Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 736872, 8 pages
http://dx.doi.org/10.1155/2013/736872

Research Article

Some Types of Generalized Fuzzy n-Fold Filters in Residuated Lattices

Zhen Ming Ma

School of Science, Linyi University, Linyi 276005, China

Correspondence should be addressed to Zhen Ming Ma; dmgywto@126.com

Received 3 November 2013; Accepted 18 December 2013

Academic Editor: Yansheng Liu

Fuzzy filters and their generalized types have been extensively studied in the literature. In this paper, a one-to-one correspondence between the set of all generalized fuzzy filters and the set of all generalized fuzzy congruences is established, a quotient residuated lattice with respect to generalized fuzzy filter is induced, and several types of generalized fuzzy n-fold filters such as generalized fuzzy n-fold positive implicative (fantastic and Boolean) filters are introduced; examples and results are provided to demonstrate the relations among these filters.

1. Introduction

Residuated lattices, introduced by Ward and Dilworth in [1], are very basic algebraic structures among algebras associated with logical systems. In fact, many algebras have been proposed as the semantical systems of logical systems, for example, Boolean algebras, MV-algebras, BL-algebras, lattice implication algebras, MTL-algebras, NM-algebras, and R_0-algebras, and so forth, and they are all particular cases of residuated lattices.

Filters are tools of extreme importance in studying these logical algebras and the completeness of nonclassical logics. A filter is also called a deductive system [2]. From logical point of view, various filters correspond to various sets of provable formulae. Filters are also particularly interesting because they are closely related to congruence relations. Hájek [3] introduced the notions of filters and prime filters in BL-algebras and proved the completeness of Basic Logic, BL. Turunen [2] proposed the notions of implicative filters and Boolean filters of BL-algebras and proved that implicative filters are equivalent to Boolean filters in BL-algebras. In [4, 5], some types of filters such as implicative filters, fantastic filters, and Boolean filters in BL-algebras were proposed and some characterizations of them with identity forms were given, and in [6], these filters were generalized to residuated lattices. In [7] some new types of filters such as IMTL-filters, strong MTL-filters, and associative filters were introduced in MTL-algebras, and it was shown in [8] that in any MTL-algebra there exists only one proper associative filter.

At present, there are two branches to generalize the existing types of filters. One is the folding theory and the other is fuzzy sets theory. In the folding approach, in [9–11], n-fold (positive) implicative filters are proposed in BL-algebras. In [12], n-fold EIMTL and n-fold IMTL-filters of MTL-algebras were defined and some relations between these filters and n-fold (positive) implicative filters, n-fold fantastic filters, and n-fold obstinate filters of MTL-algebras were investigated. In the fuzzy approach, fuzzification ideas have been applied to some fuzzy logical algebras. In [6, 13, 14], fuzzy filters in MTL-algebras, BL-algebras, and residuated lattices were studied, respectively. In particular, several types of fuzzy filters such as fuzzy Boolean filters, fuzzy fantastic filters, fuzzy implicative filters and fuzzy regular filters were introduced in [6, 15]. As a further generalization of some fuzzy notions, in BL-algebras, generalized fuzzy filters [16] which are common generalizations of $(\in, \in \vee q)$-fuzzy filters [17–19] and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filters [16] were investigated, and hemirings were characterized by their $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideals [20], $(\in, \in \vee q_k)$-fuzzy ideals [21], and fuzzy ideals with thresholds [22].

This paper continues the study of generalized fuzzy n-fold filters in residuated lattices. In Section 2, some basic concepts and properties are recalled, and some new notions

about the thresholds are introduced to represent generalized fuzzy filters like fuzzy filters which are convenient to study the properties of generalized fuzzy filters. In Section 3, a one-to-one correspondence between the set of all generalized fuzzy filters and the set of all generalized fuzzy congruences is established, and a quotient residuated lattice with respect to generalized fuzzy filter is induced. In Section 4, some types of generalized fuzzy n-fold filters such as generalized fuzzy n-fold positive implicative filters, generalized fuzzy n-fold fantastic filters, and generalized fuzzy Boolean filters are introduced, some properties of them are obtained, and examples and results are provided to show the relations among these filters. The last section concludes this paper.

2. Preliminaries

Here, we recall some basic concepts and results which will be used in the sequel. Throughout this paper, L will denote a residuated lattice, unless otherwise mentioned.

2.1. Residuated Lattices

Definition 1 (see [1]). An algebra $L = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is called a residuated lattice if

(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;

(2) $(L, \otimes, 1)$ is a commutative monoid;

(3) $(\otimes, \rightarrow)$ forms an adjoint pair; that is, $z \leq x \rightarrow y$ if and only if $x \otimes z \leq y$.

In the rest of the paper by $\neg x$ we denote $x \rightarrow 0$.

Lemma 2 (see [3]). *Let L be a residuated lattice. Then the following properties hold for all $x, y, z, w \in L$:*

(1) $x \otimes y \leq x \otimes (x \rightarrow y) \leq x \wedge y$;

(2) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) = (x \otimes y) \rightarrow z$;

(3) $x \vee y \leq (x \rightarrow y) \rightarrow y \leq \neg x \rightarrow y$;

(4) $x \leq y$ *implies* $z \rightarrow x \leq z \rightarrow y$ *and* $y \rightarrow z \leq x \rightarrow z$;

(5) $(y \rightarrow x) \otimes (z \rightarrow w) \leq (x \rightarrow z) \rightarrow (y \rightarrow w)$;

(6) $(x \rightarrow y) \otimes (y \rightarrow z) \leq x \rightarrow z$;

(7) $\neg x \vee y \leq x \rightarrow y$;

(8) $x \rightarrow y \leq (x \otimes z) \rightarrow (y \otimes z)$;

(9) $y \leq x \rightarrow (x \otimes y)$;

(10) $(y \vee z) \rightarrow x = (y \rightarrow x) \wedge (z \rightarrow x)$; *in particular,* $(y \vee z) \rightarrow y = z \rightarrow y$;

(11) $(x \leftrightarrow y) \otimes (z \leftrightarrow w) \leq (x \diamond z) \leftrightarrow (y \diamond w)$, *where* $\diamond \in \{\otimes, \vee, \wedge, \rightarrow\}$;

(12) $(x \leftrightarrow y) \otimes (y \leftrightarrow z) \leq x \leftrightarrow z$.

Definition 3 (see [3]). Let F be a nonempty subset of a residuated lattice L. Then F is called a filter if

(1) $x, y \in F$ implies $x \otimes y \in F$;

(2) $x \in F$ and $x \leq y$ imply $y \in F$.

However, a filter can also be alternatively described as for all $x, y \in L$ it holds that $1 \in F$ and $x, x \rightarrow y \in F$ implies $y \in F$ [2].

2.2. Generalized Fuzzy Filters

Definition 4 (see [19]). Let L be a BL-algebra and $\alpha, \beta \in [0, 1]$ such that $\alpha < \beta$. Then a fuzzy set f of L is called a fuzzy filter with thresholds (α, β) (generalized fuzzy filter for short) if for all $x, y \in L$ it holds that

(1) $\min\{f(x), f(y), \beta\} \leq \max\{f(x \otimes y), \alpha\}$;

(2) $x \leq y$ implies $\min\{f(x), \beta\} \leq \max\{f(y), \alpha\}$.

Definition 5 (see [16]). Let f be a generalized fuzzy filter of a BL-algebra L. Then f is called

(1) a generalized fuzzy implicative (or Boolean) filter if for all $x, y, z \in L$ it holds that

$$\begin{aligned} &\min\{f(x \longrightarrow (\neg z \longrightarrow y)), f(y \longrightarrow z), \beta\} \\ &\quad \leq \max\{f(x \longrightarrow z), \alpha\}, \end{aligned} \tag{1}$$

(2) a generalized fuzzy positive implicative filter if for all $x, y, z \in L$ it holds that

$$\begin{aligned} &\min\{f(x \longrightarrow (y \longrightarrow z)), f(x \longrightarrow y), \beta\} \\ &\quad \leq \max\{f(x \longrightarrow z), \alpha\}, \end{aligned} \tag{2}$$

(3) a generalized fuzzy fantastic filter if for all $x, y, z \in L$ it holds that

$$\begin{aligned} &\min\{f(z \longrightarrow (y \longrightarrow x)), f(z), \beta\} \\ &\quad \leq \max\{f([(x \longrightarrow y) \longrightarrow y] \longrightarrow x), \alpha\}. \end{aligned} \tag{3}$$

2.3. Properties of the Thresholds α, β and Generalized Fuzzy Filters. In this section, we develop new notions about the thresholds and investigate their properties. Particularly, by these notions, we will verify some properties of generalized fuzzy filters similar to the classical or fuzzy cases.

Here, for all $a, b, \alpha, \beta \in [0, 1]$, we denote that $\min\{a, b\} = a \wedge b$, $\max\{a, b\} = a \vee b$, and $b \leq^{(\alpha,\beta)} a$ if and only if $b \wedge \beta \leq a \vee \alpha$ and $a =^{(\alpha,\beta)} b$ if and only if $b \leq^{(\alpha,\beta)} a$ and $a \leq^{(\alpha,\beta)} b$.

However, the following lemmas will be useful in the sequel.

Lemma 6. *Let $\alpha, \beta \in [0, 1]$ such that $\alpha < \beta$. Then, for all $a, b \in [0, 1]$, $a \leq^{(\alpha,\beta)} b$ implies that*

(1) $a \leq^{(\alpha,\beta)} b \wedge \beta$;

(2) $a \vee \alpha \leq^{(\alpha,\beta)} b$.

Proof. We only prove (1), (2) can be proven in a similar way. Consider the following:

$$\begin{aligned} (b \wedge \beta) \vee \alpha &= (b \wedge \beta) \vee (\alpha \wedge \beta) \\ &= (b \vee \alpha) \wedge \beta \geq a \wedge \beta. \end{aligned} \tag{4}$$

Thus $a \leq^{(\alpha,\beta)} b \wedge \beta$. □

Lemma 7. *Let $\alpha, \beta \in [0,1]$ such that $\alpha < \beta$. Then for all $a, b \in [0,1]$ the following assertions are equivalent:*

(1) $a \leq^{(\alpha,\beta)} b$;

(2) $a \leq^{(\alpha,\beta)} a \wedge b$;

(3) $a \vee b \leq^{(\alpha,\beta)} b$.

Proof. We only prove (1)⇔(2), (1)⇔(3) can be similarly proved. Assume that $a \leq^{(\alpha,\beta)} b$. Then it holds that $a \wedge \beta \leq (b \vee \alpha) \wedge a = (a \wedge b) \vee (a \wedge \alpha) \leq (a \wedge b) \vee \alpha$. That is, $a \leq^{(\alpha,\beta)} a \wedge b$.

Conversely, assume that $a \leq^{(\alpha,\beta)} a \wedge b$. That is, $a \wedge \beta \leq (a \wedge b) \vee \alpha$. Obviously, $(a \wedge b) \vee \alpha \leq b \vee \alpha$. Thus $a \wedge \beta \leq b \vee \alpha$; that is, $a \leq^{(\alpha,\beta)} b$. □

Lemma 8. *Let $\alpha, \beta \in [0,1]$ such that $\alpha < \beta$. Then for all $a, b \in [0,1]$*

(1) $b \leq^{(\alpha,\beta)} a \vee b$;

(2) $a \wedge b \leq^{(\alpha,\beta)} a$.

Proof. Since $b \leq a \vee b$ and $a \wedge b \leq a$, it holds that $b \wedge \beta \leq b \leq a \vee b \leq (a \vee b) \vee \alpha$ and $(a \wedge b) \wedge \beta \leq a \wedge b \leq a \leq a \vee \alpha$. Thus $b \leq^{(\alpha,\beta)} a \vee b$ and $a \wedge b \leq^{(\alpha,\beta)} a$. □

Corollary 9. *Let $\alpha, \beta \in [0,1]$ such that $\alpha < \beta$. Then for all $a, b \in [0,1]$ the following assertions are equivalent:*

(1) $a \leq^{(\alpha,\beta)} b$;

(2) $a \vee b =^{(\alpha,\beta)} b$;

(3) $a \wedge b =^{(\alpha,\beta)} a$.

Proof. It follows immediately from Lemmas 7 and 8. □

For $\leq^{(\alpha,\beta)}$ and $=^{(\alpha,\beta)}$, we have the following transitivity, respectively.

Lemma 10. *Let $\alpha, \beta \in [0,1]$ such that $\alpha < \beta$. Then for all $a, b, c \in [0,1]$*

(1) *$a \leq^{(\alpha,\beta)} b$ and $b \leq^{(\alpha,\beta)} c$ imply $a \leq^{(\alpha,\beta)} c$;*

(2) *$a =^{(\alpha,\beta)} b$ and $b =^{(\alpha,\beta)} c$ imply $a =^{(\alpha,\beta)} c$.*

Proof. We only prove (1). Assume that $a \leq^{(\alpha,\beta)} b$ and $b \leq^{(\alpha,\beta)} c$. Using Lemma 6, we have $a \wedge \beta \leq (b \wedge \beta) \vee \alpha = (b \vee \alpha) \wedge \beta$ and $(b \vee \alpha) \wedge \beta \leq c \vee \alpha$. Thus $a \wedge \beta \leq c \vee \alpha$. That is, $a \leq^{(\alpha,\beta)} c$. □

Here, the notions of generalized fuzzy filters can be rewritten in residuated lattices as follows.

Definition 11. Let L be a residuated lattice and $\alpha, \beta \in [0,1]$ such that $\alpha < \beta$. Then a fuzzy set f of L is called a fuzzy filter with thresholds (α, β) (generalized fuzzy filter for short) if for all $x, y \in L$ it holds that

(1) $f(x) \wedge f(y) \leq^{(\alpha,\beta)} f(x \otimes y)$;

(2) $x \leq y$ implies $f(x) \leq^{(\alpha,\beta)} f(y)$.

Theorem 12. *Let f be a fuzzy set of L. Then f is a generalized fuzzy filter if and only if for all $x, y \in L$ it holds that*

(1) $f(x) \leq^{(\alpha,\beta)} f(1)$;

(2) $f(x) \wedge f(x \to y) \leq^{(\alpha,\beta)} f(y)$.

The items in Definition 11 and Theorem 12 will be frequently used, so we do not cite them every time.

Moreover, the following properties of generalized fuzzy filters will be used in the sequel.

Proposition 13. *Let f be a generalized fuzzy filter of L. Then for all $x, y \in L$ it holds that*

(1) *$f(x \to y) =^{(\alpha,\beta)} f(1)$ implies $f(x) \leq^{(\alpha,\beta)} f(y)$;*

(2) *$f(x \otimes y) =^{(\alpha,\beta)} f(x \wedge y) =^{(\alpha,\beta)} f(x) \wedge f(y)$.*

3. Correspondence between Generalized Fuzzy Congruences and Generalized Fuzzy Filters

Here, we define the concept of generalized fuzzy congruence to determine the relationships between generalized fuzzy filters and generalized fuzzy congruences.

Definition 14. Let L be a residuated lattice. A fuzzy set C of $L \times L$ is called a generalized fuzzy congruence of L if for all $x, y, z, w \in L$ it holds that

(1) $C(x,x) =^{(\alpha,\beta)} C(1,1)$;

(2) $C(x,y) =^{(\alpha,\beta)} C(y,x)$;

(3) $C(x,y) \wedge C(y,z) \leq^{(\alpha,\beta)} C(x,z)$;

(4) $C(x,y) \wedge C(z,w) \leq^{(\alpha,\beta)} C(x \diamond z, y \diamond w)$, where $\diamond \in \{\vee, \wedge, \otimes, \to\}$.

For a given generalized fuzzy filter f, we define $C_f : L \times L \to [0,1]$ as $C_f(x,y) = f(x \to y) \wedge f(y \to x)$ for all $x, y \in L$.

Obviously, it follows from Proposition 13(2) that $C_f(x, y) =^{(\alpha,\beta)} f(x \leftrightarrow y)$.

Proposition 15. *Let f be a generalized fuzzy filter of L. Then C_f is a generalized fuzzy congruence.*

Proof. It is trivial. □

Let C be a generalized fuzzy congruence of L and $x \in L$. Define a fuzzy set C^x of L as $C^x(y) = C(x,y)$ for all $y \in L$ which is called a generalized fuzzy congruence class of x by C. The set $L/C = \{C^x \mid x \in L\}$ is called a generalized fuzzy quotient set by C.

Proposition 16. *Let C be a generalized fuzzy congruence of L. Then C^1 is a generalized fuzzy filter.*

Proof. Consider the following special case of Definition 14(3): $C^1(x) = C^1(x) \wedge C^1(x) = C(1,x) \wedge C(x,1) \leq^{(\alpha,\beta)}$

$C(1,1) = C^1(1)$. Thus $C^1(x) \leq^{(\alpha,\beta)} C^1(1)$. It follows from Definition 14(3) and (4) that

$$\begin{aligned} C^1(x) \wedge C^1(x \longrightarrow y) &= C(1,x) \wedge C(1, x \longrightarrow y) \\ &\leq^{(\alpha,\beta)} C(1 \longrightarrow y, x \longrightarrow y) \wedge C(1, x \longrightarrow y) \\ &\leq^{(\alpha,\beta)} C(1,y) = C^1(y), \end{aligned} \tag{5}$$

and hence $C^1(x) \wedge C^1(x \rightarrow y) \leq^{(\alpha,\beta)} C^1(y)$.

Thus C^1 is a generalized fuzzy filter. □

The following lemma is useful.

Lemma 17. *Let C be a generalized fuzzy congruence of L. Then for all $x, y \in L$*

$$C(x \longleftrightarrow y, 1) =^{(\alpha,\beta)} C(x,y). \tag{6}$$

Proof. It is obvious that $C(x,y) \leq^{(\alpha,\beta)} C(x \leftrightarrow y, y \leftrightarrow y) = C(x \leftrightarrow y, 1)$.

On the other hand, using Lemma 2(3) and (11),

$$\begin{aligned} C(x \longleftrightarrow y, 1) &= C(x \longleftrightarrow y, 1) \wedge C(x,x) \\ &\leq^{(\alpha,\beta)} C((x \longleftrightarrow y) \otimes x, 1 \otimes x) \\ &= C((x \longleftrightarrow y) \otimes x, x) \wedge C(y,y) \\ &\leq^{(\alpha,\beta)} C(((x \longleftrightarrow y) \otimes x) \vee y, x \vee y) \\ &= C(y, x \vee y), \end{aligned} \tag{7}$$

and hence $C(x \leftrightarrow y, 1) \leq^{(\alpha,\beta)} C(y, x \vee y)$. Similarly, we have $C(x \leftrightarrow y, 1) \leq^{(\alpha,\beta)} C(x, x \vee y)$. Thus $C(x \leftrightarrow y, 1) \leq^{(\alpha,\beta)} C(y, x \vee y) \wedge C(x, x \vee y) \leq^{(\alpha,\beta)} C(x,y)$. Then the identity holds. □

Proposition 18. *Let C be a generalized fuzzy congruence of L. Then $C_{C^1} =^{(\alpha,\beta)} C$.*

Proof. Using Lemma 17, it holds that

$$\begin{aligned} C_{C^1}(x,y) &=^{(\alpha,\beta)} C^1(x \longleftrightarrow y) \\ &= C(1, x \longleftrightarrow y) =^{(\alpha,\beta)} C(x,y). \end{aligned} \tag{8}$$

Thus $C_{C^1} =^{(\alpha,\beta)} C$. □

Proposition 19. *Let f be a generalized fuzzy filter of L. Then $C^1_f =^{(\alpha,\beta)} f$.*

Proof. It is trivial. □

Theorem 20. *There is a one-to-one correspondence between the set of all generalized fuzzy filters and the set of all generalized fuzzy congruences.*

Proof. It follows immediately from Propositions 18 and 19. □

Lemma 21. *Let f be a generalized fuzzy filter of L. Then $C^x_f =^{(\alpha,\beta)} C^y_f$ if and only if $f(x \leftrightarrow y) =^{(\alpha,\beta)} f(1)$ for all $x, y \in L$.*

Proof. Assume that $C^x_f =^{(\alpha,\beta)} C^y_f$ for all $z \in L$. Thus $f(x \leftrightarrow z) =^{(\alpha,\beta)} f(y \leftrightarrow z)$. In particular, taking $y = z$, then $f(x \leftrightarrow y) =^{(\alpha,\beta)} f(1)$.

Conversely, it follows from Lemma 2(11) that

$$\begin{aligned} C^x_f(z) &= C^x_f(z) \wedge f(1) \\ &=^{(\alpha,\beta)} f(x \longleftrightarrow z) \wedge f(x \longleftrightarrow y) \\ &\leq^{(\alpha,\beta)} f((x \longleftrightarrow z) \otimes (x \longleftrightarrow y)) \\ &\leq^{(\alpha,\beta)} f(z \longleftrightarrow y) = C^y_f(z), \end{aligned} \tag{9}$$

and hence $C^x_f(z) \leq^{(\alpha,\beta)} C^y_f(z)$. Similarly, we have $C^y_f(z) \leq^{(\alpha,\beta)} C^x_f(z)$. Thus $C^x_f(z) =^{(\alpha,\beta)} C^y_f(z)$; that is, $C^x_f =^{(\alpha,\beta)} C^y_f$. □

Let f be a generalized fuzzy filter of L. For any $C^x_f, C^y_f \in L/C_f$, we define

$$\begin{aligned} C^x_f \vee C^y_f &= C^{x \vee y}_f, \qquad C^x_f \wedge C^y_f = C^{x \wedge y}_f, \\ C^x_f \otimes C^y_f &= C^{x \otimes y}_f, \qquad C^x_f \longrightarrow C^y_f = C^{x \rightarrow y}_f. \end{aligned} \tag{10}$$

Theorem 22. *Let f be a generalized fuzzy filter of L. Then*

$$\frac{L}{C_f} = \left\{ \frac{L}{C_f}, \wedge, \vee, \otimes, \longrightarrow, C^0_f, C^1_f \right\} \tag{11}$$

is a residuated lattice.

Proof. Routine proofs show that L/C_f is a residuated lattice. □

The above residuated lattice L/C_f is called a quotient residuated lattice induced by generalized fuzzy filter f.

4. Some Types of Generalized Fuzzy n-Fold Filters

In this section, we introduce some types of generalized fuzzy n-fold filters and investigate the relationships among them.

4.1. Generalized Fuzzy n-Fold Positive Implicative Filters

Definition 23. Let f be a generalized fuzzy filter of L. Then f is called a generalized fuzzy n-fold positive implicative filter if for all $x, y, z \in L$

$$f(x^n \longrightarrow (y \longrightarrow z)) \wedge f(x^n \longrightarrow y) \leq^{(\alpha,\beta)} f(x^n \longrightarrow z). \tag{12}$$

Example 24. Let $L = \{0, a, b, 1\}$. The operations $\otimes$ and $\rightarrow$ are defined as

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\longrightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

(13)

Let $\alpha = 0.3$, $\beta = 0.8$. Define $f : L \rightarrow [0, 1]$ as $f(0) = 0.4$, $f(a) = 0.5$, $f(b) = 0.6$, and $f(1) = 0.7$. It is routine to verify that f is a generalized fuzzy 2-fold positive implicative filter.

Theorem 25. *Let f be a generalized fuzzy filter of L. Then the following assertions are equivalent:*

(1) *f is a generalized fuzzy n-fold positive implicative filter;*

(2) $f(x^n \rightarrow x^{2n}) =^{(\alpha,\beta)} f(1)$;

(3) $f(x^n \rightarrow x^{n+1}) =^{(\alpha,\beta)} f(1)$.

Proof. (1)⇒(2) Assume that f is a generalized fuzzy n-fold positive implicative filter. Taking $z = x^{2n}$ and $y = x^n$, it follows from Lemma 2(2) that $f(1) \leq^{(\alpha,\beta)} f(x^n \rightarrow x^{2n})$. Theorem 12(1) leads to the fact that $f(x^n \rightarrow x^{2n}) \leq^{(\alpha,\beta)} f(1)$. Thus $f(x^n \rightarrow x^{2n}) =^{(\alpha,\beta)} f(1)$.

(2)⇒(3) It is obvious that $x^{2n} \leq x^{n+1}$, and hence $x^n \rightarrow x^{2n} \leq x^n \rightarrow x^{n+1}$. Thus $f(1) =^{(\alpha,\beta)} f(x^n \rightarrow x^{2n}) \leq^{(\alpha,\beta)} f(x^n \rightarrow x^{n+1})$. Obviously, $f(x^n \rightarrow x^{n+1}) \leq^{(\alpha,\beta)} f(1)$. Thus $f(x^n \rightarrow x^{n+1}) =^{(\alpha,\beta)} f(1)$.

(3)⇒(1) It follows from Lemma 2(8) that $x^n \rightarrow x^{n+1} \leq x^{n+1} \rightarrow x^{n+2}$. By the same way, we get $x^{n+i} \rightarrow x^{n+i+1} \leq x^{n+i+1} \rightarrow x^{n+i+2}$ for all $i = 0, \ldots, n-2$. Using Lemma 2(6), it holds that $f(x^n \rightarrow x^{n+1}) \wedge f(x^{n+1} \rightarrow x^{n+2}) \wedge \cdots \wedge f(x^{2n-1} \rightarrow x^{2n}) \leq^{(\alpha,\beta)} f((x^n \rightarrow x^{n+1}) \otimes \cdots \otimes (x^{2n-1} \rightarrow x^{2n})) \leq^{(\alpha,\beta)} f(x^n \rightarrow x^{2n})$. Thus $f(1) \leq^{(\alpha,\beta)} f(x^n \rightarrow x^{2n})$. It follows from Lemma 2(5) that $f(x^n \rightarrow x^{2n}) \leq^{(\alpha,\beta)} f((x^{2n} \rightarrow z) \rightarrow (x^n \rightarrow z))$, and hence $f(1) \leq^{(\alpha,\beta)} f((x^{2n} \rightarrow z) \rightarrow (x^n \rightarrow z))$. Thus $f(1) =^{(\alpha,\beta)} f((x^{2n} \rightarrow z) \rightarrow (x^n \rightarrow z))$. It follows from Proposition 13 that $f(x^{2n} \rightarrow z) \leq^{(\alpha,\beta)} f(x^n \rightarrow z)$. Thus $f(x^n \rightarrow (y \rightarrow z)) \wedge f(x^n \rightarrow y) \leq^{(\alpha,\beta)} f(x^{2n} \rightarrow z) \leq^{(\alpha,\beta)} f(x^n \rightarrow z)$. Thus f is a generalized fuzzy n-fold positive implicative filter. □

Associated with the alternative definitions of generalized fuzzy n-fold positive implicative filters, the following properties can be verified.

Example 26. Let $L = \{0, a, b, 1\}$. The operations $\otimes$ and $\rightarrow$ are defined as

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

$\longrightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

(14)

Let $\alpha = 0.3$, $\beta = 0.8$. Define $f : L \rightarrow [0, 1]$ as $f(0) = 0.4$, $f(a) = 0.5$, $f(b) = 0.6$, and $f(1) = 0.7$. Then f is not a generalized fuzzy 2-fold positive implicative filter, because $f(b^2 \rightarrow b^3) = f(b) \neq^{(\alpha,\beta)} f(1)$.

Proposition 27. *Each generalized fuzzy n-fold positive implicative filter is a generalized fuzzy $n+1$-fold positive implicative filer.*

Proof. Assume that f is a generalized fuzzy n-fold positive implicative filter. By Theorem 25(3), it yields that $f(x^n \rightarrow x^{n+1}) =^{(\alpha,\beta)} f(1)$. Using Lemma 2(8), we have $x^n \rightarrow x^{n+1} \leq x^{n+1} \rightarrow x^{n+2}$, and hence $f(x^n \rightarrow x^{n+1}) \leq^{(\alpha,\beta)} f(x^{n+1} \rightarrow x^{n+2})$. Thus $f(1) \leq^{(\alpha,\beta)} f(x^{n+1} \rightarrow x^{n+2})$. It is obvious that $f(x^{n+1} \rightarrow x^{n+2}) \leq^{(\alpha,\beta)} f(1)$. Thus $f(x^{n+1} \rightarrow x^{n+2}) =^{(\alpha,\beta)} f(1)$. That is, f is a generalized fuzzy $n+1$-fold positive implicative filter. □

It is easy to prove by induction that every fuzzy generalized fuzzy n-fold positive implicative filter is a generalized fuzzy $n+k$-fold positive implicative filter for all integer $k \geq 1$.

The converse of the above proposition does not always hold.

Example 28. In Example 24, f is a generalized fuzzy 2-fold positive implicative filter but not a generalized fuzzy 1-fold positive filter, because $f(a \rightarrow a^2) = f(a) \neq^{(\alpha,\beta)} f(1)$.

Proposition 29 (Extension theorem for generalized fuzzy n-fold positive implicative filters). *Let f and g be two generalized fuzzy filters of L such that $f \leq^{(\alpha,\beta)} g$ and $f(1) =^{(\alpha,\beta)} g(1)$. If f is a generalized fuzzy n-fold positive implicative filter, then so is g.*

Proof. Using Theorem 25, we have $f(1) =^{(\alpha,\beta)} f(x^n \rightarrow x^{n+1})$. By assumption, it holds that $g(1) \leq^{(\alpha,\beta)} g(x^n \rightarrow x^{n+1})$. It is obvious that $g(x^n \rightarrow x^{n+1}) \leq^{(\alpha,\beta)} g(1)$. Thus $g(x^n \rightarrow x^{n+1}) =^{(\alpha,\beta)} g(1)$. Applying Theorem 25 again, g is a generalized fuzzy n-fold positive implicative filter. □

4.2. Generalized Fuzzy n-Fold Fantastic Filters

Definition 30. Let f be a generalized fuzzy filter of a residuated lattice L. Then f is called a generalized fuzzy n-fold fantastic filter if for all $x, y, z \in L$

$$f(z \longrightarrow (y \longrightarrow x)) \wedge f(z) \leq^{(\alpha,\beta)} f([(x^n \longrightarrow y) \longrightarrow y] \longrightarrow x). \tag{15}$$

Example 31. In Example 26, it is routine to verify that f is a generalized fuzzy 2-fold fantastic filter.

The following lemma will be useful.

Lemma 32. *Let L be a residuated lattice. Then for all $x, y \in L$*

$$(x \vee y)^n \longrightarrow y = x^n \longrightarrow y. \tag{16}$$

Proof. Applying Lemma 2(2) and (10) for $n-1$ times, we get

$$\begin{aligned}(x\vee y)^n \longrightarrow y &= (x\vee y)^{n-1} \longrightarrow [(x\vee y)\longrightarrow y]\\ &= (x\vee y)^{n-1}\longrightarrow (x\longrightarrow y)\\ &= x\longrightarrow \left[(x\vee y)^{n-1}\longrightarrow y\right],\\ &\vdots\\ &= x^n \longrightarrow y\end{aligned} \tag{17}$$

and hence $(x\vee y)^n \to y = x^n \to y$. □

Theorem 33. *Let f be a generalized fuzzy filter of a residuated lattice L. Then the following assertions are equivalent:*

(1) *f is a generalized fuzzy n-fold fantastic filter;*

(2) *$f(y\to x) =^{(\alpha,\beta)} f(((x^n\to y)\to y)\to x)$ for all $x, y\in L$;*

(3) *$f(((x^n\to y)\to y)\to (x\vee y)) =^{(\alpha,\beta)} f(1)$ for all $x, y\in L$.*

Proof. (1)⇒(2) Taking $z = 1$, it holds that $f(y\to x)\le^{(\alpha,\beta)} f([(x^n\to y)\to y]\to x)$. Since $y\le (x^n\to y)\to y$, we get $f([(x^n\to y)\to y]\to x)\le^{(\alpha,\beta)} f(y\to x)$. Thus $f([(x^n\to y)\to y]\to x) =^{(\alpha,\beta)} f(y\to x)$.

(2)⇒(3) Replacing x with $x\vee y$, it follows from Lemma 32 that $f(1) =^{(\alpha,\beta)} f([(x^n\to y)\to y]\to (x\vee y))$.

(3)⇒(1) Using Lemma 2(4) and (10), we have $[(x^n\to y)\to y]\to (x\vee y)\le (y\to x)\to \{[(x^n\to y)\to y]\to x\}$, and hence $f(1)\le^{(\alpha,\beta)} f((y\to x)\to \{[(x^n\to y)\to y]\to x\})$. It is obvious that $f((y\to x)\to \{[(x^n\to y)\to y]\to x\})\le^{(\alpha,\beta)} f(1)$. Thus $f((y\to x)\to \{[(x^n\to y)\to y]\to x\}) =^{(\alpha,\beta)} f(1)$. By Proposition 13(1), it yields that $f(y\to x)\le^{(\alpha,\beta)} f([(x^n\to y)\to y]\to x)$. Thus $f(z\to (y\to x))\wedge f(z)\le^{(\alpha,\beta)} f(z\otimes [z\to (y\to x)])\le^{(\alpha,\beta)} f(y\to x)\le^{(\alpha,\beta)} f([(x^n\to y)\to y]\to x)$. □

Associated with the alternative definitions of generalized fuzzy n-fold fantastic filters, the following properties can be verified.

Example 34. In Example 24, f is not a generalized fuzzy 2-fold fantastic filter, because $f([(b^2\to a)\to a]\to (a\vee b)) = f(b)\ne^{(\alpha,\beta)} f(1)$.

Proposition 35. *Each generalized fuzzy n-fold fantastic filter is an $n+1$-fold generalized fuzzy fantastic filer.*

Proof. Assume that f is a generalized fuzzy n-fold fantastic filter. Using Theorem 33(3), we get $f(((x^n\to y)\to y)\to (x\vee y)) =^{(\alpha,\beta)} f(1)$. Since $x^{n+1}\le x^n$, it follows from Lemma 2(4) that $((x^n\to y)\to y)\to (x\vee y)\le ((x^{n+1}\to y)\to y)\to (x\vee y)$, and hence $f(((x^n\to y)\to y)\to (x\vee y))\le^{(\alpha,\beta)} f(((x^{n+1}\to y)\to y)\to (x\vee y))$. Thus $f(1)\le^{(\alpha,\beta)} f(((x^{n+1}\to y)\to y)\to (x\vee y))$. It is obvious that $f(((x^{n+1}\to y)\to y)\to (x\vee y))\le^{(\alpha,\beta)} f(1)$. Thus $f(((x^{n+1}\to y)\to y)\to (x\vee y)) =^{(\alpha,\beta)} f(1)$. That is, f is an $n+1$-fold generalized fuzzy fantastic filter. □

It is easy to prove by induction that every generalized fuzzy n-fold fantastic filter is a generalized fuzzy $n+k$-fold fantastic filter for all integer $k\ge 1$.

The converse of the above proposition does not always hold.

Example 36. Let $L = \{0, a, b, 1\}$. The operations $\otimes$ and $\to$ are defined as

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	a	b	1

$\longrightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	b	b	1	1
1	0	a	b	1

(18)

Let $\alpha = 0.3$, $\beta = 0.8$. Define $f : L\to [0,1]$ as $f(0) = 0.4$, $f(a) = 0.5$, $f(b) = 0.6$, and $f(1) = 0.7$. It is routine to verify that f is a generalized fuzzy 2-fold fantastic filter, but not a generalized fuzzy 1-fold fantastic filter, because $f([(a\to 0)\to 0]\to (a\vee 0)) = f(b)\ne^{(\alpha,\beta)} f(1)$.

Proposition 37 (Extension theorem for generalized fuzzy n-fold fantastic filters). *Let f and g be two generalized fuzzy filters of L such that $f\le^{(\alpha,\beta)} g$ and $f(1) =^{(\alpha,\beta)} g(1)$. If f is a generalized fuzzy n-fold fantastic filter, then so is g.*

Proof. Using Theorem 33, the proof is similar to that of Proposition 29. □

4.3. Generalized Fuzzy n-Fold Boolean Filters

Definition 38. Let f be a generalized fuzzy filter of L. Then f is called a generalized fuzzy n-fold Boolean filter if for all $x, y, z\in L$

$$f(x\longrightarrow (\neg z^n\longrightarrow y))\wedge f(y\longrightarrow z)\le^{(\alpha,\beta)} f(x\longrightarrow z). \tag{19}$$

Example 39. Let $L = \{0, a, b, c, d, 1\}$ with $0 < a, b < c < 1$, and $0 < b < d < 1$ and let a and b, c and d be uncomparable. The operations $\otimes$ and $\to$ are defined as

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\longrightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

(20)

Let $\alpha = 0.2$, $\beta = 0.9$. Define $f : L\to [0,1]$ as $f(0) = 0.3$, $f(a) = 0.5$, $f(b) = 0.4$, $f(c) = 0.7$, $f(d) = 0.6$, and $f(1) = 0.8$. It is routine to verify that f is a generalized fuzzy 2-fold Boolean filter.

The following lemma will be useful.

Lemma 40. *Let L be a residuated lattice. Then for all $x, y \in L$*

$$\neg(x \vee \neg x^n)^n \longrightarrow (x \vee \neg x^n) = 1. \tag{21}$$

Proof. By Lemma 2(4), (5), and (8), it holds that

$$\begin{aligned}\neg(x \vee \neg x^n)^n \longrightarrow (x \vee \neg x^n) &\geq \neg(x \vee \neg x^n)^n \longrightarrow \neg x^n \\ &\geq x^n \longrightarrow (x \vee \neg x^n)^n = 1,\end{aligned} \tag{22}$$

and hence $\neg(x \vee \neg x^n)^n \rightarrow (x \vee \neg x^n) = 1$. □

Theorem 41. *Let f be a generalized fuzzy filter of L. Then the following assertions are equivalent:*

(1) *f is a generalized fuzzy n-fold Boolean filter;*

(2) *$f(\neg z^n \rightarrow z) =^{(\alpha,\beta)} f(z)$ for all $z \in L$;*

(3) *$f(\neg z^n \vee z) =^{(\alpha,\beta)} f(1)$ for all $z \in L$.*

Proof. (1)⇒(2) Assume that f is a generalized fuzzy n-fold Boolean filter. Taking $x = 1$ and $y = z$, we have $f(\neg z^n \rightarrow z) \leq^{(\alpha,\beta)} f(z)$. Since $z \leq \neg z^n \rightarrow z$, it holds that $f(z) \leq^{(\alpha,\beta)} f(\neg z^n \rightarrow z)$. Thus $f(\neg z^n \rightarrow z) =^{(\alpha,\beta)} f(z)$.

(2)⇒(3) Replacing z with $\neg z^n \vee z$, it follows from Lemma 40 that $f(\neg z^n \vee z) =^{(\alpha,\beta)} f(1)$.

(3)⇒(1) Using Lemma 2(5), (3), (6), (4), and (2), we get

$$\begin{aligned} f(z \vee \neg z^n) &\leq^{(\alpha,\beta)} f((\neg z^n \longrightarrow z) \longrightarrow z) \\ &\leq^{(\alpha,\beta)} f([(\neg z^n \longrightarrow y) \otimes (y \longrightarrow z)] \longrightarrow z) \\ &= f((\neg z^n \longrightarrow y) \longrightarrow [(y \longrightarrow z) \longrightarrow z]) \\ &\leq^{(\alpha,\beta)} f([x \longrightarrow (\neg z^n \longrightarrow y)] \\ &\qquad \longrightarrow \{x \longrightarrow [(y \longrightarrow z) \longrightarrow z]\}) \\ &= f([x \longrightarrow (\neg z^n \longrightarrow y)] \\ &\qquad \longrightarrow [(y \longrightarrow z) \longrightarrow (x \longrightarrow z)]), \end{aligned} \tag{23}$$

and hence $f(1) \leq^{(\alpha,\beta)} f([x \rightarrow (\neg z^n \rightarrow y)] \rightarrow [(y \rightarrow z) \rightarrow (x \rightarrow z)])$. It is obvious that $f([x \rightarrow (\neg z^n \rightarrow y)] \rightarrow [(y \rightarrow z) \rightarrow (x \rightarrow z)]) \leq^{(\alpha,\beta)} f(1)$. Thus $f([x \rightarrow (\neg z^n \rightarrow y)] \rightarrow [(y \rightarrow z) \rightarrow (x \rightarrow z)]) =^{(\alpha,\beta)} f(1)$. It follows from Proposition 13 that $f(x \rightarrow (\neg z^n \rightarrow y)) \leq^{(\alpha,\beta)} f((y \rightarrow z) \rightarrow (x \rightarrow z))$. Thus $f(x \rightarrow (\neg z^n \rightarrow y)) \wedge f(y \rightarrow z) \leq^{(\alpha,\beta)} f((y \rightarrow z) \rightarrow (x \rightarrow z)) \wedge f(y \rightarrow z) \leq^{(\alpha,\beta)} f(x \rightarrow z)$. □

Associated with the alternative definitions of generalized fuzzy n-fold Boolean filters, the following properties can be verified.

Proposition 42. *Each generalized fuzzy n-fold Boolean filter is an $n+1$-fold generalized fuzzy Boolean filer.*

Proof. Assume that f is a generalized fuzzy n-fold Boolean filter. It follows from Theorem 41(3) that $f(z \vee \neg z^n) =^{(\alpha,\beta)} f(1)$. Since $z \vee \neg z^n \leq z \vee \neg z^{n+1}$, it yields that $f(z \vee \neg z^n) \leq^{(\alpha,\beta)} f(z \vee \neg z^{n+1})$, and hence $f(1) \leq^{(\alpha,\beta)} f(z \vee \neg z^{n+1})$. It is obvious that $f(z \vee \neg z^{n+1}) \leq^{(\alpha,\beta)} f(1)$. Thus $f(z \vee \neg z^{n+1}) =^{(\alpha,\beta)} f(1)$. That is, f is an $n+1$-fold generalized fuzzy Boolean filter. □

It is easy to prove by induction that every generalized fuzzy n-fold Boolean filter is a generalized fuzzy $n+k$-fold Boolean filter for all integer $k \geq 1$.

The converse of the above proposition does not always hold.

Example 43. In Example 39, f is a generalized fuzzy 2-fold Boolean filter, but not a generalized fuzzy 1-fold Boolean filter, because $f(b \vee \neg b) = f(c) \neq^{(\alpha,\beta)} f(1)$.

In order to investigate the relationships among these three types defined above, the following lemma is useful.

Lemma 44. *Let L be a residuated lattice. Then for all $x \in L$*

$$x \vee \neg x^n \leq x^n \longrightarrow x^{n+1}. \tag{24}$$

Proof. Using Lemma 2(1), (2), and (10), we have

$$\begin{aligned} &(x \vee \neg x^n) \longrightarrow (x^n \longrightarrow x^{n+1}) \\ &\quad = [x \longrightarrow (x^n \longrightarrow x^{n+1})] \wedge [\neg x^n \longrightarrow (x^n \longrightarrow x^{n+1})] \\ &\quad = 1 \wedge [(\neg x^n \otimes x^n) \longrightarrow x^{n+1}] \geq 1, \end{aligned} \tag{25}$$

and hence $x \vee \neg x^n \leq x^n \rightarrow x^{n+1}$. □

Theorem 45. *Let f be a generalized fuzzy filter of L. Then f is a generalized fuzzy n-fold Boolean filter if and only if f is both a generalized fuzzy n-fold fantastic filter and a generalized fuzzy n-fold positive implicative filter.*

Proof. Assume that f is a generalized fuzzy n-fold Boolean filter. It follows from Lemma 44 that $f(1) \leq^{(\alpha,\beta)} f(x \vee \neg x^n) \leq^{(\alpha,\beta)} f(x^n \rightarrow x^{n+1})$. It is obvious that $f((x^n \rightarrow x^{n+1}) \leq^{(\alpha,\beta)} f(1)$. Thus $f(x^n \rightarrow x^{n+1}) =^{(\alpha,\beta)} f(1)$. That is, f is a generalized fuzzy n-fold positive implicative filter.

Using Lemma 2(2), (3), (4) and (6), we have

$$\begin{aligned} f(x \vee \neg x^n) &\leq^{(\alpha,\beta)} f((\neg x^n \longrightarrow x) \longrightarrow x) \\ &\leq^{(\alpha,\beta)} f((\neg x^n \longrightarrow y) \longrightarrow [(y \longrightarrow x) \longrightarrow x]), \end{aligned} \tag{26}$$

and hence $f(1) \leq^{(\alpha,\beta)} f((\neg x^n \rightarrow y) \rightarrow [(y \rightarrow x) \rightarrow x])$. Replacing x with $x \vee y$, we get $f(1) \leq^{(\alpha,\beta)} f([\neg(x \vee y)^n \rightarrow y] \rightarrow (x \vee y))$. By Lemma 2(3), (4), and Lemma 32, it holds that

$$\begin{aligned} &f([\neg(x \vee y)^n \longrightarrow y] \longrightarrow (x \vee y)) \\ &\quad \leq^{(\alpha,\beta)} f(\{[(x \vee y)^n \longrightarrow y] \longrightarrow y\} \longrightarrow (x \vee y)) \\ &\quad = f([(x^n \longrightarrow y) \longrightarrow y] \longrightarrow (x \vee y)), \end{aligned} \tag{27}$$

and hence $f(1) \leq^{(\alpha,\beta)} f([(x^n \to y) \to y] \to (x \vee y))$. It is obvious that $f([(x^n \to y) \to y] \to (x \vee y)) \leq^{(\alpha,\beta)} f(1)$. Thus $f([(x^n \to y) \to y] \to (x \vee y)) =^{(\alpha,\beta)} f(1)$. That is, f is a generalized fuzzy n-fold fantastic filter.

Conversely, it follows from Theorem 25(2) that $f(x^n \to x^{2n}) =^{(\alpha,\beta)} f(1)$. Since $x^n \to x^{2n} \leq \neg x^{2n} \to \neg x^n = (x^n \to \neg x^n) \to \neg x^n$, it holds that $f(1) \leq^{(\alpha,\beta)} f((x^n \to \neg x^n) \to \neg x^n)$. Using Theorem 33, we get $f(((x^n \to y) \to y) \to (x \vee y)) =^{(\alpha,\beta)} f(1)$. In particular, taking $y = \neg x^n$, it yields that $f(((x^n \to \neg x^n) \to \neg x^n) \to (x \vee \neg x^n)) =^{(\alpha,\beta)} f(1)$. It follows from Proposition 13 that $f((x^n \to \neg x^n) \to \neg x^n) \leq^{(\alpha,\beta)} f(x \vee \neg x^n)$. Thus $f(1) \leq^{(\alpha,\beta)} f(x \vee \neg x^n)$. It is obvious that $f(x \vee \neg x^n) \leq^{(\alpha,\beta)} f(1)$. Thus f is a generalized fuzzy n-fold Boolean filter. □

Obviously, the generalized fuzzy filters in Examples 24 and 26 are not generalized fuzzy 2-fold Boolean filters, because they are not generalized fuzzy 2-fold positive implicative filter and generalized fuzzy 2-fold fantastic filter, respectively.

Proposition 46 (Extension theorem for generalized fuzzy n-fold Boolean filters). *Let f and g be two generalized fuzzy filters of L such that $f \leq^{(\alpha,\beta)} g$ and $f(1) =^{(\alpha,\beta)} g(1)$. If f is a generalized fuzzy n-fold Boolean filter, then so is g.*

Proof. Using Theorem 41, the proof is similar to that of Proposition 29. □

5. Conclusions

Generalized fuzzy filters have been extensively studied in the literature. In this paper, we researched the properties of generalized fuzzy filters and introduced some types of generalized fuzzy n-fold filters in residuated lattices. We established a correspondence theorem between the set of generalized fuzzy filters and the set of generalized fuzzy congruences and induced a quotient residuated lattice with respect to generalized fuzzy filters. We also defined generalized fuzzy n-fold positive implicative, fantastic, and Boolean filters and investigated the correlations among these filters.

In our future work, we will introduce some other types of generalized fuzzy n-fold filters in residuated lattices.

Acknowledgment

This research was supported by AMEP of Linyi University, the Natural Science Foundation of Shandong Province (Grant no. ZR2013FL006).

References

[1] M. Ward and P. R. Dilworth, "Residuated lattice," *Transactions of the American Mathematical Society*, vol. 45, pp. 335–354, 1939.

[2] E. Turunen, "Boolean deductive systems of *BL*-algebras," *Archive for Mathematical Logic*, vol. 40, no. 6, pp. 467–473, 2001.

[3] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1998.

[4] M. Haveshki, A. B. Saeid, and E. Eslami, "Some types of filters in *BL* algebras," *Soft Computing*, vol. 10, no. 8, pp. 657–664, 2006.

[5] M. Kondo and W. A. Dudek, "Filter theory of *BL* algebras," *Soft Computing*, vol. 12, no. 5, pp. 419–423, 2008.

[6] Y. Zhu and Y. Xu, "On filter theory of residuated lattices," *Information Sciences*, vol. 180, no. 19, pp. 3614–3632, 2010.

[7] R. A. Borzooei, S. Khosravi Shoar, and R. Ameri, "Some types of filters in *MTL*-algebras," *Fuzzy Sets and Systems*, vol. 187, no. 1, pp. 92–102, 2012.

[8] M. Haveshki, "A note on some types of filters in *MTL*-algebras," *Fuzzy Sets and Systems*, 2013.

[9] M. Haveshki and E. Eslami, "n-fold filters in *BL*-algebras," *Mathematical Logic Quarterly*, vol. 54, no. 2, pp. 176–186, 2008.

[10] E. Turunen, N. Tchikapa, and C. Lele, "n-Fold implicative basic logic is Gödel logic," *Soft Computing*, vol. 16, no. 1, pp. 177–181, 2012.

[11] E. Turunen, N. Tchikapa, and C. Lele, "Erratum to: n-Fold implicative basic logic is Gödel logic, Soft Comput," *Soft Computing*, vol. 16, no. 1, p. 183, 2012.

[12] O. Zahiri and H. Farahani, "n-Fold filters of *MTL*-algebras," *Afrika Matematika*, 2013.

[13] Y. B. Jun, Y. Xu, and X. H. Zhang, "Fuzzy filters of *MTL*-algebras," *Information Sciences*, vol. 175, no. 1-2, pp. 120–138, 2005.

[14] L. Liu and K. Li, "Fuzzy filters of *BL*-algebras," *Information Sciences*, vol. 173, no. 1–3, pp. 141–154, 2005.

[15] L. Lianzhen and L. Kaitai, "Fuzzy Boolean and positive implicative filters of *BL*-algebras," *Fuzzy Sets and Systems*, vol. 152, no. 2, pp. 333–348, 2005.

[16] X. Ma, J. Zhan, and W. A. Dudek, "Some kinds of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filters of B L-algebras," *Computers and Mathematics with Applications*, vol. 58, no. 2, pp. 248–256, 2009.

[17] Y. B. Jun, Y. U. Cho, E. H. Roh, and J. Zhan, "General types of $(\in, \in \vee q)$-fuzzy filters in *BL*-algebras," *Neural Computing and Applications*, vol. 20, no. 3, pp. 335–343, 2011.

[18] X. Ma and J. Zhan, "On $(\in, \in \vee q)$-fuzzy filters of *BL*-algebras," *Journal of Systems Science and Complexity*, vol. 21, no. 1, pp. 144–158, 2008.

[19] J. Zhan and Y. Xu, "Some types of generalized fuzzy filters of *BL*-algebras," *Computers and Mathematics with Applications*, vol. 56, no. 6, pp. 1604–1616, 2008.

[20] T. Mahmood, "Hemirings Characterized by the properties of their $(\overline{\in}, \overline{\in} \vee \overline{q}\kappa)$-fuzzy ideals," *Iranian Journal of Science and Technology*, vol. 37, pp. 265–275, 2013.

[21] M. Shabir and T. Mahmood, "Characterizations of hemirings by $(\in, \in \vee q\kappa)$-fuzzy ideals," *Computers and Mathematics with Applications*, vol. 61, no. 4, pp. 1059–1078, 2011.

[22] M. Shabir and T. Mahmood, "Hemirings characterized by the properties of their fuzzy ideals with thresholds," *Quasigroups and Related Systems*, vol. 18, pp. 195–212, 2010.