Research Article

Solving Integral Representations Problems for the Stationary Schrödinger Equation

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When solutions of the stationary Schrödinger equation in a half-space belong to the weighted Lebesgue classes, we give integral representations of them, which imply known representation theorems of classical harmonic functions in a half-space.

1. Introduction and Results

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the sets of all real numbers and of all positive real numbers, respectively. Let $\mathbf{R}^n$ ($n \geq 2$) denote the n -dimensional Euclidean space with points $x = (x', x_n)$, where $x' = (x_1, x_2, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. The unit sphere and the upper half unit sphere in $\mathbf{R}^n$ are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}_+^{n-1}$, respectively. The boundary and closure of an open set D of $\mathbf{R}^n$ are denoted by ∂D and $\bar{D}$, respectively. The upper half space is the set $H = \{(x', x_n) \in \mathbf{R}^n; x_n > 0\}$, whose boundary is ∂H .

We identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$ and $\mathbf{R}^{n-1}$ with $\mathbf{R}^{n-1} \times \{0\}$, writing typical points $x, y \in \mathbf{R}^n$ as $x = (x', x_n)$, $y = (y', y_n)$, where $y' = (y_1, y_2, \dots, y_{n-1}) \in \mathbf{R}^{n-1}$ and putting

$$\begin{aligned} x \cdot y &= \sum_{j=1}^n x_j y_j, & |x| &= \sqrt{x \cdot x}, \\ \Theta &= \frac{x}{|x|}, & \Phi &= \frac{y}{|y|}. \end{aligned} \quad (1)$$

For $x \in \mathbf{R}^n$ and $r > 0$, let $B(x, r)$ denote the open ball with center at x and radius r ($r > 0$) in $\mathbf{R}^n$. $S_r = \partial B(O, r)$, where O is the origin of $\mathbf{R}^n$. For a set E , $E \subset \mathbf{R}_+ \cup \{0\}$, we denote $\{x \in H; |x| \in E\}$ and $\{x \in \partial H; |x| \in E\}$ by HE and ∂HE , respectively. By Hr we denote $H \cap S_r$. We denote by dS_r the $(n-1)$ -dimensional volume elements induced by the Euclidean metric on S_r .

Let $\mathcal{A}_b$ denote the class of nonnegative radial potentials $b(x)$, that is, $0 \leq b(x) = b(|x|)$, $x \in H$, such that $b \in L_{\text{loc}}^a(H)$ with some $a > n/2$ if $n \geq 4$ and with $a = 2$ if $n = 2$ or $n = 3$.

This paper is devoted to the stationary Schrödinger equation:

$$SSE_b h(x) = -\Delta h(x) + b(x)h(x) = 0, \quad (2)$$

where $x \in H$, Δ is the Laplace operator and $b \in \mathcal{A}_b$. Note that solutions of (2) are (classical) harmonic functions in the case $b = 0$. Under these assumptions the operator SSE_b can be extended in the usual way from the space $C_0^\infty(H)$ to an essentially self-adjoint operator on $L^2(H)$ (see [1]). We will denote it by SSE_b as well. This last one has a Green function $G_b(x, y)$. Here $G_b(x, y)$ is positive on H and its inner normal derivative $\partial G_b(x, y)/\partial n(y') \geq 0$. We denote this derivative by $P_b(x, y')$, which is called the Poisson b -kernel with respect to H . If $G(x, y)$ and $P(x, y')$ are denoted by the Green function and Poisson kernel of the Laplace operator in H , respectively, then

$$P(x, y') = -\frac{\partial G(x, y)}{\partial y_n} \Big|_{y_n=0} = \frac{2x_n}{w_n} \frac{1}{|x - y'|^n}, \quad (3)$$

where $x = (x', x_n)$, $y = (y', y_n)$ and w_n is the area of the unit sphere in $\mathbf{R}^n$.

Let g be a continuous function on ∂H . We say that h is a solution of the Dirichlet problem for the stationary Schrödinger operator SSE_b on H with g , if

$$\begin{aligned} h &\in C^2(H) \cap C^0(\overline{H}), \\ SSE_b h &= 0 \quad \text{in } H, \\ h &= g \quad \text{on } \partial H. \end{aligned} \quad (4)$$

Note that h is a solution of the classical Dirichlet problem for the Laplace operator Δ on H with g in the case $b = 0$.

Let Λ be a Laplace-Beltrami operator (spherical part of the Laplace) on the unit sphere. It is known (see, e.g., [2, page 41]) that the eigenvalue problem

$$\begin{aligned} \Lambda \varphi(\Theta) + \tau \varphi(\Theta) &= 0 \quad \Theta \in \mathbf{S}_+^{n-1}, \\ \varphi(\Theta) &= 0 \quad \Theta \in \partial \mathbf{S}_+^{n-1} \end{aligned} \quad (5)$$

has the eigenvalues $\tau_j = j(j + n - 2)$, where $j = 0, 1, 2, \dots$. Corresponding eigenfunctions are denoted by $\varphi_{j\nu}$ ($1 \leq \nu \leq \nu_j$), where ν_j is the multiplicity of τ_j . We norm the eigenfunctions in $L^2(\mathbf{S}_+^{n-1})$ and $\varphi_1 = \varphi_{11} > 0$.

Let $P_j(r)$ and $Q_j(r)$ stand, respectively, for the increasing and nonincreasing, as $r \rightarrow +\infty$, solutions of the equation:

$$\begin{aligned} -T''(r) - \frac{n-1}{r}T'(r) \\ + \left(\frac{\tau_j}{r^2} + b(r) \right) T(r) &= 0, \quad 0 < r < \infty, \end{aligned} \quad (6)$$

normalized under the condition $P_j(1) = Q_j(1) = 1$.

We shall also consider the class $\mathcal{B}_b$, consisting of the potentials $b \in \mathcal{A}_b$ such that there exists a finite limit $\lim_{r \rightarrow \infty} r^2 b(r) = s \in [0, \infty)$; moreover, $r^{-1}|r^2 b(r) - s| \in L(1, \infty)$. If $b \in \mathcal{B}_b$, then solutions of (2) are continuous (see [3]).

In the rest of paper, we assume that $b \in \mathcal{B}_b$ and we shall suppress this assumption for simplicity. Further, we use the standard notations $u^+ = \max\{u, 0\}$, $u^- = -\min\{u, 0\}$, and $[d]$ is the integer part of d and $d = [d] + \{d\}$, where d is a positive real number.

Denote

$$\lambda_{j,s}^\pm = \frac{2 - n \pm \sqrt{(n-2)^2 + 4(s + \tau_j)}}{2} \quad (j = 0, 1, 2, 3, \dots). \quad (7)$$

Remark 1. $\lambda_{j,0}^+ = j$ in the case $b = 0$, where $j = 0, 1, 2, 3, \dots$

It is known (see [4]) that in the case under consideration the solutions to (6) have the asymptotics

$$P_j(r) \sim d_1 r^{\lambda_{j,s}^+}, \quad Q_j(r) \sim d_2 r^{\lambda_{j,s}^-}, \quad \text{as } r \rightarrow \infty, \quad (8)$$

where d_1 and d_2 are some positive constants.

If $b \in \mathcal{A}_b$, it is known that the following expansion for the Green function $G_b(x, y)$ (see [5, Chapter 11], [6]):

$$\begin{aligned} G_b(x, y) &= \sum_{j=0}^{\infty} \frac{1}{\chi'(1)} P_j(\min(|x|, |y|)) \\ &\times Q_j(\max(|x|, |y|)) \left(\sum_{\nu=1}^{\nu_j} \varphi_{j\nu}(\Theta) \varphi_{j\nu}(\Phi) \right), \end{aligned} \quad (9)$$

where $|x| \neq |y|$ and $\chi'(1) = w(Q_1(r), P_1(r))|_{r=1}$ is its Wronskian. The series converges uniformly if either $|x| \leq k|y|$ or $|y| \leq k|x|$, where $0 < k < 1$.

For a nonnegative integer m and two points $x, y \in H$, we define a modified Green function:

$$G(b, m)(x, y) = G_b(x, y) - V(b, m)(x, y), \quad (10)$$

where

$$\begin{aligned} V(b, m)(x, y) &= \begin{cases} 0 & \text{if } |y| < 1, \\ \sum_{j=0}^m \frac{1}{\chi'(1)} P_j(|x|) Q_j(|y|) \\ \times \left(\sum_{\nu=1}^{\nu_j} \varphi_{j\nu}(\Theta) \varphi_{j\nu}(\Phi) \right) & \text{if } 1 \leq |y| < \infty. \end{cases} \end{aligned} \quad (11)$$

If the generalized Poisson kernel $P(b, m)(x, y')$ with respect to H is defined by

$$P(b, m)(x, y') = \frac{\partial G(b, m)(x, y)}{\partial n(y')}, \quad (12)$$

then we have $P(b, 0)(x, y') = P_b(x, y')$ and $P(0, m)(x, y')$ coincides with ones in Finkelstein and Scheinberg [7], Siegel and Talvila [8], Deng [9], Qiao and Deng [10], and Qiao [11] (see [5, Chapter 11]).

Put

$$U(b, m; u)(x) = \int_{\partial H} P(b, m)(x, y') u(y') dy', \quad (13)$$

where $u(y')$ is a continuous function on ∂H .

For real numbers $\beta \geq 1$, we denote $\mathcal{E}(\beta, b)$ the class of all measurable functions $f(y)$ satisfying the following inequality:

$$\int_H \frac{|f(y)| \varphi_1}{1 + P_{[\beta]}(|y|) |y|^{n+\{\beta\}}} dy < \infty, \quad (14)$$

and the class $\mathcal{D}(\beta, b)$ consists of all measurable functions $g(y')$ ($y' \in \partial H$) satisfying

$$\int_{\partial H} \frac{|g(y')| P_1(|y'|) Q_1(|y'|)}{1 + \chi'(|y'|) P_{[\beta]}(|y'|) |y'|^{n+\{\beta\}-1}} \frac{\partial \varphi_1}{\partial n} dy' < \infty. \quad (15)$$

We will also consider the class of all continuous functions $h(y)$, which is the solution of (2), with $h^+(y) \in \mathcal{E}(\beta, b)$ and $h^+(y') \in \mathcal{D}(\beta, b)$, is denoted by $\mathcal{E}(\beta, b)$.

Remark 2. If $b = 0$ and $\beta = \alpha + 1$, then (14) and (15) are equivalent to

$$\begin{aligned} \int_H \frac{y_n |f(y)|}{1 + |y|^{n+\alpha+2}} dy &< \infty, \\ \int_{\partial H} \frac{|g(y')|}{1 + |y'|^{n+\alpha}} dy' &< \infty, \end{aligned} \quad (16)$$

respectively, from Remark 1 and (14), which yield that $\mathcal{E}(\alpha + 1, 0)$ is equivalent to $(CH)_\alpha$ in the notation of [9].

Let us recall the classical case $b = 0$. If $h(x) \geq 0$ is harmonic in H , continuous on $\bar{H}$, and $h \in \mathcal{E}(1, 0)$, then there exists a constant $d_3 \geq 0$ such that (see [12])

$$h(x) = d_3 x_n + \int_{\partial H} P(x, y') u(y') dy', \quad (17)$$

where $x = (x', x_n) \in H$.

Deng (see [9]) has constructed a similar representation to (17) for $h \in \mathcal{E}(\alpha + 1, 0)$ ($\alpha \geq 0$), which is the integral with a modified Poisson kernel derived by subtracting some special harmonic polynomials from $P(x, y')$. By virtue of this modified Poisson kernel, Qiao (see [11, 13]) and Qiao and Deng (see [14–18]) have constructed different integral representations for harmonic functions of finite order and infinite order.

Especially, Su (see [6, 19]) recently writes solutions to the half-space Dirichlet problem with respect to the stationary Schrödinger operator SSE_b . Now we state our main results as follows.

Theorem 3. If $h \in \mathcal{E}(\beta, b)$, then $h \in \mathcal{D}(\beta, b)$.

Theorem 4. If $h \in \mathcal{E}(\beta, b)$, m is an integer such that $P_m(|y|) < P_{[\beta]}(|y|)|y|^{[\beta]} \leq P_{m+1}(|y|)(|y| \geq 1)$, and then the following properties hold.

(I) If $\beta = 1$, then there exists a constant d_4 such that

$$h(x) = d_4 P_1(|x|) \varphi_1(\Theta) + U(b, 0; h)(x) \quad (18)$$

for $x \in H$.

(II) If $\beta > 1$, then we have $h(x) = U(b, m; h)(x) + u(x)$, where

$$u(x) = \sum_{j=0}^m \left(\sum_{v=1}^{v_j} d_{jv} \varphi_{jv}(\Theta) \right) P_j(|x|) \quad (19)$$

vanishing continuously on ∂H , $x = (x', x_n) \in H$ and d_{jv} are constants.

2. Lemmas

The following Lemma plays an important role in our discussions, which is due to Levin and Kheyfits (see [5, page 356]).

Lemma 5. If $R > 1$ and $h(y)$ is a solution of (2) on a domain containing $H(1, R)$, then

$$\begin{aligned} \int_{HR} \frac{\chi'(R)}{P_1(R)} h \varphi_1(\Phi) dS_R + \int_{\partial H(1, R)} h(y') \frac{\partial \varphi_1}{\partial n} W(|y'|) dy' \\ + d_5 + d_6 \frac{Q_1(R)}{P_1(R)} = 0, \end{aligned} \quad (20)$$

where

$$\begin{aligned} W(|y'|) &= Q_1(|y'|) - \frac{Q_1(R)}{P_1(R)} P_1(|y'|), \\ d_5 &= \int_{H1} h \varphi_1(\Phi) Q_1'(r) - Q_1(1) \varphi_1(\Phi) \frac{\partial h}{\partial n} dS_1, \\ d_6 &= \int_{H1} P_1(1) \varphi_1(\Phi) \frac{\partial h}{\partial n} - h \varphi_1(\Phi) P_1'(1) dS_1. \end{aligned} \quad (21)$$

Lemma 6 (see [6, Corollary 1.6]). If u is a continuous function on ∂H satisfying

$$\int_{\partial H} \frac{|u(y')|}{1 + |y'|^{\lambda_{m+1,s}^+ + n - 1}} dy' < \infty, \quad (22)$$

then $U(b, m; u)(x)$ is a solution of the Dirichlet problem for SSE_b on H with h and

$$\lim_{|x| \rightarrow \infty, x \in H} |x|^{-\lambda_{m+1,s}^+} U(b, m; u)(x) = 0. \quad (23)$$

Lemma 7 (see [6, Lemma 2.1] or [20, Theorem 1]). If $u(x)$ is a solution of (2) on H satisfying

$$\lim_{|x| \rightarrow \infty, x \in H} |x|^{-\lambda_{m+1,s}^+} u^+(x) = 0, \quad (24)$$

then (19) holds.

3. Proof of Theorem 3

We apply the formula (20) to $h = h^+ - h^-$ in $H(1, R)$:

$$\begin{aligned} m_+(R) + \int_{\partial H(1, R)} h^+(y') W(|y'|) \frac{\partial \varphi_1}{\partial n} dy' + d_5 + \frac{Q_1(R)}{P_1(R)} d_6 \\ = m_-(R) + \int_{\partial H(1, R)} h^-(y') W(|y'|) \frac{\partial \varphi_1}{\partial n} dy', \end{aligned} \quad (25)$$

where

$$m_\pm(R) = \int_{HR} \frac{\chi'(R)}{P_1(R)} h^\pm \varphi_1 dS_R. \quad (26)$$

Since $u \in \mathcal{E}(\beta, b)$, we obtain by (8)

$$\begin{aligned} & \int_1^\infty \frac{m_+(R) P_1(R)}{\chi'(R) P_{[\beta]}(R) R^{n+\{\beta\}}} dR \\ &= \int_{H(1,\infty)} \frac{h^+ \varphi_1}{P_{[\beta]}(|y|) |y|^{n+\{\beta\}}} dy \\ &\leq 2 \int_H \frac{h^+ \varphi_1}{1 + P_{[\beta]}(|y|) |y|^{n+\{\beta\}}} dy < \infty. \end{aligned} \quad (27)$$

From (15), we conclude that

$$\begin{aligned} & \int_1^\infty \frac{P_1(R)}{\chi'(R) P_{[\beta]}(R) R^{n+\{\beta\}}} \\ & \times \int_{\partial H(1,R)} h^+(y') W(|y'|) \frac{\partial \varphi_1}{\partial n} dy' dR \\ &= \int_{\partial H(1,\infty)} h^+(y') P_1(|y'|) \\ & \times \int_{|y'|}^\infty \frac{P_1(R)}{\chi'(R) P_{[\beta]}(R) R^{n+\{\beta\}}} \\ & \times \left(\frac{Q_1(|y'|)}{P_1(|y'|)} - \frac{W_1(R)}{P_1(R)} \right) dR \frac{\partial \varphi_1}{\partial n} dy' \\ &\leq M \int_{\partial H(1,\infty)} \frac{h^+(y') P_1(|y'|) Q_1(|y'|)}{\chi'(|y'|) P_{[\beta]}(|y'|) |y'|^{n+\{\beta\}-1}} \frac{\partial \varphi_1}{\partial n} dy' \\ &\leq M \int_{\partial H} \frac{h^+(y') P_1(|y'|) Q_1(|y'|)}{1 + \chi'(|y'|) P_{[\beta]}(|y'|) |y'|^{n+\{\beta\}-1}} \frac{\partial \varphi_1}{\partial n} dy' \\ &< \infty. \end{aligned} \quad (28)$$

Combining (25), (27), and (28), we obtain

$$\begin{aligned} & \int_1^\infty \frac{P_1(R)}{\chi'(R) P_{[\beta]}(R) R^{n+(\{\beta\}/2)}} \\ & \times \int_{\partial H(1,R)} h^-(y') W(|y'|) \frac{\partial \varphi_1}{\partial n} dy' dR \\ &\leq \int_1^\infty \frac{m_+(R) P_1(R)}{\chi'(R) P_{[\beta]}(R) R^{n+(\{\beta\}/2)}} dR \\ & + \int_1^\infty \frac{P_1(R)}{\chi'(R) P_{[\beta]}(R) R^{n+(\{\beta\}/2)}} \\ & \times \int_{\partial H(1,R)} h^+(y') W(|y'|) \frac{\partial \varphi_1}{\partial n} dy' dR \\ & + \int_1^\infty \frac{1}{\chi'(R) P_{[\beta]}(R) R^{n+(\{\beta\}/2)}} \\ & \times (P_1(R) d_5 + Q_1(R) d_6) dR \\ &< \infty. \end{aligned} \quad (29)$$

Set

$$\begin{aligned} \mathcal{H}(\beta) &= \lim_{|y'| \rightarrow \infty} \frac{\chi'(|y'|) P_{[\beta]}(|y'|) |y'|^{n+\{\beta\}-1}}{Q_1(|y'|)} \\ & \times \int_{|y'|}^\infty \frac{P_1(R)}{\chi'(R) P_{[\beta]}(R) R^{n+(\{\beta\}/2)}} \\ & \times \left(\frac{Q_1(|y'|)}{P_1(|y'|)} - \frac{W_1(R)}{P_1(R)} \right) dR \\ &= \lim_{|y'| \rightarrow \infty} |y'|^{\lambda_{[\beta],s}^+ + \lambda_{1,s}^+ + n + \{\beta\} - 2} \\ & \times \int_{|y'|}^\infty \frac{1}{R^{\lambda_{[\beta],s}^+ - \lambda_{1,s}^+ + (\{\beta\}/2) + 1}} \left(\frac{1}{|y'|^{\chi_{1,s}}} - \frac{1}{R^{\chi_{1,s}}} \right) dR, \end{aligned} \quad (30)$$

where $\chi_{1,s} = \lambda_{1,s}^+ - \lambda_{1,s}^-$.

By the L'hospital's rule, we have

$$\begin{aligned} \mathcal{H}(\beta) &= \begin{cases} \frac{\chi_{1,s}}{(\lambda_{[\beta],s}^+ - \lambda_{1,s}^+)(\lambda_{[\beta],s}^+ + \lambda_{1,s}^+ + n - 2)} & \text{if } \{\beta\} = 0, \\ +\infty & \text{if } \{\beta\} \neq 0, \end{cases} \end{aligned} \quad (31)$$

which yields that there exists a positive constant M such that, for any $|y'| \geq 1$,

$$\begin{aligned} & \int_{|y'|}^\infty \frac{P_1(R)}{\chi'(R) P_{[\beta]}(R) R^{n+(\{\beta\}/2)}} W(|y'|) dR \\ &\geq \frac{MP_1(|y'|) Q_1(|y'|)}{\chi'(|y'|) P_{[\beta]}(|y'|) |y'|^{n+\{\beta\}-1}}. \end{aligned} \quad (32)$$

Then

$$\begin{aligned} & M \int_{\partial H(1,\infty)} \frac{h^-(y') P_1(|y'|) Q_1(|y'|)}{\chi'(|y'|) P_{[\beta]}(|y'|) |y'|^{n+\{\beta\}-1}} \frac{\partial \varphi_1}{\partial n} dy' \\ &\leq \int_{\partial H(1,\infty)} h^- \int_{|y'|}^\infty \frac{P_1(R)}{\chi'(R) P_{[\beta]}(R) R^{n+(\{\beta\}/2)}} \\ & \times W(|y'|) dR \frac{\partial \varphi_1}{\partial n} dy' \\ &< \infty, \end{aligned} \quad (33)$$

which shows that $h \in \mathcal{D}(\beta, b)$ from $|h| = h^+ + h^-$. Then Theorem 3 is proved.

4. Proof of Theorem 4

To prove (II), notice that $P_m(|y|) < P_{[\beta]}(|y|) |y|^{\{\beta\}} \leq P_{m+1}(|y|) (|y| \geq 1)$ and condition (15) is stronger than condition (22) from Theorem 3.

Consider the function $h(x) = U(a, m; h)(x)$. Then it follows from Lemma 6 and Theorem 3 that this is a solution of (2) in H and vanishes continuously on ∂H .

Then

$$0 \leq (h(x) - U(b, m; h)(x))^+ \leq h^+(x) + (U(b, m; h))^-(x) \quad (34)$$

for any $P \in H$. Further, (8) gives that

$$\lim_{|x| \rightarrow \infty, x \in H} |x|^{-\lambda_{m+1,s}^+} h^+(x) = 0, \quad (35)$$

which together with (34) and Lemmas 6 and 7 give the result of (II).

If $u \in \mathcal{E}(1, b)$, then $h \in \mathcal{E}(\beta, b)$ for each $\beta > 1$ and there exists a constant d_7 such that

$$h(x) = d_7 P_1(|x|) \varphi_1(\Theta) + U(b, 1; h)(x) \quad (36)$$

for all $x \in H$. So if we take

$$d_4 = d_7 - \int_{\partial H[1, \infty)} P(b, 1)(0, y') h(y') dy', \quad (37)$$

we see that (18) holds for all $x \in H$.

Thus we complete the proof of Theorem 4.

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