) \leq d(x_n, x_{n+1}) - d(x_{n+1}, x_n).$$  \hspace{1cm} (19)

Taking $n \to \infty$ in (19), we get $\lim_{n \to \infty} \varphi(d(x_n, x_{n+1})) = 0$. Since $\varphi \in \Phi$, we obtain that $\lim_{n \to \infty} d(x_n, x_{n+1}) = 0$; that is, $r = 0$.

We prove that the iterative sequence $\{x_n\}$ is Cauchy. Take $m > n$, then $x_m \preceq x_n$. From (12), we obtain that

$$d(x_{m+1}, x_{n+1}) \preceq N(x_m, x_n) - \varphi(N(x_m, x_n)),$$  \hspace{1cm} (20)

and thus,

$$0 \leq \varphi(N(x_m, x_n)) \leq N(x_m, x_n) - d(x_{m+1}, x_{n+1}),$$  \hspace{1cm} (21)

where

$$N(x_m, x_n) = \max \left\{ d(x_{m+1}, x_{n+1}), d(x_{m+1}, x_{n+1}), d(x_n, x_{n+1}), \right\} \leq \frac{d(x_{m+1}, x_{n+1}) + d(x_{n+1}, x_{n+1})}{2}.$$

Hence, by (21),

$$0 \leq \varphi(N(x_m, x_n)) \leq d(x_{m+1}, x_{n+1}) + d(x_{n+1}, x_{n+1}).$$  \hspace{1cm} (22)

This shows that $\lim_{n \to \infty} \varphi(N(x_m, x_n)) = 0$; that is, $\{x_n\}$ is Cauchy. Since $(X, d)$ is a complete metric space, there then exists $x \in X$ such that $\lim_{n \to \infty} x_n = x$. Now, we prove that $x$ is a fixed point of $f$.

If (a) holds, that is, if $f$ is continuous, then

$$x = \lim_{n \to \infty} x_n = \lim_{n \to \infty} f(x_{n-1}) = f(x).$$  \hspace{1cm} (24)

Suppose that (b) holds. By using (12), we derive that

$$0 \leq \varphi(N(x_n, x)) \leq N(x_n, x) - d(x_{n+1}, f(x)).$$  \hspace{1cm} (25)

Thus, we have

$$d(x_n, f(x)) \preceq d(x_n, x_{n+1}) + d(x_{n+1}, f(x)).$$  \hspace{1cm} (26)

So $\lim_{n \to \infty} d(x_n, f(x)) = d(x, f(x))$. Taking $n \to \infty$ in (25), we get $\lim_{n \to \infty} \varphi(N(x_n, x)) = 0$. Since $\varphi \in \Phi$, we conclude that $\lim_{n \to \infty} N(x_n, x) = 0$. So $d(x, f(x)) = 0$ and hence $x = f(x)$.

Now, we show that this fixed point $x$ of the self-mapping $f$ is unique. If for each $x, y \in F(f)$, $x$ and $y$ are comparable, then the fixed point is unique. Let $x, y$ be two fixed points of $f$. Then $N(x, y) = d(x, y)$ and from (12), we conclude that $\varphi(d(x, y)) = 0$. Thus, $d(x, y) = 0$ and hence, $x = y$. This completes the proof.

The following consequence of Theorem 4 plays a crucial role in the proof of our main result, Theorem 9.

**Theorem 5.** Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ in $X$ such that $(X, d)$ is a complete metric space. Let $f : X \to X$ be a nondecreasing mapping such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that there exists $\varphi \in \Phi$ such that

$$d(f(x), f(y)) \preceq d(x, y) - \varphi(d(x, y)),$$  \hspace{1cm} (27)
for each \( x, y \in X \) with \( x \preceq y \) (i.e., weak-\( \varphi \)-contraction).

Suppose also that either

(a) \( f \) is continuous or
(b) for every nondecreasing sequence \( \{x_n\} \) if \( x_n \to x \), then \( x_n \preceq x \) for all \( n \in \mathbb{N} \).

Then \( f \) has a fixed point. Moreover, if for each \( x, y \in F(f) \) there exists \( z \in X \) which is comparable to \( x \) and \( y \), then the fixed point of \( f \) is unique.

**Remark 6.** In Theorem 4, if the condition “every \( x, y \in F(f) \) is comparable” is replaced by the condition “for each \( x, y \in F(f) \) there exists \( z \in X \) which is comparable to \( x \) and \( y \)”, then we cannot conclude that the fixed point is unique. The following example illustrates our claim.

**Example 7.** Let \( X = \{x, y, z, w\} \) be endowed with the relation \( \preceq \) given as follows:

\[
\begin{align*}
& x \preceq z, \quad x \preceq w, \\
& y \preceq z, \quad y \preceq w,
\end{align*}
\]

and \( a \preceq a \) for each \( a \in X \). Obviously, \((X, \preceq)\) is a partially ordered set. Also, we may endow \( X \) with the following metric:

\[
\begin{align*}
& d(x, z) = d(x, w) = d(y, z) = d(y, w) = 1, \\
& d(z, w) = 2,
\end{align*}
\]

and \( d(a, a) = 0 \) for each \( a \in X \). Define \( f : X \to X \) by \( f(x) = x, f(y) = y, f(z) = w, \) and \( f(w) = z \). Obviously, the mapping \( f \) is nondecreasing and

\[
d(f(a), f(b)) \leq d(a, b) - \varphi(d(a, b)),
\]

for all \( a, b \in X \) with \( a \preceq b \), where \( \varphi(t) = (1/3)t \). Also \( F(f) = \{x, y\} \), but \( x \preceq z \) and \( y \preceq z \).

**Remark 8.** If \( \beta \in \mathcal{S} \), then \( \varphi(t) = t - \beta(t) \in \Phi \). But if \( \varphi \notin \Phi \), then we can not conclude that the function

\[
\beta(t) = \begin{cases} 
1 - \frac{\varphi(t)}{t}, & t > 0 \\
0, & t = 0 
\end{cases}
\]

belongs to \( \mathcal{S} \). Consider, for example,

\[
\varphi(t) = \begin{cases} 
\frac{1}{2}t, & 0 \leq t < 1 \\
\frac{1}{2}, & 1 \leq t.
\end{cases}
\]

which illustrates our claim. As a result, Theorem 5 is a proper extension of Theorem 3.

**4. Applications**

**4.1. Solving a Boundary Value Problem.** In this paragraph, we prove the existence of a solution of the problem (1).

**Theorem 9.** Consider problem (1) with \( f \) being continuous. Suppose that there exists \( \lambda > 0 \) such that for \( x, y \in \mathbb{R} \) with \( y \geq x \)

\[
0 \leq f(t, y) + \lambda y - [f(t, x) + \lambda x] \\
\leq \lambda [(y - x) - \varphi(y - x)],
\]

where \( \varphi \in \Phi \) and \( t \mapsto t - \varphi(t) \) is nondecreasing. Then the existence of a lower solution for (1) provides the existence of a unique solution for (1).

**Proof.** Define \( \zeta = \zeta_0 / T \). Then, problem (1) becomes as follows

\[
\begin{align*}
& u'(t) = f(t, u(t)) & \text{if } t \in [0, T], \\
& u(0) = u(T) + \zeta T.
\end{align*}
\]

Suppose \( y(t) = u(t) + \zeta t \). So \( y'(t) = u'(t) + \zeta \) and hence problem (34) can be rewritten as

\[
y'(t) = h(t, y(t)) & \text{if } t \in [0, T], \\
y(0) = y(T).
\]

where \( h : I \times \mathbb{R} \to \mathbb{R} \) is defined by \( h(t, z) = f(t, z - \xi t) + \zeta t \) and \( I = [0, T] \). Obviously, \( h \) is continuous. Also the lower solution of (34) is replaced by the lower solution of (35). Now we prove that the problem (35) has a unique solution. Obviously, if \( x, y \in \mathbb{R} \) and \( y \geq x \), then for every \( t \in I, y - \xi t \geq x - \xi t \) and hence from (33),

\[
0 \leq f(t, y - \xi t) + \lambda (y - \xi t) - [f(t, x - \xi t) + \lambda (x - \xi t)] \\
\leq \lambda [(y - \xi t) - (x - \xi t)] - \varphi((y - \xi t) - (x - \xi t))].
\]

Inequality (36) implies that if \( x, y \in \mathbb{R} \),

\[
0 \leq h(t, y) + \lambda y - [h(t, x) + \lambda x] \leq \lambda [(y - x) - \varphi(y - x)].
\]

Problem (35) is equivalent to the following integral equation:

\[
y(t) = \int_0^T G(t, s) \left[ h(s, y(s)) + \lambda y(s) \right] ds,
\]

where

\[
G(t, s) = \begin{cases} 
\frac{e^\lambda(t+s-t)}{e^\lambda - 1}, & 0 \leq s < t \leq T, \\
\frac{e^\lambda(t-t)}{e^\lambda - 1}, & 0 \leq t < s \leq T.
\end{cases}
\]

Let \( C(I, \mathbb{R}) \) be the set of continuous functions defined on \( I = [0, T] \). Consider \( F : C(I, \mathbb{R}) \to C(I, \mathbb{R}) \) given by

\[
(Fy)(t) = \int_0^T G(t, s) \left[ h(s, y(s)) + \lambda y(s) \right] ds.
\]

Note that if \( y \in C(I, \mathbb{R}) \) is a fixed point of \( F \), then \( y \in C^1(I, \mathbb{R}) \) is a solution of (35). Now, we check that hypotheses of Theorem 5 are satisfied.
Take $X = C(I, \mathbb{R})$. The space $X$ can be equipped with a partial order $\leq$ given by
\[ x, y \in C(I, \mathbb{R}), \quad x \leq y \iff x(t) \leq y(t), \quad \forall t \in I. \] (41)

Also, $X$ can be equipped with the following metric:
\[ x, y \in C(I, \mathbb{R}), \quad d(x, y) = \sup_{t \in I} |x(t) - y(t)|. \] (42)

We have that $(X, d)$ is complete. For every $y \geq x$ and for every $t \in I$, we have $y - t\zeta \geq x - t\zeta$ and by hypothesis,
\[ f(t, y - t\zeta) + \lambda(y - t\zeta) \geq f(t, x - t\zeta) + \lambda(x - t\zeta). \] (43)

Therefore,
\[ h(t, y) + \lambda y \geq h(t, x) + \lambda x, \] (44)

and since $G(t, s) > 0$ for $(t, s) \in I \times I$, hence
\[ (Fy)(t) \geq (Fx)(t), \] (45)

for all $x, y \in C(I, \mathbb{R})$ with $y \geq x$.

Also, for all $x, y \in C(I, \mathbb{R})$ with $y \geq x$, we find (using the fact that $t \mapsto t - \varphi(t)$ is non-decreasing)
\[ d(Fy,Fx) = \sup_{t \in I} \left| (Fy)(t) - (Fx)(t) \right| \]
\[ \leq \sup_{t \in I} \int_0^T G(t, s) \times \left| h(s, y(s)) + \lambda y(s) - h(s, x(s)) - \lambda x(s) \right| ds \]
\[ \leq \sup_{t \in I} \int_0^T G(t, s) \lambda \left| (y(s) - x(s)) - \varphi(y(s) - x(s)) \right| ds \]
\[ \leq \lambda \left( \sup_{t \in I} \frac{1}{e^{AT} - 1} \left( \frac{1}{\lambda} e^{\lambda(T+t)} + \frac{1}{\lambda} e^{\lambda(T-t)} \right) \right) \]
\[ = \lambda \left[ d(y, x) - \varphi(d(y, x)) \right] \sup_{t \in I} \int_0^T G(t, s) ds \]
\[ = \lambda \left[ d(y, x) - \varphi(d(y, x)) \right] \frac{1}{e^{AT} - 1} \]
\[ = \lambda \left[ d(y, x) - \varphi(d(y, x)) \right] \frac{1}{e^{AT} - 1}. \] (46)

Finally, let $\alpha(t)$ be a lower solution for (35). We can show that $\alpha \leq F\alpha$ by a method similar to that in [1, 2]. Also, $X$ is totally ordered. Hence, due to Theorem 5, $F$ has a unique fixed point. Therefore, problem (35) has a unique solution $y \in C^1(I, \mathbb{R})$. Thus, $x(t) = y(t) - \zeta t$ is the unique solution of (34) and this completes the proof.

**Remark 10.** If the mapping $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfies the condition (33), then for $x, y \in \mathbb{R}$ with $y \geq x$ and for $t \in [0, T]$,\n\[ -\lambda(y - x) \leq f(t, y) - f(t, x) \leq -\lambda\varphi(y - x) \leq 0. \] (47)

Hence, for all $x, y \in \mathbb{R}$ and all $t \in [0, T]$, we have
\[ \left| f(t, y) - f(t, x) \right| \leq \lambda \left| y - x \right|. \] (48)

Therefore, by using Banach contraction principle, for every $\eta \in \mathbb{R}$, the problem
\[ u'(t) = f(t, u(t)) \quad \text{if} \quad t \in [0, T] \]
\[ u(0) = \eta \] (49)

has a unique solution $u_\eta \in C^1([0, T])$. So there exists a unique $\eta \in \mathbb{R}$ such that $u_\eta$ is a solution of (1).

Now let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a mapping such that for all $x, y \in \mathbb{R}$ and all $t \in [0, T]$,\n\[ \left| f(t, y) - f(t, x) \right| \leq R \left| y - x \right|, \] (50)

for some $R > 0$. We know that for every $\eta \in \mathbb{R}$, problem (49) has a unique solution $u_\eta \in C^1([0, T])$.

**Question 1.** It is natural to ask whether there is an $\eta \in \mathbb{R}$ where $u_\eta$ is a solution of problem (2), i.e., $(u_\eta(0) = u_\eta(T))$?

The following example shows that the above question is not true.

**Example 12.** Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be defined by $f(t, x) = t + |x|$. Obviously, (50) holds for $R = 1$. Let $u \in C^1([0, T])$ be a solution for problem (2). From $u'(t) = f(t, u(t)) = t + |u(t)|$, we conclude that $u'(t) > 0$ for all $t > 0$. Hence, $u$ is monotone non-decreasing. Using $u(0) = u(T)$, we conclude that $u \equiv 0$. Since $u'(t) = f(t, u(t)) = t + |u(t)|$ and $u \equiv 0$, then $t = 0$ for all $t \in [0, T]$ and this is a contradiction. So, problem (2) has no solution.

**Example 12.** Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be defined by $f(t, x) = \exp(t - (1/2)x)$ and let $\varphi : [0, +\infty) \to [0, +\infty)$ be defined by $\varphi(t) = (1/3)t$. Take $\lambda = 1$. One can show that inequality (33) holds. Suppose that $\alpha : \mathbb{R} \to \mathbb{R}$ is defined by $\alpha(t) = 0$. Obviously, $\alpha$ is a lower solution of problem (2). Hence, problem (2) has a unique solution, which is
\[ u(t) = \frac{2}{3} \exp(t) + C \exp \left( \frac{-1}{2}t \right), \] (51)

where $C = 2\exp(T) - 1/3(1 - \exp((-1/2)t))$.

4.2. **Solving Some Polynomials.** In this paragraph, we prove the existence and uniqueness of a solution of some polynomials.

**Theorem 13.** Let $a_0, a_1, \ldots, a_{k-1} \in [0, +\infty)$ be such that $a_1 + a_2 + \cdots + a_{k-1} < 1$ and $a_0 \geq 1$. Then,
\[ y^k = a_{k-1}y^{k-1} + a_{k-2}y^{k-2} + \cdots + a_1y + a_0 \] (52)

has a unique solution on $[\sqrt{a_0}, +\infty)$.
Proof. Suppose that \( f : [a_0, +\infty) \to [a_0, +\infty) \) is defined by
\[
f(x) = a_{k-1} \sqrt[k]{x^{k-1}} + a_{k-2} \sqrt[k]{x^{k-2}} + \cdots + a_1 \sqrt[k]{x} + a_0.
\] (53)
If \( x \leq y \), then \( f(x) \leq f(y) \). So \( f \) is nondecreasing. Also for \( x, y \in [a_0, +\infty) \) with \( x \leq y \), we derive that
\[
0 \leq f(y) - f(x) = a_{k-1} \left( \sqrt[k]{y^{k-1}} - \sqrt[k]{x^{k-1}} \right) + a_{k-2} \left( \sqrt[k]{y^{k-2}} - \sqrt[k]{x^{k-2}} \right) + \cdots + a_1 \sqrt[k]{y - x}.
\] (54)
Therefore, from (54), we get
\[
0 \leq f(y) - f(x) \leq (y - x) - \varphi(y - x),
\] (55)
where \( \varphi(t) = \left[ 1 - (a_{k-1} + a_{k-2} + \cdots + a_1) \right] t \). Also \( a_k \leq f(a_k) \). Thus, using Theorem 5, the mapping \( f \) has a unique fixed point \( x \in [a_0, +\infty) \). Moreover, the sequence \( \{f^n(a_k)\} \) converges to this fixed point. Note that the space \( X \) is taken to be \( [a_0, +\infty) \), which is equipped with the usual Euclidean metric and the usual partial order.

On the other hand, there exists a unique \( y \in [\sqrt[k]{a_0}, +\infty) \) such that \( y^k = x \). So, from \( x = f(x) \), we have \( y^k = f(y^k) \) and therefore we find
\[
y^k = a_{k-1} y^{k-1} + a_{k-2} y^{k-2} + \cdots + a_1 y + a_0.
\] (56)
Also the sequence \( \{\sqrt[k]{f^n(a_k)}\} \) converges to \( y \) and this completes the proof. \( \square \)

References


