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Research Article

On Uncertainty Principle for Quaternionic Linear Canonical Transform

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We generalize the linear canonical transform (LCT) to quaternion-valued signals, known as the quaternionic linear canonical transform (QLCT). Using the properties of the LCT we establish an uncertainty principle for the QLCT. This uncertainty principle prescribes a lower bound on the product of the effective widths of quaternion-valued signals in the spatial and frequency domains. It is shown that only a 2D Gaussian signal minimizes the uncertainty.

1. Introduction

The classical uncertainty principle of harmonic analysis states that a nontrivial function and its Fourier transform (FT) cannot both be sharply localized. The uncertainty principle plays an important role in signal processing [1–11] and physics [12–21]. In quantum mechanics an uncertainty principle asserts that one cannot make certain of the position and velocity of an electron (or any particle) at the same time. That is, increasing the knowledge of the position decreases the knowledge of the velocity or momentum of an electron. In quaternionic analysis some papers combine the uncertainty relations and the quaternionic Fourier transform (QFT) [22–24].

The QFT plays a vital role in the representation of (hypercomplex) signals. It transforms a real (or quaternionic) 2D signal into a quaternion-valued frequency domain signal. The four components of the QFT separate four cases of symmetry into real signals instead of only two as in the complex FT. In [25] the authors used the QFT to proceed color image analysis. The paper [26] implemented the QFT to design a color image digital watermarking scheme. The authors in [27] applied the QFT to image preprocessing and neural computing techniques for speech recognition. Recently, certain asymptotic properties of the QFT were analyzed and straightforward generalizations of classical Bochner-Minlos theorems to the framework of quaternionic analysis were derived [28, 29]. In this paper, we study the uncertainty principle for the QLCT and the generalization of the QFT to the Hamiltonian quaternionic algebra.

The classical LCT being a generalization of the FT, was first proposed in the 1970s by Collins [30] and Moshinsky and Quesne [31]. It is an effective processing tool for chirp signal analysis, such as the parameter estimation, sampling progress for nonbandlimited signals with nonlinear Fourier atoms [32], and the LCT filtering [33–35]. The windowed LCT [36], with a local window function, can reveal the local LCT-frequency contents, and it enjoys high concentrations and eliminates cross terms. The analogue of the Poisson summation formula, sampling formulas, series expansions, Paley-Wiener theorem, and uncertainly relations is studied in [36, 37]. In view of numerous applications, one is particularly interested in higher-dimensional analogues to Euclidean space. The LCT was first extended to the Clifford analysis setting in [38]. It was used to study the generalized prolate spheroidal wave functions and their connection to energy concentration problems [38]. In the present work, we study the QLCT which transforms a quaternionic 2D signal into a quaternion-valued frequency domain signal. Some important properties of the QLCT are analyzed. An uncertainty principle for the QLCT is established. This uncertainty principle prescribes a lower bound on the product of the effective widths of quaternion-valued signals in the spatial and frequency domains. To the best of our knowledge, the study of a Heisenberg-type uncertainty principle for the

QLCT has not been carried out yet. The results in this paper are new in the literature. The main motivation of the present study is to develop further general numerical methods for differential equations and to investigate localization theorems for summation of Fourier series in the quaternionic analysis setting. Further investigations and extensions of this topic will be reported in a forthcoming paper.

The paper is organized as follows. Section 2 gives a brief introduction to some general definitions and basic properties of quaternionic analysis. The LCT of 2D quaternionic signal is introduced and studied in Section 3. Some important properties such as Parseval's and inversion theorems are obtained. In Section 4, we introduce and discuss the concept of QLCT and demonstrate some important properties that are necessary to prove the uncertainty principle for the QLCT. The classical Heisenberg uncertainty principle is generalized for the QLCT in Section 5. This principle prescribes a lower bound on the product of the effective widths of quaternion-valued signals in the spatial and frequency domains. Some conclusions are drawn in Section 6.

2. Preliminaries

The quaternionic algebra was invented by Hamilton in 1843 and is denoted by $\mathbb{H}$ in his honor. It is an extension of the complex numbers to a 4D algebra. Every element of $\mathbb{H}$ is a linear combination of a real scalar and three orthogonal imaginary units (denoted, resp., by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$) with real coefficients

$$\mathbb{H} := \{q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R}\}, \quad (1)$$

where the elements $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ obey Hamilton's multiplication rules

$$\begin{aligned} &\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1; \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \\ &\mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \end{aligned} \quad (2)$$

For every quaternionic number $q = q_0 + \underline{q}$, $\underline{q} = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$, the scalar and nonscalar parts of q are defined as $\mathrm{Sc}(q) := q_0$ and $\mathrm{NSc}(q) := \underline{q}$, respectively.

Every quaternion $q = q_0 + \underline{q}$ has a *quaternionic conjugate* $\overline{q} = q_0 - \underline{q}$. This leads to a *norm* of $q \in \mathbb{H}$ defined as

$$|q| := \sqrt{q\overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \quad (3)$$

Let $|q|$ and θ ($\in \mathbb{R}$) be polar coordinates of the point $(q_0, \underline{q}) \in \mathbb{H}$ that corresponds to a *nonzero* quaternion $q = q_0 + \underline{q}$. q can be written in *polar form* as

$$q = |q| (\cos\theta + \underline{e}\sin\theta), \quad (4)$$

where $q_0 = |q|\cos\theta$, $|\underline{q}| = |q|\sin\theta$, $\theta = \arctan(|\underline{q}|/q_0)$, and $\underline{e} = \underline{q}/|\underline{q}|$. If $q \equiv 0$, the coordinate θ is undefined; so it is always understood that $q \neq 0$ whenever $\theta = \arg q$ is discussed.

The symbol $e^{\underline{e}\theta}$, or $\exp(\underline{e}\theta)$, is defined by means of an infinite series (or *Euler's formula*) as

$$e^{\underline{e}\theta} := \sum_{n=0}^{\infty} \frac{(\underline{e}\theta)^n}{n!} = \cos\theta + \underline{e}\sin\theta, \quad (5)$$

where θ is to be measured in radians. It enables us to write the polar form (4) in *exponential form* more compactly as

$$q = |q| e^{\underline{e}\theta} = |q| \exp\left(\frac{\underline{q}}{|\underline{q}|} \arctan\left(\frac{|\underline{q}|}{q_0}\right)\right). \quad (6)$$

Quaternions can be used for three- or four-entry vector analyses. Recently, quaternions have also been used for color image analysis. For $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \in \mathbb{H}$, we can use q_1, q_2, and q_3 to represent, respectively, the R, G, and B values of a color image pixel and set $q_0 = 0$.

For $p = 1$ and 2, the quaternion modules $L^p(\mathbb{R}^2; \mathbb{H})$ are defined as

$$\begin{aligned} L^p(\mathbb{R}^2; \mathbb{H}) := \Big\{ f \mid f : \mathbb{R}^2 \longrightarrow \mathbb{H}, \|f\|_{L^p(\mathbb{R}^2;\mathbb{H})} \\ := \int_{\mathbb{R}^2} |f(x_1, x_2)|^p dx_1 dx_2 < \infty \Big\}. \end{aligned} \quad (7)$$

For two quaternionic signals $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$ the quaternionic space can be equipped with a Hermitian inner product,

$$\langle f, g\rangle_{L^2(\mathbb{R}^2;\mathbb{H})} := \int_{\mathbb{R}^2} f(x_1, x_2)\, \overline{g(x_1, x_2)} dx_1 dx_2, \quad (8)$$

whose associated norm is

$$\|f\|_{L^2(\mathbb{R}^2;\mathbb{H})} := \left(\int_{\mathbb{R}^2} |f(x_1, x_2)|^2 dx_1 dx_2\right)^{1/2}. \quad (9)$$

As a consequence of the inner product (9), we obtain the *quaternionic Cauchy-Schwarz inequality*

$$\begin{aligned} \left|\mathrm{Sc}\left(\langle f, g\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}\right)\right| &\le \left|\langle f, g\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}\right| \\ &\le \|f\|_{L^2(\mathbb{R}^2;\mathbb{H})} \|g\|_{L^2(\mathbb{R}^2;\mathbb{H})} \end{aligned} \quad (10)$$

for any $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$.

In [39, 40] a Clifford-valued generalized function theory is developed. In the following, we adopt the definition that T is called a *tempered distribution*, if T is a continuous linear functional from $\mathbb{S} := \mathbb{S}(\mathbb{R}^2)$ to $\mathbb{H}$, where $\mathbb{S}(\mathbb{R}^2)$ is the Schwarz class of rapidly decreasing functions. The set of all tempered distributions is denoted by $\mathbb{S}'$. If $T \in \mathbb{S}'$, we denote this value for a test function ϕ by writing

$$T[\phi] := \int_{\mathbb{R}} T(x_1, x_2)\, \phi(x_1, x_2)\, dx_1 dx_2, \quad (11)$$

using square brackets. (In the literature one often sees the notation $\langle T, \phi\rangle$, but we shall avoid this, since it does not completely share the properties of the inner product.)

This is equivalent to the one defined in [39] using modules and enables us to define Fourier transforms on tempered distributions, by the formula

$$\widehat{T}[\phi] = T\left[\widehat{\phi}\right], \quad \forall \phi \in \mathbb{S}, \quad (12)$$

which is just to perform Fourier transform

$$\widehat{\phi}(\omega_1, \omega_2) = \int_{\mathbb{R}^2} \phi(x_1, x_2)\, e^{\mathbf{i}(x_1\omega_1 + x_2\omega_2)} dx_1 dx_2 \quad (13)$$

on each of the components of the distribution. We will use the following results:

$$\hat{1}(\omega_1,\omega_2) = (2\pi)^2\delta(\omega_1,\omega_2), \tag{14}$$

$$\left(\widehat{\mathbf{i}^{-|\alpha|}D^{\alpha}\delta}\right)(\omega_1,\omega_2) = \omega_1^{\alpha_1}\omega_2^{\alpha_2}, \tag{15}$$

where $\alpha = (\alpha_1,\alpha_2)$, $|\alpha| = \alpha_1+\alpha_2$, $D^{\alpha} = (\partial/\partial x_1)^{\alpha_1}(\partial/\partial x_2)^{\alpha_2}$, and δ is the usual Dirac delta function.

In the following we introduce the LCT for 2D quaternionic signals.

3. LCTs of 2D Quaternionic Signals

The LCT was first introduced in the 70s and is a four-parameter class of linear integral transform, which includes among its many special cases the FT, the fractional Fourier transform (FRFT), the Fresnel transform, the Lorentz transform, and scaling operations. In a way, the LCT has more degrees of freedom and is more flexible than the FT and the FRFT, but with similar computation cost as the conventional FT [41]. Due to the mentioned advantages, it is natural to generalize the classical LCT to the quaternionic algebra.

3.1. Definition. Using the definition of the LCT [33, 42], we extend the LCT to the 2D quaternionic signals. Let us define the *left-sided* and *right-sided LCTs* of 2D quaternionic signals.

Definition 1 (left-sided and right-sided LCTs). Let $A_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \in \mathbb{R}^{2\times 2}$ be a matrix parameter such that $\det(A_i) = 1$, for $i = 1,2$. The left-sided and right-sided LCTs of 2D quaternionic signals $f \in L^1(\mathbb{R}^2;\mathbb{H})$ are defined by

$$\begin{aligned} L_l^{\mathbf{i}}(f)(u_1,x_2) &:= \begin{cases} \dfrac{1}{\sqrt{\mathbf{i}2\pi b_1}}\displaystyle\int_{\mathbb{R}} e^{\mathbf{i}((a_1/2b_1)x_1^2-(1/b_1)x_1u_1+(d_1/2b_1)u_1^2)} f(x_1,x_2)\,dx_1, & b_1 \neq 0; \\ \sqrt{d_1}e^{\mathbf{i}(c_1d_1/2)u_1^2} f(d_1u_1,x_2), & b_1 = 0, \end{cases} \\ L_r^{\mathbf{j}}(f)(x_1,u_2) &:= \begin{cases} \displaystyle\int_{\mathbb{R}} f(x_1,x_2)\dfrac{1}{\sqrt{\mathbf{j}2\pi b_2}} e^{\mathbf{j}((a_2/2b_2)x_2^2-(1/b_2)x_2u_2+(d_2/2b_2)u_2^2)}dx_2, & b_2 \neq 0; \\ f(x_1,d_2u_2)\sqrt{d_2}e^{\mathbf{j}(c_2d_2/2u_2^2)}, & b_2 = 0, \end{cases} \end{aligned} \tag{16}$$

respectively.

Note that, for $b_i = 0$ $(i = 1,2)$, the LCT of a signal is essentially a chirp multiplication and it is of no particular interest for our objective in this work. Hence, without loss of generality, we set $b_i \neq 0$ in the following sections unless stated otherwise. Therefore

$$L_l^{\mathbf{i}}(f)(u_1,x_2) = \int_{\mathbb{R}} K_{A_1}^{\mathbf{i}}(x_1,u_1) f(x_1,x_2)\,dx_1, \quad b_1 \neq 0, \tag{17}$$

$$L_r^{\mathbf{j}}(f)(x_1,u_2) = \int_{\mathbb{R}} f(x_1,x_2) K_{A_2}^{\mathbf{j}}(x_2,u_2)\,dx_2, \quad b_2 \neq 0, \tag{18}$$

where the kernel functions

$$K_{A_1}^{\mathbf{i}}(x_1,u_1) := \frac{1}{\sqrt{\mathbf{i}2\pi b_1}} e^{\mathbf{i}((a_1/2b_1)x_1^2-(1/b_1)x_1u_1+(d_1/2b_1)u_1^2)}, \tag{19}$$

$$K_{A_2}^{\mathbf{j}}(x_2,u_2) := \frac{1}{\sqrt{\mathbf{j}2\pi b_2}} e^{\mathbf{i}((a_2/2b_2)x_2^2-(1/b_2)x_2u_2+(d_2/2b_2)u_2^2)}, \tag{20}$$

respectively.

3.2. Properties. The following proposition summarizes some important properties of the kernel functions $K_{A_1}^{\mathbf{i}}$ (and $K_{A_2}^{\mathbf{j}}$) of the left-sided (and right-sided) LCTs which will be useful to study the properties of LCTs, such as the Plancherel theorem.

Proposition 2. *Let the kernel function K_A be defined by* (19) *or* (20). *Then*

(i) $K_A(-x,\omega) = K_A(x,-\omega)$;

(ii) $K_A(-x,-\omega) = K_A(x,\omega)$;

(iii) $\overline{K_A(x,\omega)} = K_B(x,\omega)$, *where* $B = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$;

(iv) $\int_{\mathbb{R}} K_{A_1}(x,\omega)K_{A_2}(\omega,y)d\omega = K_{A_1A_2}(x,y)$, *where A_1A_2 corresponds to matrix product.*

The proofs of properties (i) to (iii) follow from definitions (19) and (20). The proof of property (iv) can be found in [33, 35].

Note that some properties of the LCT for 2D quaternionic signals follow from the one-dimensional case [35, 42].

Proposition 3. *Let $L_{A_i}(f)$ $(i = 1,2)$ be defined by $L_l^{\mathbf{i}}$ in* (17) *or $L_r^{\mathbf{j}}$ in* (18), *respectively. If $f,g \in L^1 \bigcap L^2(\mathbb{R}^2;\mathbb{H})$, then the following properties hold.*

(i) *Additivity:*

$$\begin{aligned} L_{A_2}(L_{A_1})(f) &= L_{A_2A_1}(f), \quad \text{for } L_{A_i} := L_l^{\mathbf{i}}, \\ L_{A_2}(L_{A_1})(f) &= L_{A_1A_2}(f), \quad \text{for } L_{A_i} := L_r^{\mathbf{j}}. \end{aligned} \tag{21}$$

(ii) *Reversibility:*

$$L_{A_i^{-1}}\left(L_{A_i}\right)(f)=f,\quad \text{for } L_{A_i}:=L_l^{\mathbf{i}},$$
$$L_{A_i^{-1}}\left(L_{A_i}\right)(f)=f,\quad \text{for } L_{A_i}:=L_r^{\mathbf{j}}. \tag{22}$$

(iii) *Plancherel Theorem (right-sided LCT): If $f, g \in \mathbb{S}$, then*

$$\langle f,g\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}=\left\langle L_r^{\mathbf{j}}(f),L_r^{\mathbf{j}}(g)\right\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{23}$$

In particular, with $f=g$, we get the Parseval theorem; that is,

$$\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}=\left\|L_r^{\mathbf{j}}(f)\right\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{24}$$

Proof. By Fubini's theorem, property (iv) of Proposition 2 establishes the additivity property (i) of left-sided LCTs,

$$\begin{aligned}
&L_{A_2}\left(L_{A_1}(f)(u_1,x_2)\right)(u_1,x_2)\\
&\quad=\int_{\mathbb{R}}K_{A_2}^{\mathbf{i}}(u_1,y_1)\,L_{A_1}(f)(u_1,x_2)\,du_1\\
&\quad=\int_{\mathbb{R}}\left(\int_{\mathbb{R}}K_{A_2}^{\mathbf{i}}(u_1,y_1)\,K_{A_1}^{\mathbf{i}}(x_1,u_1)\,du_1\right)f(x_1,x_2)\,dx_1\\
&\quad=L_{A_2A_1}(f)(y_1,x_2).
\end{aligned} \tag{25}$$

The proof of the right-sided LCT $L_{A_i}:=L_r^{\mathbf{j}}$ is similar.

Reversibility property (ii) is an immediate consequence of additivity property (i) once we observe that $A_1=A_i$ and $A_2=A_i^{-1}$.

To verify property (iii), applying Fubini's theorem, it suffices to see that

$$\begin{aligned}
&\left\langle L_r^{\mathbf{j}}(f),L_r^{\mathbf{j}}(g)\right\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}\\
&\quad=\int_{\mathbb{R}^2}L_r^{\mathbf{j}}(f)(u_1,x_2)\,\overline{L_r^{\mathbf{j}}(g)(u_1,x_2)}\,du_1dx_2\\
&\quad=\int_{\mathbb{R}^4}f(x_1,x_2)\,K_{A_1}^{\mathbf{j}}(x_1,u_1)\\
&\qquad\times\overline{K_{A_1}^{\mathbf{j}}(y_1,u_1)}\,\overline{g(y_1,x_2)}\,du_1dx_1dy_1dx_2\\
&\quad=\frac{1}{2\pi b_1}\int_{\mathbb{R}^4}f(x_1,x_2)\,e^{\mathbf{j}(a_1/2b_1)(x_1^2-y_1^2)}\\
&\qquad\times e^{-\mathbf{j}(1/b_1)u_1(x_1-y_1)}\,\overline{g(y_1,x_2)}\,du_1dx_1dy_1dx_2\\
&\quad=\int_{\mathbb{R}^3}f(x_1,x_2)\,e^{\mathbf{j}(a_1/2b_1)(x_1^2-y_1^2)}\\
&\qquad\times\delta(y_1-x_1)\,\overline{g(y_1,x_2)}\,dx_1dy_1dx_2\\
&\quad=\langle f,g\rangle_{L^2(\mathbb{R}^2;\mathbb{H})},
\end{aligned} \tag{26}$$

where we have used (14). □

Notice that the left-sided and right-sided LCTs of quaternionic signals are unitary operators on $L^2(\mathbb{R}^2;\mathbb{H})$. In signal analysis, it is interpreted in the sense that (right-sided) LCT of quaternionic signal preserves the energy of a signal.

Remark 4. Note that the Plancherel theorem is not valid for the two-sided or left-sided LCT of 2D quaternionic signal. For this reason, we study the right-sided LCT of 2D quaternionic signals in the following.

It is worth noting that when $A_1=A_2=\left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]$, the left-sided and right-sided LCTs of f reduce to the left-sided and right-sided FTs of f. That is,

$$\begin{aligned}
L_l^{\mathbf{i}}(f)(u_1,x_2)&=\frac{1}{\sqrt{2\pi\mathbf{i}}}\int_{\mathbb{R}}e^{-\mathbf{i}x_1u_1}f(x_1,x_2)\,dx_1\\
&=\frac{1}{\sqrt{2\pi\mathbf{i}}}F_l^{\mathbf{i}}(f)(u_1,x_2),\\
L_r^{\mathbf{j}}(f)(x_1,u_2)&=\frac{1}{\sqrt{2\pi\mathbf{j}}}\int_{\mathbb{R}}f(x_1,x_2)\,e^{-\mathbf{j}x_2u_2}dx_2\\
&=\frac{1}{\sqrt{2\pi\mathbf{j}}}F_r^{\mathbf{j}}(f)(x_1,u_2),
\end{aligned} \tag{27}$$

respectively. Here

$$\begin{aligned}
F_l^{\mathbf{i}}(f)(u_1,x_2)&:=\int_{\mathbb{R}}e^{-\mathbf{i}x_1u_1}f(x_1,x_2)\,dx_1,\\
F_r^{\mathbf{j}}(f)(x_1,u_2)&:=\int_{\mathbb{R}}f(x_1,x_2)\,e^{-\mathbf{j}x_2u_2}dx_2
\end{aligned} \tag{28}$$

are the left-sided FT and right-sided FT of f, respectively.

We now formulate the linear canonical integral representation of a 2D quaternionic signal f.

Theorem 5 (linear canonical inversion theorem). *Suppose that $f\in L^1(\mathbb{R}^2;\mathbb{H})$, that f is continuous except for a finite number of finite jumps in any finite interval, and that $f(s,t)=(1/2)(f(s,t+)+f(s,t-))$ for all t and s. Then*

$$f(s,t_0)=\lim_{\alpha\to\infty}\int_{-\alpha}^{\alpha}L_r^{\mathbf{i}}(f)(s,\omega)\,K_{A^{-1}}^{\mathbf{i}}(\omega,t_0)\,d\omega \tag{29}$$

for every t_0 and s where f has (generalized) left and right partial derivatives. In particular, if f is piecewise smooth (i.e., continuous and with a piecewise continuous derivative), then the formula holds for all t_0 and uniformly in s.

Proof. Put

$$I(s,t_0;\alpha):=\int_{-\alpha}^{\alpha}L_r^{\mathbf{i}}(f)(s,\omega)\,K_{A^{-1}}^{\mathbf{i}}(\omega,t_0)\,d\omega \tag{30}$$

and rewrite this expression by inserting the definition of $L_r^{\mathbf{i}}(f)$,

$$\begin{aligned}
&I(s,t_0;\alpha)\\
&\quad=\int_{-\alpha}^{\alpha}\left(\int_{\mathbb{R}}f(s,t)\,K_A^{\mathbf{i}}(t,\omega)\,dt\right)K_{A^{-1}}^{\mathbf{i}}(\omega,t_0)\,d\omega
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \int_{-\alpha}^{\alpha} f(s,t) K_A^{\mathbf{i}}(t,\omega) K_{A^{-1}}^{\mathbf{i}}(\omega,t_0)\, d\omega\, dt \\
&= \int_{\mathbb{R}} f(s,t) \left[\frac{1}{2\pi b} e^{\mathbf{i}(a/2b)(t^2-t_0^2)} \int_{-\alpha}^{\alpha} e^{\mathbf{i}(1/b)(t_0-t)\omega} d\omega \right] dt \\
&= \frac{1}{4\pi} e^{-\mathbf{i}(a/2b)t_0^2} \int_{\mathbb{R}} f(s,t) \left(e^{\mathbf{i}(a/2b)t^2} \frac{\sin((\alpha/b)(t_0-t))}{t_0-t} \right) dt \\
&= \frac{1}{4\pi} e^{-\mathbf{i}(a/2b)t_0^2} \int_{\mathbb{R}} f(s,t_0-u) \\
&\quad \times \left(e^{\mathbf{i}(a/2b)(t_0-u)^2} \frac{\sin((\alpha/b)u)}{u} \right) du.
\end{aligned} \tag{31}
$$

Switching the order of integration is permitted, because the improper double integral is absolutely convergent over the strip $(t,\omega) \in \mathbb{R} \times [-\alpha,\alpha]$, and in the last step we have put $t_0 - t = u$. Using the formula

$$
\int_0^{\infty} e^{\mathbf{i}(a/2b)(t_0-u)^2} \frac{\sin((\alpha/b)u)}{u} du = 2\pi e^{\mathbf{i}(a/2b)t_0^2}, \quad \text{for } \alpha, b > 0, \tag{32}
$$

we can write

$$
\begin{aligned}
&\frac{1}{2\pi} e^{-\mathbf{i}(a/2b)t_0^2} \int_0^{\infty} f(s,t_0-u) \\
&\qquad \times \left(e^{\mathbf{i}(a/2b)(t_0-u)^2} \frac{\sin((\alpha/b)u)}{u} \right) du \\
&\quad - f(s,t_0-) \\
&= \frac{1}{2\pi} e^{-\mathbf{i}(a/2b)t_0^2} \int_0^{\infty} (f(s,t_0-u) - f(s,t_0-)) \\
&\qquad \times \left(e^{\mathbf{i}(a/2b)(t_0-u)^2} \frac{\sin((\alpha/b)u)}{u} \right) du.
\end{aligned} \tag{33}
$$

Now let $\epsilon > 0$ be given. Since we have assumed that $f \in L^1(\mathbb{R}^2;\mathbb{H})$, there exists a number β such that

$$
\frac{1}{2\pi} \int_{\beta}^{\infty} |f(s,t_0-u)|\, du < \epsilon. \tag{34}
$$

Changing the variable, we find that

$$
\begin{aligned}
&\int_{\beta}^{\infty} e^{\mathbf{i}(a/2b)(t_0-u)^2} \frac{\sin((\alpha/b)u)}{u} du \\
&\quad = \int_{\alpha\beta/b}^{\infty} e^{\mathbf{i}(a/2b)(t_0-bx/\alpha)^2} \frac{\sin x}{x} dx \longrightarrow 0, \\
&\qquad \text{as } \frac{\alpha}{b} \longrightarrow \infty.
\end{aligned} \tag{35}
$$

The last integral in (33) can be split into three terms:

$$
\begin{aligned}
&\frac{1}{2\pi} e^{-\mathbf{i}(a/2b)t_0^2} \int_0^{\beta} \frac{f(s,t_0-u) - f(s,t_0-)}{u} \\
&\qquad \times \left(e^{\mathbf{i}(a/2b)(t_0-u)^2} \sin((\alpha/b)u) \right) du \\
&\quad + \frac{1}{2\pi} e^{-\mathbf{i}(a/2b)t_0^2} \int_{\beta}^{\infty} f(s,t_0-u) \\
&\qquad \times \left(e^{\mathbf{i}(a/2b)(t_0-u)^2} \frac{\sin((\alpha/b)u)}{u} \right) du \\
&\quad - \frac{1}{2\pi} e^{-\mathbf{i}(a/2b)t_0^2} f(s,t_0-) \\
&\quad \times \int_{\beta}^{\infty} \left(e^{\mathbf{i}(a/2b)(t_0-u)^2} \frac{\sin((\alpha/b)u)}{u} \right) du \\
&= I_1 + I_2 - I_3.
\end{aligned} \tag{36}
$$

The term I_3 tends to zero as $b > 0$ and $\alpha \to \infty$, because of (35). The term I_2 can be estimated:

$$
\begin{aligned}
|I_2| &= \left| \frac{1}{2\pi} e^{-\mathbf{i}(a/2b)t_0^2} \int_{\beta}^{\infty} f(s,t_0-u) \right. \\
&\qquad \left. \times \left(e^{\mathbf{i}(a/2b)(t_0-u)^2} \frac{\sin((\alpha/b)u)}{u} \right) du \right| \\
&\le \frac{1}{2\pi} \int_{\beta}^{\infty} |f(s,t_0-u)|\, du \le \epsilon.
\end{aligned} \tag{37}
$$

In the term I_1 we have the function $g(s,u) = (f(s,t_0-u) - f(s,t_0-))/u$. This is continuous except for jumps in the interval $\mathbb{R} \times (0,\beta)$, and it has the finite limit $g(s,0+) = (\partial/\partial t) f_L(s,t_0)$ as $u \searrow 0$; this means that g is bounded uniformly in s and thus integrable on the interval. By the Riemann-Lebesgue lemma, we conclude that $I_1 \to 0$ as $\beta \to \infty$. All this together gives, since ϵ can be taken as small as we wish,

$$
\begin{aligned}
&\frac{1}{2\pi} e^{-\mathbf{i}(a/2b)t_0^2} \\
&\quad \times \int_0^{\infty} f(s,t_0-u) \\
&\qquad \times \left(e^{\mathbf{i}(a/2b)(t_0-u)^2} \frac{\sin((\alpha/b)u)}{u} \right) du \longrightarrow f(s,t_0-), \\
&\qquad \text{as } \frac{\alpha}{b} \longrightarrow \infty.
\end{aligned} \tag{38}
$$

A parallel argument implies that the corresponding integral over $(-\infty,0)$ tends to $f(s,t_0+)$ uniformly in s. Taking the mean value of these two results, we have completed the proof of the theorem. □

Remark 6. If $L_r^{\mathbf{i}}(f) \in L^1(\mathbb{R}^2;\mathbb{H})$, then (29) can be written as the absolutely convergent integral

$$f(s,t_0) = \int_{\mathbb{R}} L_r^{\mathbf{i}}(f)(s,\omega) K_{A^{-1}}^{\mathbf{i}}(\omega,t_0)\, d\omega. \tag{39}$$

The following lemma gives the relationship between the left-(right-) sided LCTs and Left-(right-) sided FTs of f.

Lemma 7. *Let* $A_i = \left[\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right] \in \mathbb{R}^{2\times 2}$ *be a matrix parameter such that* $\det(A_i) = 1$, *for* $i = 1,2$. *Let* $f \in L^1(\mathbb{R}^2;\mathbb{H})$; *then one has*

$$L_l^{\mathbf{i}}(f)(u_1,x_2) = e^{\mathbf{i}(d_1/2b_1)u_1^2} F_l^{\mathbf{i}}\left(\frac{1}{\sqrt{\mathbf{i}2\pi b_1}} e^{\mathbf{i}(a_1/2b_1)(\cdot)^2} f(\cdot,x_2)\right) \times \left(\frac{u_1}{b_1}, x_2\right), \tag{40}$$

$$\begin{aligned} L_r^{\mathbf{j}}(f)(x_1,u_2) &= \left(F_r^{\mathbf{j}}\left(f(x_1,\cdot)\frac{1}{\sqrt{\mathbf{j}2\pi b_2}} e^{\mathbf{j}(a_2/2b_2)(\cdot)^2}\right) \times \left(x_1, \frac{u_2}{b_2}\right)\right) e^{\mathbf{j}(d_2/2b_2)u_2^2}. \end{aligned} \tag{41}$$

Proof. By the definition of $L_l^{\mathbf{i}}(f)$ in (17), a direct computation shows that

$$\begin{aligned} L_l^{\mathbf{i}}(f)(u_1,x_2) &= \int_{\mathbb{R}} \frac{1}{\sqrt{\mathbf{i}2\pi b_1}} e^{\mathbf{i}((a_1/2b_1)x_1^2 - (1/b_1)x_1u_1 + (d_1/2b_1)u_1^2)} f(x_1,x_2)\, dx_1 \\ &= e^{\mathbf{i}(d_1/2b_1)u_1^2} \times \left(\int_{\mathbb{R}} e^{-\mathbf{i}x_1(u_1/b_1)} \left(\frac{1}{\sqrt{\mathbf{i}2\pi b_1}} e^{\mathbf{i}(a_1/2b_1)x_1^2} f(x_1,x_2)\right) dx_1\right) \\ &= e^{\mathbf{i}(d_1/2b_1)u_1^2} F_l^{\mathbf{i}}\left(\frac{1}{\sqrt{\mathbf{i}2\pi b_1}} e^{\mathbf{i}(a_1/2b_1)(\cdot)^2} f(\cdot,x_2)\right)\left(\frac{u_1}{b_1}, x_2\right). \end{aligned} \tag{42}$$

Similarly, by the definition of $L_r^{\mathbf{j}}(f)$ in (18), we obtain (41). □

The LCT can be further generalized into the offset linear canonical transform (offset LCT) [33, 43, 44]. It has two extra parameters which represent the space and frequency offsets. The basic theories of the LCT have been developed including uncertainty principles [20, 45], convolution theorem [42, 46], the Hilbert transform [11, 47], sampling theory [32, 42], and discretization [41, 48, 49], which enrich the theoretical system of the LCT. On the other hand, since the LCT has three free parameters, it is more flexible and has found many applications in radar system analysis, filter design, phase retrieval, pattern recognition, and many other areas [35, 42].

4. QLCTs of 2D Quaternionic Signals

4.1. Definition. This section leads to the *quaternionic linear canonical transforms* (QLCTs). Due to the noncommutative property of multiplication of quaternions, there are many different types of QLCTs: two-sided QLCTs, left-sided QLCTs, and right-sided QLCTs.

Definition 8 (two-sided QLCTs). Let $A_i = \left[\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right] \in \mathbb{R}^{2\times 2}$ be a matrix parameter such that $\det(A_i) = 1$, $b_i \neq 0$ for $i = 1,2$. The two-sided QLCTs of signals $f \in L^1(\mathbb{R}^2;\mathbb{H})$ are the functions $\mathscr{L}^{\mathbf{i},\mathbf{j}}(f) : \mathbb{R}^2 \to \mathbb{H}$ given by

$$\mathscr{L}^{\mathbf{i},\mathbf{j}}(f)(u_1,u_2) := \int_{\mathbb{R}^2} K_{A_1}^{\mathbf{i}}(x_1,u_1) f(x_1,x_2) K_{A_2}^{\mathbf{j}}(x_2,u_2)\, dx_1 dx_2, \tag{43}$$

where $u_1, u_2 = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$, with $K_{A_1}^{\mathbf{i}}(x_1,u_1)$ and $K_{A_2}^{\mathbf{j}}(x_2,u_2)$ given by (19) and (20), respectively.

Definition 9 (left-sided QLCTs). Let $A_i = \left[\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right] \in \mathbb{R}^{2\times 2}$ be a matrix parameter such that $\det(A_i) = 1$, $b_i \neq 0$ for $i = 1,2$. The left-sided QLCTs of signals $f \in L^1(\mathbb{R}^2;\mathbb{H})$ are the functions $\mathscr{L}_l^{\mathbf{i},\mathbf{j}}(f) \in L(\mathbb{R}^2;\mathbb{H})$ given by

$$\mathscr{L}_l^{\mathbf{i},\mathbf{j}}(f)(u_1,u_2) := \int_{\mathbb{R}^2} K_{A_1}^{\mathbf{i}}(x_1,u_1) K_{A_2}^{\mathbf{j}}(x_2,u_2) f(x_1,x_2)\, dx_1 dx_2, \tag{44}$$

where the kernels $K_{A_1}^{\mathbf{i}}$ and $K_{A_2}^{\mathbf{j}}$ are given by (19) and (20), respectively.

Due to the validity of the Plancherel theorem, we study the right-sided QLCTs of 2D quaternionic signals in this paper.

Definition 10 (right-sided QLCTs). Let $A_i = \left[\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right] \in \mathbb{R}^{2\times 2}$ be a matrix parameter such that $\det(A_i) = 1$, $b_i \neq 0$ for $i = 1,2$. The left-sided QLCTs of signals $f \in L^1(\mathbb{R}^2;\mathbb{H})$ are the functions $\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f) \in L^1(\mathbb{R}^2;\mathbb{H})$ given by

$$\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)(u_1,u_2) := \int_{\mathbb{R}^2} f(x_1,x_2) K_{A_1}^{\mathbf{i}}(x_1,u_1) K_{A_2}^{\mathbf{j}}(x_2,u_2)\, dx_1 dx_2, \tag{45}$$

where $K_{A_1}^{\mathbf{i}}$ and $K_{A_2}^{\mathbf{j}}$ are given by (19) and (20), respectively.

It is significant to note that when $A_1 = A_2 = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, the QLCT of f reduces to the QFT of f. We denote it by

$$\mathscr{F}_r^{\mathbf{i},\mathbf{j}}(f)(u_1,u_2) := \int_{\mathbb{R}^2} f(x_1,x_2) e^{-\mathbf{i}x_1u_1} e^{-\mathbf{j}x_2u_2} dx_1 dx_2. \tag{46}$$

Remark 11. In fact, the right-sided QLCTs defined above can be generalized as follows:

$$\begin{aligned}&\mathscr{L}_r^{\mathbf{e}_1,\mathbf{e}_2}(f)(u_1,u_2)\\&\quad=\int_{\mathbb{R}^2} f(x_1,x_2)\,K_{A_1}^{\mathbf{e}_1}(x_1,u_1)\,K_{A_2}^{\mathbf{e}_2}(x_2,u_2)\,dx_1dx_2,\end{aligned}\tag{47}$$

where $\mathbf{e}_1=\mathbf{e}_{1,\mathbf{i}}\mathbf{i}+\mathbf{e}_{1,\mathbf{j}}\mathbf{j}+\mathbf{e}_{1,\mathbf{k}}\mathbf{k}$ and $\mathbf{e}_2=\mathbf{e}_{2,\mathbf{i}}\mathbf{i}+\mathbf{e}_{2,\mathbf{j}}\mathbf{j}+\mathbf{e}_{2,\mathbf{k}}\mathbf{k}$ so that

$$\begin{aligned}&\mathbf{e}_{1,\mathbf{i}}^2+\mathbf{e}_{1,\mathbf{j}}^2+\mathbf{e}_{1,\mathbf{k}}^2=\mathbf{e}_{2,\mathbf{i}}^2+\mathbf{e}_{2,\mathbf{j}}^2+\mathbf{e}_{2,\mathbf{k}}^2=1\\&\qquad\left(\text{i.e., }\mathbf{e}_1^2=\mathbf{e}_2^2=-1\right)\\&\mathbf{e}_{1,\mathbf{i}}\mathbf{e}_{2,\mathbf{i}}+\mathbf{e}_{1,\mathbf{j}}\mathbf{e}_{2,\mathbf{j}}+\mathbf{e}_{1,\mathbf{k}}\mathbf{e}_{2,\mathbf{k}}=0.\end{aligned}\tag{48}$$

Equation (45) is the special case of (47) in which $\mathbf{e}_1=\mathbf{i}$ and $\mathbf{e}_2=\mathbf{j}$.

Remark 12. For $b_i\neq 0$ $(i=1,2)$ and $f\in L^1(\mathbb{R}^2;\mathbb{R})$, the (right-sided) QLCT of a 2D signal $f\in L^1(\mathbb{R}^2;\mathbb{H})$ in (45) has the closed-form representation:

$$\begin{aligned}&\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)(\omega_1,\omega_2)\\&\quad:=\Phi_0(\omega_1,\omega_2)+\Phi_1(\omega_1,\omega_2)\\&\qquad+\Phi_2(\omega_1,\omega_2)+\Phi_3(\omega_1,\omega_2),\end{aligned}\tag{49}$$

where we put the integrals

$$\begin{aligned}&\Phi_0(\omega_1,\omega_2)\\&\quad=\int_{\mathbb{R}^2} f(x_1,x_2)\,\frac{1}{2\pi\sqrt{\mathbf{k}b_1b_2}}\\&\qquad\times\cos\left(\frac{a_1}{2b_1}x_1^2-\frac{1}{b_1}x_1\omega_1+\frac{d_1}{2b_1}\omega_1^2\right)\\&\qquad\times\cos\left(\frac{a_2}{2b_2}x_2^2-\frac{1}{b_2}x_2\omega_2+\frac{d_2}{2b_2}\omega_2^2\right)dx_1dx_2,\\&\Phi_1(\omega_1,\omega_2)\\&\quad=\int_{\mathbb{R}^2} f(x_1,x_2)\,\frac{\sqrt{\mathbf{i}}}{2\pi\sqrt{\mathbf{j}b_1b_2}}\\&\qquad\times\sin\left(\frac{a_1}{2b_1}x_1^2-\frac{1}{b_1}x_1\omega_1+\frac{d_1}{2b_1}\omega_1^2\right)\\&\qquad\times\cos\left(\frac{a_2}{2b_2}x_2^2-\frac{1}{b_2}x_2\omega_2+\frac{d_2}{2b_2}\omega_2^2\right)dx_1dx_2,\\&\Phi_2(\omega_1,\omega_2)\\&\quad=\int_{\mathbb{R}^2} f(x_1,x_2)\,\frac{\sqrt{\mathbf{j}}}{2\pi\sqrt{\mathbf{i}b_1b_2}}\\&\qquad\times\cos\left(\frac{a_1}{2b_1}x_1^2-\frac{1}{b_1}x_1\omega_1+\frac{d_1}{2b_1}\omega_1^2\right)\\&\qquad\times\sin\left(\frac{a_2}{2b_2}x_2^2-\frac{1}{b_2}x_2\omega_2+\frac{d_2}{2b_2}\omega_2^2\right)dx_1dx_2,\\&\Phi_3(\omega_1,\omega_2)\\&\quad=\int_{\mathbb{R}^2} f(x_1,x_2)\,\frac{\sqrt{\mathbf{k}}}{2\pi\sqrt{b_1b_2}}\\&\qquad\times\sin\left(\frac{a_1}{2b_1}x_1^2-\frac{1}{b_1}x_1\omega_1+\frac{d_1}{2b_1}\omega_1^2\right)\\&\qquad\times\sin\left(\frac{a_2}{2b_2}x_2^2-\frac{1}{b_2}x_2\omega_2+\frac{d_2}{2b_2}\omega_2^2\right)dx_1dx_2.\end{aligned}\tag{50}$$

These equations clearly show how the QLCTs separate real signals into four quaternion components, that is, the even-even, odd-even, even-odds and odd-odd components of f.

Let us give an example to illustrate expression (45).

Example 13. Consider the quaternionic distribution signal, that is, the QLCT kernel of (45)

$$f(x_1,x_2)=K_{A_2}^{-\mathbf{j}}(x_2,u_0)\,K_{A_1}^{-\mathbf{i}}(x_1,v_0).\tag{51}$$

It is easy to see that the QLCT of f is a *Dirac* quaternionic function; that is,

$$\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)(u_1,u_2)=(2\pi)^2\delta(u_1,u_2-\underline{t}),\quad \underline{t}=t_1\mathbf{e}_1+t_2\mathbf{e}_2.\tag{52}$$

4.2. Properties. This subsection describes important properties of the QLCTs that will be used to establish the uncertainty principles for the QLCTs.

We now establish a relation between the right-sided LCTs and the right-sided QLCTs of 2D quaternion-valued signals.

Lemma 14. *Let* $A_i=\left[\begin{smallmatrix}a_i & b_i\\ c_i & d_i\end{smallmatrix}\right]\in\mathbb{R}^{2\times 2}$ *be a matrix parameter such that* $\det(A_i)=1$, $b_i\neq 0$ *for* $i=1,2$. *For* $f\in L^1(\mathbb{R}^2;\mathbb{H})$, *one has*

$$\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)(u_1,u_2)\,L_r^{\mathbf{j}}\left(L_r^{\mathbf{i}}(f)\right)(u_1,u_2).\tag{53}$$

Proof. By using the defnition of right-sided QLCTs (45),

$$\begin{aligned}
&\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)(u_1,u_2)\\
&\quad=\int_{\mathbb{R}^2} f(x_1,x_2)K_{A_1}^{\mathbf{i}}(x_1,u_1)K_{A_2}^{\mathbf{j}}(x_2,u_2)\,dx_1dx_2\\
&\quad=\int_{\mathbb{R}} L_r^{\mathbf{i}}(f)(u_1,x_2)K_{A_2}^{\mathbf{j}}(x_2,u_2)\,dx_2\\
&\quad=L_r^{\mathbf{j}}\left(L_r^{\mathbf{i}}(f)\right)(u_1,u_2).
\end{aligned}\tag{54}$$

□

We then establish the Plancherel theorems, specific to the right-sided QLCTs.

Theorem 15 (the Plancherel theorems of QLCTs). *For $i = 1,2$, let $f_i \in \mathbb{S}$; the inner product* (8) *of two quaternionic module functions and their QLCTs is related by*

$$\langle f_1,f_2\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}=\left\langle \mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f_1),\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f_2)\right\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}.\tag{55}$$

In particular, with $f_1 = f_2 = f$, we get the Parseval identity; that is,

$$\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}=\left\|\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)\right\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}.\tag{56}$$

Proof. By the inner product (8) and definition of right-sided QLCTs (45), a straightforward computation and Fubini's theorem show that

$$\begin{aligned}
&\left\langle \mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f_1),\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f_2)\right\rangle\\
&=\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} f_1(x_1,x_2)K_{A_1}^{\mathbf{i}}(x_1,u_1)K_{A_2}^{\mathbf{j}}(x_2,u_2)\,dx_1dx_2\right)\\
&\quad\times\left(\overline{\int_{\mathbb{R}^2} f_2(y_1,y_2)K_{A_1}^{\mathbf{i}}(y_1,u_1)K_{A_2}^{\mathbf{j}}(y_2,u_2)\,dy_1dy_2}\right)du_1du_2\\
&=\int_{\mathbb{R}^6} f_1(x_1,x_2)K_{A_1}^{\mathbf{i}}(x_1,u_1)\\
&\quad\times\left(K_{A_2}^{\mathbf{j}}(x_2,u_2)\overline{K_{A_2}^{\mathbf{j}}(y_2,u_2)}\right)\\
&\quad\times\overline{f_2(y_1,y_2)K_{A_1}^{\mathbf{i}}(y_1,u_1)}dx_1dx_2dy_1dy_2du_1du_2\\
&=\int_{\mathbb{R}^6} f_1(x_1,x_2)K_{A_1}^{\mathbf{i}}(x_1,u_1)\\
&\quad\times\left(\frac{1}{2\pi b}e^{\mathbf{j}(a_2/2b_2)(x_2^2-y_2^2)}e^{-\mathbf{j}(1/b_2)u_2(x_2-y_2)}\right)\\
&\quad\times\overline{f_2(y_1,y_2)K_{A_1}^{\mathbf{i}}(y_1,u_1)}du_1dx_1dx_2dy_1dy_2du_2\\
&=\int_{\mathbb{R}^5} f_1(x_1,x_2)K_{A_1}^{\mathbf{i}}(x_1,u_1)\\
&\quad\times\left(e^{\mathbf{j}(a_2/2b_2)(x_2^2-y_2^2)}\delta(y_2-x_2)\right)\\
&\quad\times\overline{f_2(y_1,y_2)K_{A_1}^{\mathbf{i}}(y_1,u_1)}dx_1dx_2dy_1dy_2du_2\\
&=\left\langle L_r^{\mathbf{i}}(f_1),L_r^{\mathbf{i}}(f_2)\right\rangle=\langle f_1,f_2\rangle,
\end{aligned}\tag{57}$$

where we have used the Plancherel theorem of right-sided LCTs (24) and formula (14). □

Remark 16. Note that the Plancherel theorem is not valid for the two-sided or left-sided QLCT of quaternionic signals. For this reason, we choose to apply the right-sided QLCT of 2D quaternionic signals in the present paper.

Theorem 15 shows that the total signal energy computed in the spatial domain equals the total signal energy in the quaternionic domain. The Parseval theorem allows the energy of a quaternion-valued signal to be considered on either the spatial domain or the quaternionic domain and the change of domains for convenience of computation.

To proceed with, we prove the following derivative properties.

Lemma 17. *For $i = 1,2$, let $A_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i\end{bmatrix} \in \mathbb{R}^{2\times 2}$ be a matrix parameter, $b_i \neq 0$, and $a_id_i - b_ic_i = 1$. If $f \in \mathbb{S}$, then*

$$\begin{aligned}
&\int_{\mathbb{R}^2} u_i^2\left|\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)(u_1,u_2)\right|^2du_1du_2\\
&\quad=b_i^2\int_{\mathbb{R}^2}\left|\frac{\partial}{\partial x_i}f(x_1,x_2)\right|^2dx_1dx_2.
\end{aligned}\tag{58}$$

Proof. For $i = 1$, using (14), (15), and Fubini's theorem, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} u_1^2\left|\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)(u_1,u_2)\right|^2du_1du_2\\
&=\int_{\mathbb{R}^2} u_1^2\left(\int_{\mathbb{R}^2} f(s_1,s_2)K_{A_1}^{\mathbf{i}}(s_1,u_1)K_{A_2}^{\mathbf{j}}(s_2,u_2)\,ds_1ds_2\right)\\
&\quad\times\left(\overline{\int_{\mathbb{R}^2} f(x_1,x_2)K_{A_1}^{\mathbf{i}}(x_1,u_1)K_{A_2}^{\mathbf{j}}(x_2,u_2)\,dx_1dx_2}\right)du_1du_2\\
&=\int_{\mathbb{R}^6} u_1^2 f(s_1,s_2)K_{A_1}^{\mathbf{i}}(s_1,u_1)\left(K_{A_2}^{\mathbf{j}}(s_2,u_2)\overline{K_{A_2}^{\mathbf{j}}(x_2,u_2)}\right)\\
&\quad\times\overline{K_{A_1}^{\mathbf{i}}(x_1,u_1)}\,\overline{f(x_1,x_2)}du_2ds_1ds_2dx_1dx_2du_1\\
&=\int_{\mathbb{R}^5} u_1^2 f(s_1,s_2)K_{A_1}^{\mathbf{i}}(s_1,u_1)\left(e^{\mathbf{j}(a_2/2b_2)(s_2^2-x_2^2)}\delta(x_2-s_2)\right)\\
&\quad\times\overline{K_{A_1}^{\mathbf{i}}(x_1,u_1)}\,\overline{f(x_1,x_2)}ds_1ds_2dx_1dx_2du_1\\
&=\int_{\mathbb{R}^4} f(s_1,s_2)\left(u_1^2K_{A_1}^{\mathbf{i}}(s_1,u_1)\overline{K_{A_1}^{\mathbf{i}}(x_1,u_1)}\right)\\
&\quad\times\overline{f(x_1,x_2)}du_1ds_1dx_1dx_2
\end{aligned}$$

$$
\begin{aligned}
&= -b_1^2 \int_{\mathbb{R}^3} f(s_1, x_2) \\
&\quad \times \left(e^{\mathbf{i}(a_1/2b_1)(s_1^2 - x_1^2)} \frac{\partial^2}{\partial x_1^2} \delta(x_1 - s_1) \right) \overline{f(x_1, x_2)} ds_1 dx_1 dx_2 \\
&= -b_1^2 \int_{\mathbb{R}^2} f(x_1, x_2) \frac{\partial^2}{\partial x_1^2} \overline{f(x_1, x_2)} dx_1 dx_2 \\
&= b_1^2 \int_{\mathbb{R}^2} \left| \frac{\partial}{\partial x_1} f(x_1, x_2) \right|^2 dx_1 dx_2.
\end{aligned} \tag{59}
$$

To prove the case $i = 2$, we argue in the same spirit as in the proof of the case $i = 1$. Applying (14), (15), and Fubini's theorem, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2} u_2^2 \left| \mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)(u_1, u_2) \right|^2 du_1 du_2 \\
&= \int_{\mathbb{R}^2} u_2^2 \left(\int_{\mathbb{R}^2} f(s_1, s_2) K_{A_1}^{\mathbf{i}}(s_1, u_1) K_{A_2}^{\mathbf{j}}(s_2, u_2) ds_1 ds_2 \right) \\
&\times \left(\overline{\int_{\mathbb{R}^2} f(x_1, x_2) K_{A_1}^{\mathbf{i}}(x_1, u_1) K_{A_2}^{\mathbf{j}}(x_2, u_2) dx_1 dx_2} \right) du_1 du_2 \\
&= \int_{\mathbb{R}^6} f(s_1, s_2) K_{A_1}^{\mathbf{i}}(s_1, u_1) \left(u_2^2 K_{A_2}^{\mathbf{j}}(s_2, u_2) \overline{K_{A_2}^{\mathbf{j}}(x_2, u_2)} \right) \\
&\quad \times \overline{f(x_1, x_2) K_{A_1}^{\mathbf{i}}(x_1, u_1)} du_2 ds_1 ds_2 dx_1 dx_2 du_1 \\
&= -b_2^2 \int_{\mathbb{R}^5} f(s_1, s_2) K_{A_1}^{\mathbf{i}}(s_1, u_1) \\
&\quad \times \left(e^{\mathbf{j}(a_2/2b_2)(s_2^2 - x_2^2)} \frac{\partial^2}{\partial x_2^2} \delta(x_2 - s_2) \right) \\
&\quad \times \overline{f(x_1, x_2) K_{A_1}^{\mathbf{i}}(x_1, u_1)} ds_1 ds_2 dx_1 dx_2 du_1 \\
&= -b_2^2 \int_{\mathbb{R}^5} f(s_1, x_2) K_{A_1}^{\mathbf{i}}(s_1, u_1) \\
&\quad \times \overline{\left(\frac{\partial^2}{\partial x_2^2} f(x_1, x_2) \right) K_{A_1}^{\mathbf{i}}(x_1, u_1)} ds_1 dx_1 dx_2 du_1 \\
&= -b_2^2 \int_{\mathbb{R}^4} f(s_1, x_2) \left(K_{A_1}^{\mathbf{i}}(s_1, u_1) \overline{K_{A_1}^{\mathbf{i}}(x_1, u_1)} \right) \\
&\quad \times \overline{\frac{\partial^2}{\partial x_2^2} f(x_1, x_2)} du_1 ds_1 dx_1 dx_2 \\
&= -b_2^2 \int_{\mathbb{R}^3} f(s_1, x_2) \left(e^{\mathbf{i}(a_1/2b_1)(s_1^2 - x_1^2)} \delta(x_1 - s_1) \right) \\
&\quad \times \overline{\frac{\partial^2}{\partial x_2^2} f(x_1, x_2)} ds_1 dx_1 dx_2 \\
&= -b_2^2 \int_{\mathbb{R}^2} f(x_1, x_2) \frac{\partial^2}{\partial x_2^2} \overline{f(x_1, x_2)} dx_1 dx_2 \\
&= b_2^2 \int_{\mathbb{R}^2} \left| \frac{\partial}{\partial x_2} f(x_1, x_2) \right|^2 dx_1 dx_2.
\end{aligned} \tag{60}
$$

□

Some properties of the QLCT are summarized in Table 1. Let f_1 and $f_2 \in \mathbb{S}$, the constants α and $\beta \in \mathbb{R}$, $A_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \in \mathbb{R}^{2\times 2}$, $b_i \neq 0$, and $a_i d_i - b_i c_i = 1$.

5. Uncertainty Principles for QLCTs

In signal processing much effort has been placed in the study of the classical Heisenberg uncertainty principle during the last years. Shinde and Gadre [9] established an uncertainty principle for fractional Fourier transforms that provides a lower bound on the uncertainty product of real signal representations in both time and frequency domains. Korn [50] proposed Heisenberg-type uncertainty principles for Cohen transforms which describe lower limits for the time frequency concentration. In the meantime, Hitzer et al. [51–54] investigated a directional uncertainty principle for the Clifford-Fourier transform, which describes how the variances (in arbitrary but fixed directions) of a multivector-valued function and its Clifford-Fourier transform are related. On our knowledge, a systematic work on the investigation of uncertainty relations using the QLCT of a multivector-valued function has not been carried out.

In the following we explicitly prove and generalize the classical uncertainty principle to quaternionic module functions using the QLCTs. We also give an explicit proof for the Gaussian quaternionic functions (the Gabor filters) to be indeed the only functions that minimize the uncertainty. We further emphasize that our generalization is nontrivial because the multiplication of quaternions and the quaternionic linear canonical kernel are both noncommutative. For this purpose we introduce the following definition.

Definition 18. For $k = 1, 2$, let $f, x_k f \in L^2(\mathbb{R}^2; \mathbb{H})$ and $\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)$, $u_k \mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f) \in L^2(\mathbb{R}^2; \mathbb{H})$. Then the effective spatial width or spatial uncertainty Δx_k of f is evaluated by

$$
\Delta x_k := \sqrt{\mathrm{Var}_k(f)}, \tag{61}
$$

where $\mathrm{Var}_k(f)$ is the variance of the energy distribution of f along the x_k-axis defined by

$$
\begin{aligned}
\mathrm{Var}_k(f) &:= \frac{\|x_k f\|_{L^2(\mathbb{R}^2;\mathbb{H})}^2}{\|f\|_{L^2(\mathbb{R}^2;\mathbb{H})}^2} \\
&= \frac{\int_{\mathbb{R}^2} x_k^2 |f(x_1, x_2)|^2 dx_1 dx_2}{\int_{\mathbb{R}^2} |f(x_1, x_2)|^2 dx_1 dx_2}.
\end{aligned} \tag{62}
$$

Similarly, in the quaternionic domain we define the effective spectral width as

$$
\Delta u_k := \sqrt{\mathrm{Var}_k\left(\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)\right)}, \tag{63}
$$

Table 1: Properties of the QLCT.

Property	Function	QLCT
Real linearity	$\alpha f_1(x_1,x_2)+\beta f_2(x_1,x_2)$	$\alpha\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)(u_1,u_2)+\beta\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f_2)(u_1,u_2)$
Formula		
Plancherel	$\langle f_1,f_2\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}=$	$\left\langle\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f_1),\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f_2)\right\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}$
Parseval	$\lVert f\rVert_{L^2(\mathbb{R}^2;\mathbb{H})}=$	$\left\lVert\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)\right\rVert_{L^2(\mathbb{R}^2;\mathbb{H})}$
Derivatives	$\int_{\mathbb{R}^2}u_i^2\left\lvert\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)(u_1,u_2)\right\rvert^2du_1du_2=$	$b_i^2\int_{\mathbb{R}^2}\left\lvert\frac{\partial}{\partial x_i}f(x_1,x_2)\right\rvert^2dx_1dx_2$

where $\mathrm{Var}_k(\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f))$ is the variance of the frequency spectrum of f along the u_k frequency axis given by

$$\begin{aligned}\mathrm{Var}_k\left(\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)\right)&=\frac{\left\lVert u_k\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)\right\rVert^2_{L^2(\mathbb{R}^2;\mathbb{H})}}{\left\lVert\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)\right\rVert^2_{L^2(\mathbb{R}^2;\mathbb{H})}}\\&=\frac{\int_{\mathbb{R}^2}u_k^2\left\lvert\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)(u_1,u_2)\right\rvert^2du_1du_2}{\int_{\mathbb{R}^2}\left\lvert\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)(u_1,u_2)\right\rvert^2du_1du_2}.\end{aligned}\tag{64}$$

Example 19. Let us consider a 2D Gaussian quaternionic function (Figures 1, 2, 3, and 4) of the form

$$f(x_1,x_2)=Ce^{-(\alpha_1x_1^2+\alpha_2x_2^2)},\tag{65}$$

where $C=C_{i0}+\mathbf{i}C_{i1}+\mathbf{j}C_{i2}+\mathbf{k}C_{i3}\in\mathbb{H}$, for $i=1,2$, are quaternionic constants and $\alpha_1,\alpha_2\in\mathbb{R}$ are positive real constants.

Then the QLCT of f is given by

$$\begin{aligned}&\mathscr{L}^{\mathbf{i},\mathbf{j}}(f)(u_1,u_2)\\&=C\left(\int_{\mathbb{R}}K_{A_1}^{\mathbf{i}}(x_1,u_1)e^{-\alpha_1x_1^2}dx_1\right)\\&\quad\times\left(\int_{\mathbb{R}}e^{-\alpha_2x_2^2}K_{A_2}^{\mathbf{j}}(x_2,u_2)dx_2\right)\\&=C\sqrt{\frac{2b_1\pi}{2\alpha_1b_1-a_1\mathbf{i}}}\\&\quad\times e^{(((4\alpha_1b_1-2a_1\mathbf{i})d_1+2)/(2b_1(4\alpha_1b_1-2a_1\mathbf{i})))u_1^2}\\&\quad\times e^{(((4\alpha_2b_2-2a_2\mathbf{j})d_2+2)/(2b_2(4\alpha_2b_2-2a_2\mathbf{j})))u_2^2}\\&\quad\times\sqrt{\frac{2b_2\pi}{2\alpha_2b_2-a_2\mathbf{j}}}.\end{aligned}\tag{66}$$

This shows that the QLCT of the Gaussian quaternionic function is another Gaussian quaternionic function.

Figures 1 and 2 visualize the quaternionic Gaussian function, for $\alpha_1=\alpha_2=3$, and $\alpha_1=3$ and $\alpha_2=1$ in the spatial domain. Figures 3 and 4 visualize the quaternionic Gaussian function, for $\alpha_1=1$ and $\alpha_2=3$, and $\alpha_1=\alpha_2=1/2$ in the spatial domain.

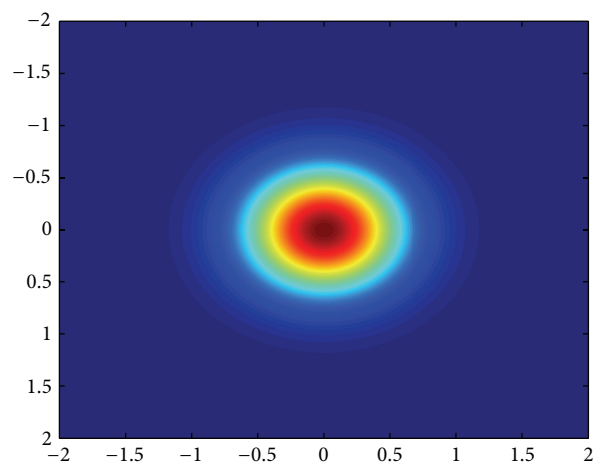

Figure 1

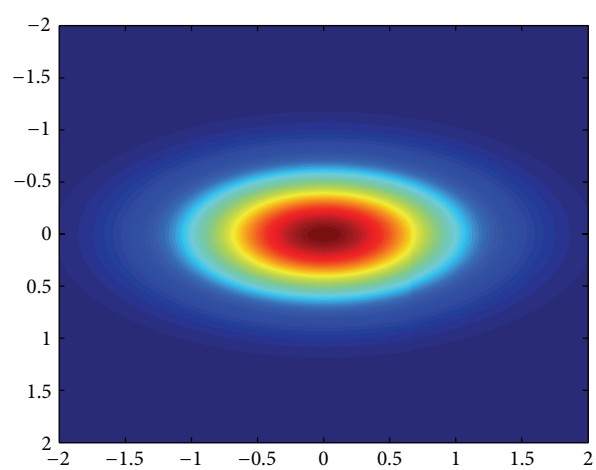

Figure 2

Now let us begin the proofs of two uncertainty relations.

Theorem 20. *For $k=1,2$, let $f\in\mathbb{S}$. Then the following uncertainty relations are fulfilled:*

$$\begin{aligned}\Delta x_1\Delta u_1&\geq\frac{b_1}{2},\\\Delta x_2\Delta u_2&\geq\frac{b_2}{2}.\end{aligned}\tag{67}$$

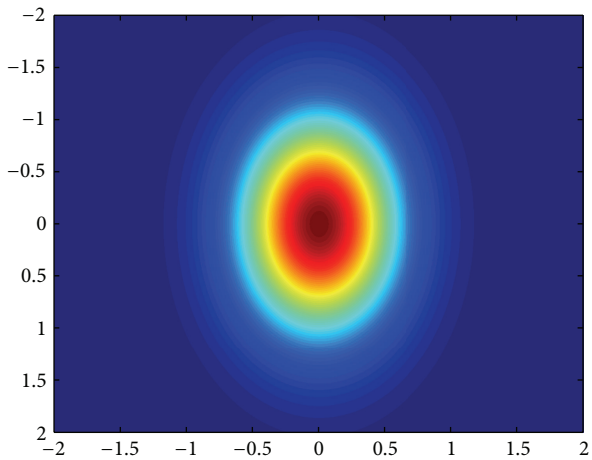

Figure 3

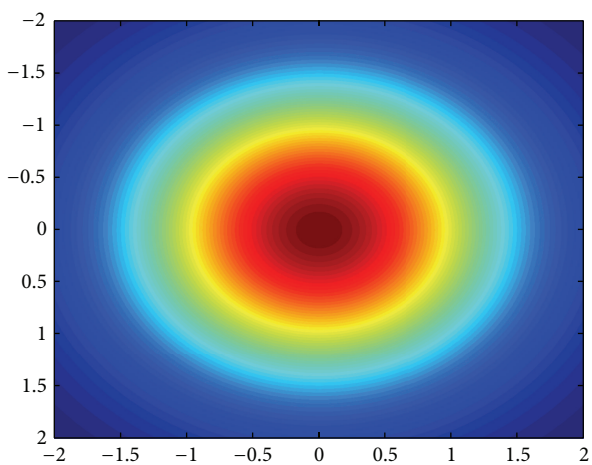

Figure 4

The combination of the two spatial uncertainty principles above leads to the uncertainty principle for the 2D quaternionic signal $f(x_1, x_2)$ of the form

$$\Delta x_1 \Delta x_2 \Delta u_1 \Delta u_2 \geq \frac{b_1 b_2}{4}. \tag{68}$$

Equality holds in (68) *if and only if f is a 2D Gaussian function; that is.*

$$f(x_1, x_2) = \beta e^{-(C_1 x_1^2 + C_2 x_2^2)/2}, \tag{69}$$

where C_1, C_2 are positive real constants and $\beta = \|f\|_{L^2(\mathbb{R}^2;\mathbb{H})} (C_1 C_2/\pi^2)^{1/4}$.

Proof. Applying (58) in Lemma 17 and using the Schwarz inequality (10), we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^2} x_k^2 |f(x_1, x_2)|^2 dx_1 dx_2\right) \\
&\quad \times \left(\int_{\mathbb{R}^2} u_k^2 \left|\mathcal{L}_r^{\mathbf{i},\mathbf{j}}(f)(u_1, u_2)\right|^2 du_1 du_2\right) \\
&= \left(\int_{\mathbb{R}^2} x_k^2 |f(x_1, x_2)|^2 dx_1 dx_2\right) \\
&\quad \times \left(b_k^2 \int_{\mathbb{R}^2} \left|\frac{\partial}{\partial x_k} f(x_1, x_2)\right|^2 dx_1 dx_2\right) \\
&\geq b_k^2 \left|\int_{\mathbb{R}^2} x_k \overline{f(x_1, x_2)} \frac{\partial}{\partial x_k} f(x_1, x_2)\, dx_1 dx_2\right|^2.
\end{aligned} \tag{70}$$

Using the exponential form of a 2D quaternionic signal (6), let

$$f(x_1, x_2) = f_0(x_1, x_2) + \underline{f}(x_1, x_2) = |f(x_1, x_2)| e^{\underline{e}\alpha}, \tag{71}$$

where $\underline{e} = \underline{f}(x_1, x_2)/|\underline{f}(x_1, x_2)|$ and $\alpha = \arctan(|\underline{f}(x_1, x_2)|/f_0(x_1, x_2))$; then

$$\begin{aligned}
&x_k \overline{f(x_1, x_2)} \frac{\partial}{\partial x_k} f(x_1, x_2) \\
&\quad = x_k |f(x_1, x_2)| e^{-\underline{e}\alpha} \frac{\partial}{\partial x_k} \left(|f(x_1, x_2)| e^{\underline{e}\alpha}\right) \\
&\quad = x_k |f(x_1, x_2)| e^{-\underline{e}\alpha} \\
&\qquad \times \left[\left(\frac{\partial}{\partial x_k} |f(x_1, x_2)|\right) e^{\underline{e}\alpha} + |f(x_1, x_2)| \left(\frac{\partial}{\partial x_k} e^{\underline{e}\alpha}\right)\right] \\
&\quad = x_k |f(x_1, x_2)| \left(\frac{\partial}{\partial x_k} |f(x_1, x_2)|\right) \\
&\qquad + x_k |f(x_1, x_2)|^2 \left(\frac{\partial}{\partial x_k} (\underline{e}\alpha)\right) \\
&\quad = \frac{1}{2} \frac{\partial}{\partial x_k} \left(x_k |f(x_1, x_2)|^2\right) - \frac{1}{2} |f(x_1, x_2)|^2 \\
&\qquad + x_k |f(x_1, x_2)|^2 \left(\frac{\partial}{\partial x_k} (\underline{e}\alpha)\right).
\end{aligned} \tag{72}$$

Therefore,

$$\begin{aligned}
&b_k^2 \left|\int_{\mathbb{R}^2} x_k \overline{f(x_1, x_2)} \frac{\partial}{\partial x_k} f(x_1, x_2)\, dx_1 dx_2\right|^2 \\
&\quad = b_k^2 \left|\int_{\mathbb{R}^2} \frac{1}{2} \frac{\partial}{\partial x_k} \left(x_k |f(x_1, x_2)|^2\right) - \frac{1}{2} |f(x_1, x_2)|^2 \right. \\
&\qquad \left. + x_k |f(x_1, x_2)|^2 \left(\frac{\partial}{\partial x_k} (\underline{e}\alpha)\right) dx_1 dx_2\right|^2.
\end{aligned} \tag{73}$$

The first term is a perfect differential and integrates to zero. The second term gives minus one half of the energy $\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}$.

Hence

$$\left(\int_{\mathbb{R}^2} x_k^2 |f(x_1,x_2)|^2 dx_1 dx_2\right) \times \left(\int_{\mathbb{R}^2} u_k^2 \left|\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)(u_1,u_2)\right|^2 du_1 du_2\right) \geq b_k^2 \left|-\frac{1}{2}\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}\right|^2 = \frac{b_k^2}{4}\|f\|^4_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{74}$$

By definitions of Δx_k, Δu_k, and Parseval theorem (56), we have

$$\begin{aligned}(\Delta x_k \Delta u_k)^2 &= \left(\left(\int_{\mathbb{R}^2} x_k^2 |f(x_1,x_2)|^2 dx_1 dx_2\right) \times \left(\int_{\mathbb{R}^2} u_k^2 \left|\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)(u_1,u_2)\right|^2 du_1 du_2\right)\right) \\ &\quad \times \left(\left(\int_{\mathbb{R}^2} |f(x_1,x_2)|^2 dx_1 dx_2\right) \times \left(\int_{\mathbb{R}^2} \left|\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)(u_1,u_2)\right|^2 du_1 du_2\right)\right)^{-1} \\ &= \left(\left(\int_{\mathbb{R}^2} x_k^2 |f(x_1,x_2)|^2 dx_1 dx_2\right) \times \left(\int_{\mathbb{R}^2} u_k^2 \left|\mathscr{L}_r^{\mathbf{i},\mathbf{j}}(f)(u_1,u_2)\right|^2 du_1 du_2\right)\right) \times \left(\|f\|^4_{L^2(\mathbb{R}^2;\mathbb{H})}\right)^{-1} \\ &\geq \frac{b_k^2}{4}\end{aligned} \tag{75}$$

and therefore we have the uncertainty principle as given by (67) and (68).

We finally show that the equality in (67) and (68) is satisfied if and only if f is a Gaussian quaternionic function.

Since the minimum value for the uncertainty product is $b_k/2$, we can ask what signals have that minimum value. The Schwarz inequality (10) becomes an equality when the two functions are proportional to each other. Hence, we take $g = -Cf$, where C is a quaternionic constant and the -1 has been inserted for convenience. We therefore have

$$\frac{\partial}{\partial x_k} f(x_1,x_2) = -C_k x_k f(x_1,x_2). \tag{76}$$

This is a necessary condition for the uncertainty product to be the minimum. But it is not sufficient since we must also have the term

$$\int_{\mathbb{R}^2} x_k |f(x_1,x_2)|^2 \left(\frac{\partial}{\partial x_k}(\underline{e}\alpha)\right) dx_1 dx_2 = 0, \tag{77}$$

because by (73), we see that is the only way we can actually get the value of $b_k/2$.

Since C_k is arbitrary we can write it in terms of its scalar and nonscalar parts, $C_k := \mathrm{Sc}(C_k) + \mathrm{NSc}(C_k)$. The solution of (76) is hence

$$\begin{aligned} f(x_1,x_2) &= \beta e^{-(C_1 x_1^2 + C_2 x_2^2)/2} \\ &= \beta e^{-(\mathrm{Sc}(C_1)x_1^2 + \mathrm{Sc}(C_2)x_2^2)/2} \times e^{-(\mathrm{NSc}(C_1)x_1^2 + \mathrm{NSc}(C_2)x_2^2)/2}, \end{aligned} \tag{78}$$

for some constant β. Since $\underline{e}\alpha = -(\mathrm{NSc}(C_1)x_1^2 + \mathrm{NSc}(C_2)x_2^2)/2$, it follows that

$$\frac{\partial}{\partial x_k}(\underline{e}\alpha) = -\mathrm{NSc}(C_k)\, x_k. \tag{79}$$

We have

$$\begin{aligned} \int_{\mathbb{R}^2} x_k |f(x_1,x_2)|^2 \left(\frac{\partial}{\partial x_k}(\underline{e}\alpha)\right) dx_1 dx_2 &= -\mathrm{NSc}(C_k)\,\beta^2 \times \int_{\mathbb{R}^2} x_k^2 e^{-(\mathrm{Sc}(C_1)x_1^2 + \mathrm{Sc}(C_2)x_2^2)} dx_1 dx_2. \end{aligned} \tag{80}$$

The only way this can be zero is if $\mathrm{NSc}(C_k) = 0$ and hence C_k must be a real number. We then have

$$f(x_1,x_2) = \beta e^{-(C_1 x_1^2 + C_2 x_2^2)/2}, \tag{81}$$

where C_1, C_2 are positive real constants since $f \in \mathbb{S}$ and we have included the appropriate normalization $\beta = \|f\|_{L^2(\mathbb{R}^2;\mathbb{H})}(C_1 C_2/\pi^2)^{1/4}$. □

Since the 2D Gaussian function $f(x_1,x_2)$ of (81) achieves the minimum width-bandwidth product, it is theoretically a very good prototype waveform. One can therefore construct a basic waveform using spatially or frequency-scaled versions of $f(x_1,x_2)$ to provide multiscale spectral resolution. Such a wavelet basis construction derived from a Gaussian quaternionic function prototype waveform has been realized, for example, in the quaternionic wavelet transforms in [55]. The optimal space-frequency localization is also another reason why 2D Clifford-Gabor bandpass filters were suggested in [56].

6. Conclusion

In this paper we developed the definition of QLCT. The various properties of QLCT such as partial derivative, the Plancherel, and Parseval theorems are discussed. Using the well-known properties of the classical LCT, we established an uncertainty principle for the QLCT. This uncertainty principle states that the product of the variances of quaternion-valued signals in the spatial and frequency domains has a lower bound. It is shown that only a 2D Gaussian signal minimizes the uncertainty. With the help of this principle, we hope to contribute to the theory and applications of signal processing through this investigation and to develop

further general numerical methods for differential equations. The results in this paper are new in the literature. Further investigations on this topic are now under investigation and will be reported in a forthcoming paper.

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