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Research Article

A Uniqueness Theorem for Bessel Operator from Interior Spectral Data

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Inverse problem for the Bessel operator is studied. A set of values of eigenfunctions at some internal point and parts of two spectra are taken as data. Uniqueness theorems are obtained. The approach that was used in investigation of problems with partially known potential is employed.

1. Introduction

Inverse spectral analysis involves the problem of restoring a linear operator from some of its spectral parameters. Currently, inverse problems are being studied for certain special classes of ordinary differential operators. The simplest of these is the Sturm-Liouville operator $Ly = -y'' + q(x)y$. For the case where it is considered on the whole line or half line, the Sturm-Liouville operator together with the function $q(x)$ has been called a potential. In this direction, Borg [1] gave important results. He showed that, in general, one spectrum does not determine a Sturm-Liouville operator, so the result of Ambarzumyan [2] is an exception to the general rule. In the same paper, Borg showed that two spectra of a Sturm-Liouville operator determine it uniquely. Later, Levinson [3], Levitan [4], and Hochstadt [5] showed that when the boundary condition and one possible reduced spectrum are given, then the potential is uniquely determined. Using spectral data, that is, the spectral function, spectrum, and norming constant, different methods have been proposed for obtaining the potential function in a Sturm-Liouville problem. Such problems were subsequently investigated by other authors [4–6]. On the other hand, inverse problems for regular and singular Sturm-Liouville operators have been extensively studied by [7–15].

The inverse problem for interior spectral data of the differential operator consists in reconstruction of this operator from the known eigenvalues and some information on eigenfunctions at some internal point. Similar problems for the Sturm-Liouville operator and discontinuous Sturm-Liouville problem were formulated and studied in [16, 17].

The main goal of the present work is to study the inverse problem of reconstructing the singular Sturm-Liouville operator on the basis of spectral data of a kind: one spectrum and some information on eigenfunctions at the internal point.

Consider the following singular Sturm-Liouville operator L satisfying (1)–(3):

$$Ly = -y'' + \left[\frac{\ell(\ell+1)}{x^2} + q(x)\right] y = \lambda y, \quad 0 < x < 1 \tag{1}$$

with boundary conditions,

$$y(0) = 0, \tag{2}$$

$$y'(1,\lambda) + Hy(1,\lambda) = 0, \tag{3}$$

where $q(x)$ is a real-valued function and $q \in L_2(0,1)$, λ spectral parameter, $\ell \in \mathbb{N}_0$, $H \in \mathbb{R}$. The operator L is self adjoint on the $L_2(0,1)$ and has a discrete spectrum $\{\lambda_n\}$.

Let us introduce the second singular Sturm-Liouville operator $\widetilde{L}$ satisfying

$$\widetilde{L}y = -y'' + \left[\frac{\ell(\ell+1)}{x^2} + \widetilde{q}(x)\right] y = \lambda y, \quad 0 < x < 1, \quad (4)$$

subject to the same boundary conditions (2), (3), where $\widetilde{q}(x)$ is a real-valued function and $\widetilde{q} \in L_2(0,1)$. The operator $\widetilde{L}$ is self adjoint on the $L_2(0,1)$ and has a discrete spectrum $\{\widetilde{\lambda}_n\}$.

2. Main Results

Before giving some results concerning the Bessel equation, we should give its physical properties. The total energy of the particle is given by $E = p^2/2M = \hbar^2 k^2/2M = k^2$, where p is its initial or final momentum, and k the corresponding wave number, $\hbar$ planck constant, M particle's mass, and E energy. The reduced radial Schrödinger equation for the partial wave of angular momentum ℓ then reads [18]

$$\frac{d^2}{dr^2}\Psi_1(k,r) + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right)\Psi_1(k,r) = V(r)\Psi_1(k,r). \quad (5)$$

When $V = 0$, the above equation reduces to the classical Bessel equation in the form

$$\frac{d^2}{dr^2}\Psi_1(k,r) + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right)\Psi_1(k,r) = 0. \quad (6)$$

This equation has the solution $J_\ell(r)$, called the Bessel function.

Eigenvalues of the problem (1)–(3) are the roots of (3). This spectral characteristic satisfies the following asymptotic expression [19, 20]:

$$\lambda_n = \left(n + \frac{\ell}{2}\right)^2 \pi^2 + \int_0^1 q(x)\,dx - l(l+1) + a_n, \quad (7)$$

where the series $\sum_{n=1}^{\infty} a_n^2 < \infty$. Next, we present the main results in this paper. When $b = 1/2$, we get the following uniqueness Theorem 1.

Theorem 1. *If for every $n \in \mathbb{N}$ one has*

$$\lambda_n = \widetilde{\lambda}_n, \qquad \frac{y_n'(1/2)}{y_n(1/2)} = \frac{\widetilde{y}_n'(1/2)}{\widetilde{y}_n(1/2)} \quad (8)$$

then

$$q(x) = \widetilde{q}(x) \quad \text{a.e on the interval } (0,1). \quad (9)$$

In the case $b \neq 1/2$, the uniqueness of $q(x)$ can be proved if we require the knowledge of a part of the second spectrum.

Let $\{m(n)\}_{n\in\mathbb{N}}$ be a sequence of natural numbers with a property

$$m(n) = \frac{n}{\sigma}(1+\varepsilon_n), \quad 0 < \sigma \le 1, \ \varepsilon_n \longrightarrow 0. \quad (10)$$

Lemma 2. *Let $\{m(n)\}_{n\in\mathbb{N}}$ be a sequence of natural numbers satisfying* (10) *and $b \in (0,1/2)$ are so chosen that $\sigma > 2b$. If for any $n \in \mathbb{N}$*

$$\lambda_{m(n)} = \widetilde{\lambda}_{m(n)}, \qquad \frac{y_{m(n)}'(b)}{y_{m(n)}(b)} = \frac{\widetilde{y}_{m(n)}'(b)}{\widetilde{y}_{m(n)}(b)} \quad (11)$$

then

$$q(x) = \widetilde{q}(x) \quad \text{a.e on } (0,b]. \quad (12)$$

Let $\{l(n)\}_{n\in\mathbb{N}}$ and $\{r(n)\}_{n\in\mathbb{N}}$ be a sequence of natural numbers such that

$$l(n) = \frac{n}{\sigma_1}(1+\varepsilon_{1,n}), \quad 0 < \sigma_1 \le 1, \ \varepsilon_{1,n} \longrightarrow 0, \quad (13)$$

$$r(n) = \frac{n}{\sigma_2}(1+\varepsilon_{2,n}), \quad 0 < \sigma_2 \le 1, \ \varepsilon_{2,n} \longrightarrow 0 \quad (14)$$

and let μ_n be the eigenvalues of (1), (2), and (15) and $\widetilde{\mu}_n$ be the eigenvalues of (4), (2), and (15)

$$y'(1,\lambda) + H_1 y(1,\lambda) = 0, \quad H \neq H_1. \quad (15)$$

Using Mochizuki and Trooshin's method from Lemma 2 and Theorem 1, we will prove that the following Theorem 3 holds.

Theorem 3. *Let $\{l(n)\}_{n\in\mathbb{N}}$ and $\{r(n)\}_{n\in\mathbb{N}}$ be a sequence of natural numbers satisfying* (13) *and* (14), *and $1/2 < b < 1$ are so chosen that $\sigma_1 > 2b - 1$, $\sigma_2 > 2 - 2b$. If for any $n \in \mathbb{N}$ one has*

$$\lambda_n = \widetilde{\lambda}_n, \qquad \mu_{l(n)} = \widetilde{\mu}_{l(n)}, \qquad \frac{y_{r(n)}'(b)}{y_{r(n)}(b)} = \frac{\widetilde{y}_{r(n)}'(b)}{\widetilde{y}_{r(n)}(b)} \quad (16)$$

then

$$q(x) = \widetilde{q}(x) \quad \text{a.e on } (0,1). \quad (17)$$

3. Proof of the Main Results

In this section, we present the proofs of main results in this paper.

Proof of Theorem 1. Before proving Theorem 1, we will mention some results, which will be needed later. We get the initial value problems

$$-y'' + \left[\frac{\ell(\ell+1)}{x^2} + q(x)\right] y = \lambda y, \quad 0 < x < 1, \quad (18)$$

$$y(0) = 0, \quad (19)$$

$$-\widetilde{y}'' + \left[\frac{\ell(\ell+1)}{x^2} + \widetilde{q}(x)\right] \widetilde{y} = \lambda \widetilde{y}, \quad 0 < x < 1, \quad (20)$$

$$\widetilde{y}(0) = 0. \quad (21)$$

As known from [18], Bessel's functions of the first kind of order $\nu = \ell - 1/2$ are

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{\nu+2k}}{2^{\nu+2k} k! \Gamma(\nu+k+1)} \quad (22)$$

and asymptotic formulas for large argument

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\left\{\cos\left[x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right] + O\left(\frac{1}{x}\right)\right\},$$
$$J'_\nu(x) = -\sqrt{\frac{2}{\pi x}}\left\{\sin\left[x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right] + O\left(\frac{1}{x}\right)\right\}. \tag{23}$$

It can be shown [19] that there exists a kernel $H(x,t)(\widetilde{H}(x,t))$ continuous in the triangle $0 \le t \le x \le 1$ such that by using the transformation operator every solution of (18), (19) and (20), (21) can be expressed in the form [8, 21],

$$y(x,\lambda) = \frac{\sqrt{x}}{\left(\sqrt{\lambda}\right)^\nu} J_\nu\left(\sqrt{\lambda}x\right) + \int_0^x H(x,t)\frac{\sqrt{t}}{\left(\sqrt{\lambda}\right)^\nu} J_\nu\left(\sqrt{\lambda}t\right)dt, \tag{24}$$

$$\widetilde{y}(x,\lambda) = \frac{\sqrt{x}}{\left(\sqrt{\lambda}\right)^\nu} J_\nu\left(\sqrt{\lambda}x\right) + \int_0^x \widetilde{H}(x,t)\frac{\sqrt{t}}{\left(\sqrt{\lambda}\right)^\nu} J_\nu\left(\sqrt{\lambda}t\right)dt, \tag{25}$$

respectively, where the kernel $H(x,t)$ $(\widetilde{H}(x,t))$ is the solution of the equation

$$\frac{\partial^2 H(x,t)}{\partial x^2} + \frac{\ell(\ell+1)}{x^2}H(x,t) = \frac{\partial^2 H(x,t)}{\partial t^2} + \left(\frac{\ell(\ell+1)}{t^2} + q(t)\right)H(x,t) \tag{26}$$

subject to the boundary conditions

$$2\frac{dH(x,x)}{dx} = q(x),$$
$$\lim_{t\to 0} H(x,t)\,t^{\nu-1/2} = 0, \qquad \left[J'_\nu(t,\lambda) = O\left(t^{\nu-1/2}\right)\right]. \tag{27}$$

After the transformations

$$\xi = \frac{1}{4}(x+t)^2, \qquad \eta = \frac{1}{4}(x-t)^2,$$
$$H(x,t) = (\xi-\eta)^{-\nu+1/2}U(\xi,\eta), \tag{28}$$

we obtain the following problem:

$$\frac{\partial^2 U}{\partial\xi\partial\eta} - \frac{1}{4(\xi-\eta)}\frac{\partial U}{\partial\xi} + \frac{1}{4(\xi-\eta)}\frac{\partial U}{\partial\eta} = \frac{1}{4\sqrt{\xi\eta}}q\left(\sqrt{\xi}+\sqrt{\eta}\right)U,$$
$$U(\xi,\xi) = 0,$$
$$\frac{\partial U}{\partial\xi} + \frac{\alpha}{\xi}U = \frac{1}{4}q\left(\sqrt{\xi}\right)\xi^{\nu-1}, \quad \alpha = -\nu + \frac{1}{2}. \tag{29}$$

This problem can be solved by using the Riemann method [21].

Multiplying (18) by $\widetilde{y}(x,\lambda)$ and (20) by $y(x,\lambda)$, subtracting and integrating from 0 to 1/2, we obtain

$$\int_0^{1/2}\left(q(x) - \widetilde{q}(x)\right)y(x,\lambda)\,\widetilde{y}(x,\lambda)\,dx = \left[\widetilde{y}(x,\lambda)y'(x,\lambda) - y(x,\lambda)\widetilde{y}'(x,\lambda)\right]\Big|_0^{1/2}. \tag{30}$$

The functions $y(x,\lambda)$ and $\widetilde{y}(x,\lambda)$ satisfy the same initial conditions (19) and (21), that is,

$$\widetilde{y}(0,\lambda)\,y'(0,\lambda) - y(0,\lambda)\,\widetilde{y}'(0,\lambda) = 0. \tag{31}$$

Let

$$Q(x) = q(x) - \widetilde{q}(x), \tag{32}$$

$$K(\lambda) = \int_0^{1/2} Q(x)\,y(x,\lambda)\,\widetilde{y}(x,\lambda)\,dx. \tag{33}$$

If the properties of $y(x,\lambda)$ and $\widetilde{y}(x,\lambda)$ are considered, the function $K(\lambda)$ is an entire function.

Therefore the condition of Theorem 1 implies

$$\widetilde{y}\left(\frac{1}{2},\lambda_n\right)y'\left(\frac{1}{2},\lambda_n\right) - y\left(\frac{1}{2},\lambda_n\right)\widetilde{y}'\left(\frac{1}{2},\lambda_n\right) = 0 \tag{34}$$

and hence

$$K(\lambda_n) = 0, \quad n \in \mathbb{N}. \tag{35}$$

In addition, using (24) and (33) for $0 < x < 1$,

$$|K(\lambda)| \le M\frac{1}{\lambda^\nu}, \tag{36}$$

where M is constant.

Introduce the function

$$W(\lambda) = y'(1,\lambda) + Hy(1,\lambda). \tag{37}$$

By using the asymptotic forms of y and y', we obtain

$$W(\lambda) = \sqrt{\lambda}\sin\left(\sqrt{\lambda} - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(1). \tag{38}$$

The zeros of $W(\lambda)$ are the eigenvalues of L and hence it has only simple zeros λ_n because of the seperated boundary conditions. From (38), $W(\lambda)$ is an entire function of order 1/2 of λ. Since the set of zeros of the entire function $W(\lambda)$ is contained in the set of zeros of $K(\lambda)$, we see that the function

$$\Psi(\lambda) = \frac{K(\lambda)}{W(\lambda)} \tag{39}$$

is an entire function on the parameter λ. From (36), (38), and (39), we get

$$|\Psi(\lambda)| = O\left(\frac{1}{\lambda^{\nu+1/2}}\right). \tag{40}$$

So, for all λ, from the Liouville theorem,

$$\Psi(\lambda) = 0, \tag{41}$$

or

$$K(\lambda) = 0. \tag{42}$$

It was proved in [19] that there exists absolutely continuous function $\widetilde{\widetilde{H}}(x,\tau)$ such that we have

$$\begin{aligned} y(x,\lambda)\,\tilde{y}(x,\lambda) = \frac{1}{2}\Bigg\{1 + \cos 2\left[\sqrt{\lambda}x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right] \\ + \int_0^x \widetilde{\widetilde{H}}(x,\tau) \\ \times \cos 2\left[\sqrt{\lambda}\tau - \frac{\nu\pi}{2} - \frac{\pi}{4}\right] d\tau\Bigg\}, \end{aligned} \tag{43}$$

where

$$\begin{aligned} \widetilde{\widetilde{H}}(x,t) = 2\left[H(x, x-2\tau) + \widetilde{H}(x, x-2\tau)\right] \\ + 2\left[\int_{-x+2\tau}^{x} H(x,s)\,\widetilde{H}(x, s-2\tau)\,ds \right. \\ \left. + \int_{-x}^{x-2\tau} H(x,s)\,\widetilde{H}(x, s+2\tau)\,ds\right]. \end{aligned} \tag{44}$$

We are now going to show that $Q(x) = 0$ a.e. on $(0, 1/2]$. From (33), (43) we have

$$\begin{aligned} \frac{1}{2}\int_0^{1/2} Q(x)\Bigg\{1 + \cos 2\left[\sqrt{\lambda}x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right] \\ + \int_0^x \widetilde{\widetilde{H}}(x,\tau) \\ \times \cos 2\left[\sqrt{\lambda}\tau - \frac{\nu\pi}{2} - \frac{\pi}{4}\right] d\tau\Bigg\}\,dx = 0. \end{aligned} \tag{45}$$

This can be written as

$$\begin{aligned} \int_0^{1/2} Q(x)\,dx + \int_0^{1/2} \cos 2\left[\sqrt{\lambda}\tau - \frac{\nu\pi}{2} - \frac{\pi}{4}\right] \\ \times \left[Q(\tau) + \int_0^{1/2} Q(x) \right. \\ \left. \times \widetilde{\widetilde{H}}(x,\tau)\,dx\right] d\tau = 0. \end{aligned} \tag{46}$$

Let $\lambda \to \infty$ along the real axis, by the Riemann-Lebesgue lemma, one should have

$$\begin{aligned} \int_0^{1/2} Q(x)\,dx = 0, \\ \int_0^{1/2} \cos 2\left[\sqrt{\lambda}\tau - \frac{\nu\pi}{2} - \frac{\pi}{4}\right] \\ \times \left[Q(\tau) + \int_\tau^{1/2} Q(x)\,\widetilde{\widetilde{H}}(x,\tau)\,dx\right] d\tau = 0. \end{aligned} \tag{47}$$

Thus from the completeness of the functions cos, it follows that

$$Q(\tau) + \int_\tau^{1/2} Q(x)\,\widetilde{\widetilde{H}}(x,\tau)\,dx = 0, \quad 0 < x < \frac{1}{2}. \tag{48}$$

But this equation is a homogeneous Volterra integral equation and has only the zero solution. Thus we have obtained

$$Q(x) = q(x) - \tilde{q}(x) = 0, \tag{49}$$

or

$$\tilde{q}(x) = q(x) \tag{50}$$

almost everywhere on $(0, 1/2]$. Therefore Theorem 1 is proved. □

Theorem 4. *To prove that $q(x) = 0$ on $[1/2, 1)$ almost everywhere, we should repeat the above arguments for the supplementary problem*

$$Ly = -y'' + \left[\frac{\ell(\ell+1)}{(1-x)^2} + q(1-x)\right] y, \quad 0 < x < 1 \tag{51}$$

subject to the boundary conditions

$$\begin{aligned} y(1) = 0, \\ y'(0,\lambda) + Hy(0,\lambda) = 0. \end{aligned} \tag{52}$$

Consequently

$$q(x) = \tilde{q}(x) \quad \text{a.e on the interval } (0,1). \tag{53}$$

Next, we show that Lemma 2 holds.

Proof of Lemma 2. As in the proof of Theorem 1 we can show that

$$\begin{aligned} G(\rho) = \int_0^b Q(x)\,y(x,\lambda)\,\tilde{y}(x,\lambda)\,dx \\ = \left[\tilde{y}(x,\lambda)\,y'(x,\lambda) - y(x,\lambda)\,\tilde{y}'(x,\lambda)\right]\Big|_{x=b}, \end{aligned} \tag{54}$$

where $\rho = \sqrt{\lambda} = re^{i\theta}$ and $Q(x) = q(x) - \tilde{q}(x)$. From the assumption

$$\frac{y'_{m(n)}(b)}{y_{m(n)}(b)} = \frac{\tilde{y}'_{m(n)}(b)}{\tilde{y}_{m(n)}(b)} \tag{55}$$

together with the initial condition at 0 it follows that,

$$G(\rho_{m(n)}) = 0, \quad n \in \mathbb{N}. \tag{56}$$

Next, we will show that $G(\rho) = 0$ on the whole ρ plane. The asymptotics (23) imply that the entire function $G(\rho)$ is a function of exponential type $\leq 2b$.

Define the indicator of function $G(\rho)$ by

$$h(\theta) = \lim_{r\to\infty} \sup \frac{\ln\left|G\left(re^{i\theta}\right)\right|}{r}. \tag{57}$$

Since $|\operatorname{Im}\sqrt{\lambda}| = r|\sin\theta|, \theta = \arg\sqrt{\lambda}$ from (23) it follows that

$$h(\theta) \le 2b|\sin\theta|. \tag{58}$$

Let us denote by $n(r)$ the number of zeros of $G(\rho)$ in the disk $\{|\rho| \le r\}$. According to [22] set of zeros of every entire function of the exponential type, not identically zero, satisfies the inequality

$$\lim_{r\to\infty}\inf\frac{n(r)}{r} \le \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\,d\theta, \tag{59}$$

where $n(r)$ is the number of zeros of $G(\rho)$ in the disk $|\rho| \le r$. By (58),

$$\frac{1}{2\pi}\int_0^{2\pi} h(\theta)\,d\theta \le \frac{b}{\pi}\int_0^{2\pi}|\sin\theta|\,d\theta = \frac{4b}{\pi}. \tag{60}$$

From the assumption and the known asymptotic expression (7) of the eigenvalues $\sqrt{\lambda_n}$ we obtain

$$n(r) \ge 2\sum_{(\pi n/\sigma)[1+O(1/n)]<r} 1 = \frac{2}{\pi}\sigma r(1+o(1)), \quad r\longrightarrow\infty. \tag{61}$$

For the case $\sigma > 2b$,

$$\lim_{r\to\infty}\frac{n(r)}{r} \ge \frac{2}{\pi}\sigma > \frac{4b}{\pi} = 2b\int_0^{2\pi}|\sin\theta|\,d\theta \ge \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\,d\theta. \tag{62}$$

The inequalities (59) and (62) imply that $G(\rho) = 0$ on the whole ρ plane.

Similar to the proof of Theorem 1, we have

$$q(x) = \tilde{q}(x) \quad \text{a.e on the interval } (0,b]. \tag{63}$$

This completes the proof of Lemma 2. □

Now we prove that Theorem 3 is valid.

Proof of Theorem 3. From

$$\lambda_{r(n)} = \tilde{\lambda}_{r(n)}, \qquad \frac{y'_{r(n)}(b)}{y_{r(n)}(b)} = \frac{\tilde{y}'_{r(n)}(b)}{\tilde{y}_{r(n)}(b)}, \tag{64}$$

where $\{r(n)\}_{n\in\mathbb{N}}$ satisfies (14) and $\sigma_2 > 2-2b$. Similar to the proof of Lemma 2, we get

$$q(x) = \tilde{q}(x) \quad \text{a.e on } [b,1). \tag{65}$$

Thus, it needs to be proved that $q(x) = \tilde{q}(x)$ a.e on $(0,b]$. The eigenfunctions $y_n(x,\lambda_n)$ and $\tilde{y}_n(x,\lambda_n)$ satisfy the same boundary condition at 1. It means that

$$y_n(x,\lambda_n) = \xi_n\tilde{y}_n(x,\lambda_n) \tag{66}$$

on $[b,1]$ for any $n\in\mathbb{N}$ where ξ_n are constants.

Let $\rho_n = \sqrt{\lambda_n}$, $s_n = \sqrt{\mu_n}$. From (54) and (66) we obtain

$$\begin{aligned} G(\rho_n) &= 0, \quad n\in\mathbb{N},\\ G(s_{l_n}) &= 0, \quad n\in\mathbb{N}. \end{aligned} \tag{67}$$

We are going to show that inequality (59) fails and consequently, the entire function of exponential type $G(\rho)$ vanishes on the whole ρ-plane. The ρ_n and s_n have the same asymptotics (7). Counting the number of ρ_n and s_n located inside the disc of radius r, we have

$$1+\frac{2}{\pi}r\left[1+O\left(\frac{1}{n}\right)\right] \tag{68}$$

of ρ_n's and

$$1+\frac{2}{\pi}r\sigma_1\left[1+O\left(\frac{1}{n}\right)\right]. \tag{69}$$

of s_n's.

This means that

$$\begin{aligned} n(r) &= 2+\frac{2}{\pi}\left[r(\sigma_1+1)+O\left(\frac{1}{n}\right)\right],\\ \lim_{r\to\infty}\frac{n(r)}{r} &= \frac{2}{\pi}(\sigma_1+1). \end{aligned} \tag{70}$$

Repeating the last part of the proof of Lemma 2, and considering the condition $\sigma_1 > 2b-1$, we can show that $G(\rho) = 0$ identically on the whole ρ-plane which implies that

$$q(x) = \tilde{q}(x) \quad \text{a.e on } (0,b] \tag{71}$$

and consequently

$$q(x) = \tilde{q}(x) \quad \text{a.e on } (0,1). \tag{72}$$

Hence the proof of Theorem 3 is completed. □

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