Research Article

Multiplicity of Positive Solutions for Semilinear Elliptic Systems

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We study the effect of the coefficient $h(x)$ of the critical nonlinearity on the number of positive solutions for semilinear elliptic systems. Under suitable assumptions for $f(x)$, $g(x)$, and $h(x)$, we should prove that for sufficiently small $\lambda, \mu > 0$, there are at least $k + 1$ positive solutions of the semilinear elliptic systems $-\Delta u = \lambda f(x)|u|^{q-2}u + (\alpha/(\alpha + \beta))h(x)|u|^{\alpha-2}u|v|^\beta$, $-\Delta v = \mu g(x)|v|^{q-2}v + (\beta/(\alpha + \beta))h(x)|u|^\alpha|v|^{\beta-2}v$, where $0 \in \Omega \subset \mathbb{R}^N$ is a bounded domain, $\alpha > 1$, $\beta > 1$, and $N/(N - 2) < q < 2 < \alpha + \beta = 2^*$ for $N > 4$.

1. Introduction and Main Results

For $N \geq 3$, $\alpha > 1$, $\beta > 1$, and $1 \leq q < 2 < \alpha + \beta = 2^* = 2N/(N - 2)$, consider the semilinear elliptic systems

$$\begin{cases} -\Delta u = \lambda f(x)|u|^{q-2}u + \frac{\alpha}{\alpha + \beta}h(x)|u|^{\alpha-2}u|v|^\beta & \text{in } \Omega, \\ -\Delta v = \mu g(x)|v|^{q-2}v + \frac{\beta}{\alpha + \beta}h(x)|u|^\alpha|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (P_{\lambda,\mu})$$

where $\lambda, \mu > 0$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$.

Let f , g , and h satisfy the following conditions.

(H1) f , g , and h are positive continuous functions in $\bar{\Omega}$ and $\max_{x \in \bar{\Omega}} h(x) = 1$.

(H2) There exist k points $a_1, a_2, \dots, a_k \in \Omega$ and some $\sigma \geq N - 2$ such that $h(a_i)$ are strict maxima and satisfy

$$h(a_i) = \max_{x \in \Omega} h(x) = 1 \quad \forall 1 \leq i \leq k \quad (1)$$

and $h(x) = h(a_i) + O(|x - a_i|^\sigma)$ as $x \rightarrow a_i$ uniformly in i .

Recent studies [1–10] have investigated the elliptic systems with subcritical or critical exponents and have proved the existence of a ground state solution or the existence of at least two positive solutions for these problems. For the case of

$N > 4$, $\alpha > 1$, $\beta > 1$, and $2 < q < \alpha + \beta = 2^* = 2N/(N - 2)$, Lin [11] constructs the k compact Palais-Smale sequences that are suitably localized in correspondence of k maximum points of h . Under assumptions (H1)-(H2), she has showed that there are at least k positive solutions of the problem $(P_{\lambda,\mu})$ for sufficiently small $\lambda, \mu > 0$. In this paper, we study the problem $(P_{\lambda,\mu})$ and complement the results of [11] to the case $1 \leq q < 2$. Under assumptions (H1)-(H2), we should prove that there exist at least $k + 1$ positive solutions of the problem $(P_{\lambda,\mu})$ for sufficiently small $\lambda, \mu > 0$.

Let $E = H_0^1(\Omega) \times H_0^1(\Omega)$ be the space with the standard norm

$$\|(u, v)\|_E = \left(\int_{\Omega} (|\nabla u|^2 + |\nabla v|^2) dx \right)^{1/2}. \quad (2)$$

Associated with the problem $(P_{\lambda,\mu})$, we consider the C^1 -functional $I_{\lambda,\mu}$, for $(u, v) \in E$,

$$\begin{aligned} I_{\lambda,\mu}(u, v) &= \frac{1}{2} \|(u, v)\|_E^2 \\ &\quad - \frac{1}{q} \int_{\Omega} (\lambda f(x)|u|^q + \mu g(x)|v|^q) dx \\ &\quad - \frac{1}{2^*} \int_{\Omega} h(x)|u|^\alpha|v|^\beta dx. \end{aligned} \quad (3)$$

The weak solution $(u, v) \in E$ of the problem $(P_{\lambda, \mu})$ is the critical point of the functional $I_{\lambda, \mu}$; that is, $(u, v) \in E$ satisfies

$$\begin{aligned} & \int_{\Omega} (\nabla u \nabla \varphi_1 + \nabla v \nabla \varphi_2) dx - \lambda \int_{\Omega} f(x) |u|^{q-2} u \varphi_1 dx \\ & - \mu \int_{\Omega} g(x) |v|^{q-2} v \varphi_2 dx - \frac{\alpha}{2^*} \int_{\Omega} h(x) |u|^{\alpha-2} u |v|^{\beta} \varphi_1 dx \\ & - \frac{\beta}{2^*} \int_{\Omega} h(x) |u|^{\alpha} |v|^{\beta-2} v \varphi_2 dx = 0 \end{aligned} \quad (4)$$

for any $(\varphi_1, \varphi_2) \in E$.

Let $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) \mid \nabla u \in (L^2(\mathbb{R}^N))^N\}$ with the norm $\|u\|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx$, and let S be the best Sobolev constant defined by

$$\begin{aligned} S &= \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{2/2^*}} \\ &= \left(\inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx}{\left(\int_{\Omega} |u|^{2^*} dx\right)^{2/2^*}} \right) > 0, \end{aligned} \quad (5)$$

and let

$$S_{\alpha, \beta} = \inf_{u, v \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} (|\nabla u|^2 + |\nabla v|^2) dx}{\left(\int_{\Omega} |u|^{\alpha} |v|^{\beta} dx\right)^{2/(\alpha+\beta)}}; \quad (6)$$

then, by [1, Theorem 5], we have

$$S_{\alpha, \beta} = \left(\left(\frac{\alpha}{\beta} \right)^{\beta/(\alpha+\beta)} + \left(\frac{\beta}{\alpha} \right)^{\alpha/(\alpha+\beta)} \right) S, \quad (7)$$

where $\alpha + \beta = 2^*$.

Set

$$\begin{aligned} \Lambda_1 &= \left(\frac{2-q}{2^*-q} \right)^{2/(2^*-2)} \left(\frac{(2^*-q) \gamma_{\infty}}{2^*-2} |\Omega|^{(2^*-q)/2^*} \right)^{-2/(2-q)} \\ &\quad \times S^{N/2+q/(2-q)} > 0, \end{aligned} \quad (8)$$

where $\gamma_{\infty} = \max\{|f|_{L^{\infty}(\Omega)}, |g|_{L^{\infty}(\Omega)}\}$.

The main results of this paper are given as follows.

Theorem 1. Assume that (H1) holds. If $\lambda, \mu > 0$ satisfy $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$, then there exists at least one positive ground state solution of the problem $(P_{\lambda, \mu})$.

Theorem 2. Under the assumptions (H1)-(H2), and $N/(N-2) < q < 2$ and $N > 4$, there exists a positive number $\Lambda^* \in (0, \Lambda_1)$ such that for $\lambda, \mu > 0$ and $\lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda^*$, the problem $(P_{\lambda, \mu})$ has $k+1$ positive solutions.

This paper is organized as follows. In Section 2, we consider the Nehari manifold

$$\mathcal{N}_{\lambda, \mu} = \{(u, v) \in E \setminus \{0\} \mid \langle I'_{\lambda, \mu}(u, v), (u, v) \rangle = 0\}, \quad (9)$$

where

$$\begin{aligned} & \langle I'_{\lambda, \mu}(u, v), (u, v) \rangle \\ &= \|(u, v)\|_E^2 - \int_{\Omega} (\lambda f(x) |u|^q + \mu g(x) |v|^q) dx \\ & \quad - \int_{\Omega} h(x) |u|^{\alpha} |v|^{\beta} dx. \end{aligned} \quad (10)$$

Note that $\mathcal{N}_{\lambda, \mu}$ contains all nontrivial weak solution of the problem $(P_{\lambda, \mu})$. Using the argument of Tarantello [12, 13], we split $\mathcal{N}_{\lambda, \mu}$ into two parts $\mathcal{N}_{\lambda, \mu}^+$ and $\mathcal{N}_{\lambda, \mu}^-$ for $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$. In Section 3, we prove Theorem 1. In Section 4, since $I_{\lambda, \mu}$ satisfies the $(PS)_{\gamma}$ -condition for $\gamma \in (-\infty, (1/N)(S_{\alpha, \beta})^{N/2} - C_0(\lambda^{2/(2-q)} + \mu^{2/(2-q)}))$, for sufficiently small λ, μ , and some restriction on q and N , we construct the k compact Palais-Smale sequences which are suitably localized in correspondence with the k maximum points of h and which converge to distinct solutions of the problem $(P_{\lambda, \mu})$ belonging to $\mathcal{N}_{\lambda, \mu}^-$. Hence, we prove Theorem 2 (one is the ground state solution belonging to $\mathcal{N}_{\lambda, \mu}^+$ and the others are in $\mathcal{N}_{\lambda, \mu}^-$).

2. Nehari Manifold

Throughout this paper, (H1) will be assumed. First, we give some notations.

Notations. We make use of the following notations.

$L^p(\Omega)$, $1 \leq p \leq \infty$, denote Lebesgue spaces; the norm L^p is denoted by $|\cdot|_{L^p(\Omega)}$ for $1 \leq p \leq \infty$.

$E = [H_0^1(\Omega)]^2$, endowed with norm $\|z\|_E^2 = \|(u, v)\|_E^2 = |\nabla u|_2^2 + |\nabla v|_2^2$.

The dual space of a Banach space E will be denoted by E^{-1} .

$|z| = |(u, v)| = (|u|, |v|)$ and $tz = t(u, v) = (tu, tv)$ for all $z \in E$ and $t \in \mathbb{R}$.

$z = (u, v)$ is said to be nonnegative in Ω if $u \geq 0$ and $v \geq 0$ in Ω .

$z = (u, v)$ is said to be positive in Ω if $u > 0$ and $v > 0$ in Ω .

$|\Omega|$ is the Lebesgue measure of Ω .

$B_r(a) = \{x \in \mathbb{R}^N \mid |x - a| < r\}$ is a ball in $\mathbb{R}^N$.

$O(\varepsilon^t)$ denotes $|O(\varepsilon^t)|/\varepsilon^t \leq C$ as $\varepsilon \rightarrow 0$ for $t \geq 0$.

$O_1(\varepsilon^t)$ means that there exist the constants $C_1, C_2 > 0$ such that $C_1 \varepsilon^t \leq O_1(\varepsilon^t) \leq C_2 \varepsilon^t$ as ε is small.

$o_n(1)$ denotes $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$.

$\gamma_{\infty} = \max\{|f|_{L^{\infty}(\Omega)}, |g|_{L^{\infty}(\Omega)}\}$.

C, C_i will denote various positive constants, the exact values of which are not important.

Let $K_{\lambda,\mu} : E \rightarrow \mathbb{R}$ be the functional defined by

$$K_{\lambda,\mu}(z) = \int_{\Omega} (\lambda f(x) |u|^q + \mu g(x) |v|^q) dx \quad \forall z = (u, v) \in E. \quad (11)$$

We know that $I_{\lambda,\mu}$ is not bounded below on E . From the following lemma, we have that $I_{\lambda,\mu}$ is bounded from below on the Nehari manifold $\mathcal{N}_{\lambda,\mu}$ defined in (9).

Lemma 3. *The energy functional $I_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$.*

Proof. If $z = (u, v) \in \mathcal{N}_{\lambda,\mu}$, then by (10), the Hölder inequality, and the Sobolev embedding theorem, we get

$$I_{\lambda,\mu}(z) = \frac{2^* - 2}{2^* 2} \|z\|_E^2 - \frac{2^* - q}{2^* q} K_{\lambda,\mu}(z) \quad (12)$$

$$\begin{aligned} &\geq \frac{1}{N} \|z\|_E^2 - \frac{2^* - q}{2^* q} \gamma_{\infty} S^{-q/2} |\Omega|^{(2^* - q)/2^*} \\ &\quad \times (\lambda^{2/(2-q)} + \mu^{2/(2-q)})^{(2-q)/2} \|z\|_E^q. \end{aligned} \quad (13)$$

Hence, we have that $I_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$. $\square$

Define

$$\Phi_{\lambda,\mu}(z) = \langle I'_{\lambda,\mu}(z), z \rangle. \quad (14)$$

Then, for $z \in \mathcal{N}_{\lambda,\mu}$,

$$\begin{aligned} &\langle \Phi'_{\lambda,\mu}(z), z \rangle \\ &= 2\|z\|_E^2 - qK_{\lambda,\mu}(z) - 2^* \int_{\Omega} h(x) |u|^{\alpha} |v|^{\beta} dx \end{aligned} \quad (15)$$

$$= (2 - q) \|z\|_E^2 - (2^* - q) \int_{\Omega} h(x) |u|^{\alpha} |v|^{\beta} dx \quad (16)$$

$$= (2^* - q) K_{\lambda,\mu}(z) - (2^* - 2) \|z\|_E^2. \quad (17)$$

We apply the method in [12]; let

$$\begin{aligned} \mathcal{N}_{\lambda,\mu}^+ &= \{z \in \mathcal{N}_{\lambda,\mu} \mid \langle \Phi'_{\lambda,\mu}(z), z \rangle > 0\}, \\ \mathcal{N}_{\lambda,\mu}^0 &= \{z \in \mathcal{N}_{\lambda,\mu} \mid \langle \Phi'_{\lambda,\mu}(z), z \rangle = 0\}, \\ \mathcal{N}_{\lambda,\mu}^- &= \{z \in \mathcal{N}_{\lambda,\mu} \mid \langle \Phi'_{\lambda,\mu}(z), z \rangle < 0\}. \end{aligned} \quad (18)$$

By using equality (17), we get that $K_{\lambda,\mu}(z) > 0$ for $z \in \mathcal{N}_{\lambda,\mu}^+$. Moreover, we have the following results.

Lemma 4. *Let Λ_1 be a constant defined as in (8). If $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$, then $\mathcal{N}_{\lambda,\mu}^0 = \emptyset$.*

Proof. Assuming the contrary, there exist $\lambda, \mu > 0$ with $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$ such that $\mathcal{N}_{\lambda,\mu}^0 \neq \emptyset$. Then, by (16) and (17), for $u \in \mathcal{N}_{\lambda,\mu}^0$, we have

$$\|z\|_E^2 = \frac{2^* - q}{2 - q} \int_{\Omega} h(x) |u|^{\alpha} |v|^{\beta} dx = \frac{2^* - q}{2^* - 2} K_{\lambda,\mu}(z). \quad (19)$$

Using (H1) and both the Hölder and the Sobolev inequalities, we get

$$\|z\|_E \geq \left(\frac{2 - q}{2^* - q} S^{2^*/2} \right)^{1/(2^* - 2)}, \quad (20)$$

$$\begin{aligned} \|z\|_E &\leq \left(\frac{2^* - q}{2^* - 2} S^{-q/2} |\Omega|^{(2^* - q)/2^*} \gamma_{\infty} \right)^{1/(2 - q)} \\ &\quad \times (\lambda^{2/(2-q)} + \mu^{2/(2-q)})^{1/2}. \end{aligned} \quad (21)$$

This implies

$$\begin{aligned} &\lambda^{2/(2-q)} + \mu^{2/(2-q)} \\ &\geq \left(\frac{2 - q}{2^* - q} \right)^{2/(2^* - 2)} \left(\frac{(2^* - q) \gamma_{\infty}}{2^* - 2} |\Omega|^{(2^* - q)/2^*} \right)^{-2/(2 - q)} \\ &\quad \times S^{N/2 + q/(2 - q)} = \Lambda_1, \end{aligned} \quad (22)$$

which is a contradiction. $\square$

For each $z \in E$ with $\int_{\Omega} h(x) |u|^{\alpha} |v|^{\beta} dx > 0$, we write

$$t_{\max} = \left(\frac{(2 - q) \|z\|_E^2}{(2^* - q) \int_{\Omega} h(x) |u|^{\alpha} |v|^{\beta} dx} \right)^{1/(2^* - 2)} > 0. \quad (23)$$

Then, the following lemma holds.

Lemma 5. *Suppose that $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$, and $z \in E$ with $\int_{\Omega} h(x) |u|^{\alpha} |v|^{\beta} dx > 0$. Then, there exist unique $0 < t^+ < t_{\max} < t^-$ such that $t^+ z \in \mathcal{N}_{\lambda,\mu}^+$, $t^- z \in \mathcal{N}_{\lambda,\mu}^-$ and*

$$\begin{aligned} I_{\lambda,\mu}(t^+ z) &= \inf_{0 \leq t \leq t_{\max}} I_{\lambda,\mu}(tz); \\ I_{\lambda,\mu}(t^- z) &= \sup_{t \geq 0} I_{\lambda,\mu}(tz). \end{aligned} \quad (24)$$

Proof. This is similar to the proof of Hsu [14, Lemma 2.7]. $\square$

Applying Lemma 4 ($\mathcal{N}_{\lambda,\mu}^0 = \emptyset$ for $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$), we write $\mathcal{N}_{\lambda,\mu} = \mathcal{N}_{\lambda,\mu}^+ \cup \mathcal{N}_{\lambda,\mu}^-$ and define

$$\begin{aligned} \theta_{\lambda,\mu} &= \inf_{z \in \mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu}(z); \\ \theta_{\lambda,\mu}^+ &= \inf_{z \in \mathcal{N}_{\lambda,\mu}^+} I_{\lambda,\mu}(z); \\ \theta_{\lambda,\mu}^- &= \inf_{z \in \mathcal{N}_{\lambda,\mu}^-} I_{\lambda,\mu}(z). \end{aligned} \quad (25)$$

The following lemma shows that the minimizers on $\mathcal{N}_{\lambda,\mu}$ are usual critical points for $I_{\lambda,\mu}$.

Lemma 6. *For the case when $\lambda \in (0, \Lambda_1)$, if z_0 is a local minimizer for $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$, then $I'_{\lambda,\mu}(z_0) = 0$ in E^{-1} .*

Proof. See Brown and Zhang [15, theorem 2.3]. $\square$

Lemma 7. (i) If $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$ and $z = (u, v) \in \mathcal{N}_{\lambda, \mu}^+$, then one has

$$K_{\lambda, \mu}(z) > 0, \quad I_{\lambda, \mu}(z) < 0. \quad (26)$$

In particular, $\theta_{\lambda, \mu} \leq \theta_{\lambda, \mu}^+ < 0$.

(ii) If $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < (q/2)^{2/(2-q)} \Lambda_1$ and $z = (u, v) \in \mathcal{N}_{\lambda, \mu}^-$, then one has $u \not\equiv 0, v \not\equiv 0$ in Ω ,

$$\|z\|_E > \left(\frac{2-q}{2^*-q} \right)^{1/(2^*-2)} S^{N/4}, \quad (27)$$

and $\theta_{\lambda, \mu}^- > d_0$ for some positive constant $d_0 = d_0(\lambda, \mu, q, N, S, \gamma_\infty, |\Omega|)$.

Proof. (i) Let $z = (u, v) \in \mathcal{N}_{\lambda, \mu}^+$. By (16) and (17), we have

$$\begin{aligned} K_{\lambda, \mu}(z) &> \frac{2^*-2}{2^*-q} \|z\|_E^2 > 0, \\ \frac{2-q}{2^*-q} \|z\|_E^2 &> \int_{\Omega} h(x) |u|^\alpha |v|^\beta dx. \end{aligned} \quad (28)$$

Then,

$$\begin{aligned} I_{\lambda, \mu}(z) &= \left(\frac{1}{2} - \frac{1}{q} \right) \|z\|_E^2 + \left(\frac{1}{q} - \frac{1}{2^*} \right) \int_{\Omega} h(x) |u|^\alpha |v|^\beta dx \\ &< \left[\left(\frac{1}{2} - \frac{1}{q} \right) + \left(\frac{1}{q} - \frac{1}{2^*} \right) \frac{2-q}{2^*-q} \right] \|z\|_E^2 \\ &= -\frac{2-q}{qN} \|z\|_E^2 < 0. \end{aligned} \quad (29)$$

By the definition of $\theta_{\lambda, \mu}$, $\theta_{\lambda, \mu}^+$, we deduce that $\theta_{\lambda, \mu} \leq \theta_{\lambda, \mu}^+ < 0$.

(ii) Let $z \in \mathcal{N}_{\lambda, \mu}^-$; by (16) and the Hölder and the Sobolev inequalities, we get

$$\frac{2-q}{2^*-q} \|z\|_E^2 < \int_{\Omega} h(x) |u|^\alpha |v|^\beta dx \leq S^{-2^*/2} \|z\|_E^{2^*}. \quad (30)$$

This implies

$$\begin{aligned} \int_{\Omega} h(x) |u|^\alpha |v|^\beta dx &> \frac{2^*-q}{2-q} \|z\|_E \\ &> \left(\frac{2-q}{2^*-q} \right)^{\frac{1}{2^*-2}} S^{\frac{N}{4}} \quad \forall z \in \mathcal{N}_{\lambda, \mu}^-. \end{aligned} \quad (31)$$

By (13) and (31), we obtain that $u \not\equiv 0, v \not\equiv 0$ in Ω , and

$$\begin{aligned} I_{\lambda, \mu}(z) &\geq \|z\|_E^q \left[\frac{1}{N} \|z\|_E^{2-q} - \left(\frac{2^*-q}{2^*q} \right) \gamma_\infty S^{-q/2} |\Omega|^{(2^*-q)/2^*} \right. \\ &\quad \times \left. \left(\lambda^{2/(2-q)} + \mu^{2/(2-q)} \right)^{(2-q)/2} \right] \\ &> \left(\frac{2-q}{2^*-q} \right)^{q/(2^*-2)} S^{qN/4} \\ &\quad \times \left[\frac{1}{N} \left(\frac{2-q}{2^*-q} \right)^{(2-q)/(2^*-2)} S^{(2-q)N/4} \right. \\ &\quad \times \left. \left(\frac{2^*-q}{2^*q} \right) \gamma_\infty S^{-q/2} |\Omega|^{(2^*-q)/2^*} \right. \\ &\quad \times \left. \left. \left(\lambda^{2/(2-q)} + \mu^{2/(2-q)} \right)^{(2-q)/2} \right] \right]. \end{aligned} \quad (32)$$

Thus, if $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < (q/2)^{2/(2-q)} \Lambda_1$, for all $z \in \mathcal{N}_{\lambda, \mu}^-$, then

$$I_{\lambda, \mu}(z) \geq d_0(\lambda, \mu, q, N, S, \gamma_\infty, |\Omega|) > 0. \quad (33)$$

$\square$

3. Existence of a Ground State Solution

First of all, we define the Palais-Smale (denote by (PS)) sequences and (PS)-condition in E for $I_{\lambda, \mu}$ as follows.

Definition 8. (i) For $\gamma \in \mathbb{R}$, a sequence $\{z_n\}$ is a $(PS)_\gamma$ -sequence in E for $I_{\lambda, \mu}$ if $I_{\lambda, \mu}(z_n) = \gamma + o_n(1)$ and $I'_{\lambda, \mu}(z_n) = o_n(1)$ strongly in E^{-1} as $n \rightarrow \infty$.

(ii) $I_{\lambda, \mu}$ satisfies the $(PS)_\gamma$ -condition in E if any $(PS)_\gamma$ -sequence $\{z_n\}$ in E for $I_{\lambda, \mu}$ contains a convergent subsequence.

Proof of Theorem 1. Using the same argument as in Wu [16, Proposition 9] or Hsu [14, Proposition 3.3], there exists a minimizing sequence $\{z_n\}$ for $I_{\lambda, \mu}$ on $\mathcal{N}_{\lambda, \mu}$ such that

$$I_{\lambda, \mu}(z_n) = \theta_{\lambda, \mu} + o_n(1), \quad I'_{\lambda, \mu}(z_n) = o_n(1) \text{ in } E^{-1}. \quad (34)$$

Since $I_{\lambda, \mu}$ is coercive on $\mathcal{N}_{\lambda, \mu}$ (see Lemma 3), we get that $\{z_n\}$ is bounded in E . Then, there exist a subsequence $\{z_n = (u_n, v_n)\}$ and $z_{\lambda, \mu}^1 = (u_{\lambda, \mu}^1, v_{\lambda, \mu}^1) \in E$ such that

$$u_n \rightharpoonup u_{\lambda, \mu}^1, \quad v_n \rightharpoonup v_{\lambda, \mu}^1 \text{ weakly in } H_0^1(\Omega),$$

$$u_n \rightarrow u_{\lambda, \mu}^1, \quad v_n \rightarrow v_{\lambda, \mu}^1 \text{ almost everywhere in } \Omega,$$

$$u_n \rightarrow u_{\lambda, \mu}^1, \quad v_n \rightarrow v_{\lambda, \mu}^1 \text{ strongly in } L^s(\Omega) \quad \forall 1 \leq s < 2^*. \quad (35)$$

This implies

$$K_{\lambda,\mu}(z_n) = K_{\lambda,\mu}(z_{\lambda,\mu}^1) + o_n(1) \quad \text{as } n \rightarrow \infty. \quad (36)$$

First, we claim that $z_{\lambda,\mu}^1$ is a nontrivial solution of $(P_{\lambda,\mu})$. By (34) and (35), it is easy to verify that $z_{\lambda,\mu}^1$ is a weak solution of $(P_{\lambda,\mu})$. From $z_n \in \mathcal{N}_{\lambda,\mu}$ and (12), we deduce that

$$K_{\lambda,\mu}(z_n) = \frac{q(2^*-2)}{2(2^*-q)} \|z_n\|_E^2 - \frac{2^*q}{2^*-q} I_{\lambda,\mu}(z_n). \quad (37)$$

Let $n \rightarrow \infty$ in (37); by (34), (36), and $\theta_{\lambda,\mu} < 0$, we get

$$K_{\lambda,\mu}(z_{\lambda,\mu}^1) \geq -\frac{2^*q}{2^*-q} \theta_{\lambda,\mu} > 0. \quad (38)$$

Thus, $z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}$ is a nontrivial solution of $(P_{\lambda,\mu})$. Now, we prove that $z_n \rightarrow z_{\lambda,\mu}^1$ strongly in E and $I_{\lambda,\mu}(z_{\lambda,\mu}^1) = \theta_{\lambda,\mu}$. By (37), if $z \in \mathcal{N}_{\lambda,\mu}$, then

$$I_{\lambda,\mu}(z) = \frac{1}{N} \|z\|_E^2 - \frac{2^*-q}{2^*q} K_{\lambda,\mu}(z). \quad (39)$$

In order to prove that $I_{\lambda,\mu}(z_{\lambda,\mu}^1) = \theta_{\lambda,\mu}$, it suffices to recall that $z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}$, by (39) and applying Fatou's lemma to get

$$\begin{aligned} \theta_{\lambda,\mu} &\leq I_{\lambda,\mu}(z_{\lambda,\mu}^1) = \frac{1}{N} \|z_{\lambda,\mu}^1\|_E^2 - \frac{2^*-q}{2^*q} K_{\lambda,\mu}(z_{\lambda,\mu}^1) \\ &\leq \liminf_{n \rightarrow \infty} \left(\frac{1}{N} \|z_n\|_E^2 - \frac{2^*-q}{2^*q} K_{\lambda,\mu}(z_n) \right) \\ &\leq \liminf_{n \rightarrow \infty} I_{\lambda,\mu}(z_n) = \theta_{\lambda,\mu}. \end{aligned} \quad (40)$$

This implies that $I_{\lambda,\mu}(z_{\lambda,\mu}^1) = \theta_{\lambda,\mu}$ and $\lim_{n \rightarrow \infty} \|z_n\|_E^2 = \|z_{\lambda,\mu}^1\|_E^2$. Let $\tilde{z}_n = z_n - z_{\lambda,\mu}^1$; then Brézis-Lieb lemma [17] implies

$$\|\tilde{z}_n\|_E^2 = \|z_n\|_E^2 - \|z_{\lambda,\mu}^1\|_E^2. \quad (41)$$

Therefore, $z_n \rightarrow z_{\lambda,\mu}^1$ strongly in E . Since $I_{\lambda,\mu}(z_{\lambda,\mu}^1) = I_{\lambda,\mu}(|z_{\lambda,\mu}^1|) = \theta_{\lambda,\mu}$ and $|z_{\lambda,\mu}^1| \in \mathcal{N}_{\lambda,\mu}^+$, by Lemma 6 we may assume that $z_{\lambda,\mu}^1$ is a nontrivial nonnegative solution of $(P_{\lambda,\mu})$. By an argument of Hsu [18, Lemma 4.2], we can deduce that $u_{\lambda,\mu}^1 \not\equiv 0$ and $v_{\lambda,\mu}^1 \not\equiv 0$ in Ω . Finally, from the maximum principle [19], we deduce that $z_{\lambda,\mu}^1$ is positive in Ω . $\square$

Remark 9. $z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^+$ and $I_{\lambda,\mu}(z_{\lambda,\mu}^1) = \theta_{\lambda,\mu} = \theta_{\lambda,\mu}^+$.

Proof. We claim that $z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^+$. On the contrary, assume that $z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^-$ ($\mathcal{N}_{\lambda,\mu}^0 = \emptyset$ for $\lambda^{2/(2-q)} + \mu^{2/(2-q)} \in (0, \Lambda_1)$); then by Lemma 5, there exist unique t_1^+ and t_1^- such that $t_1^+ z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^+$ and $t_1^- z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^-$. In particular, we have $t_1^+ < t_1^- = 1$. Since

$$\frac{d}{dt} I_{\lambda,\mu}(t_1^+ z_{\lambda,\mu}^1) = 0, \quad \frac{d^2}{dt^2} I_{\lambda,\mu}(t_1^+ z_{\lambda,\mu}^1) > 0, \quad (42)$$

there exists $t_1^+ < \bar{t} \leq t_1^-$ such that $I_{\lambda,\mu}(t_1^+ z_{\lambda,\mu}^1) < I_{\lambda,\mu}(\bar{t} z_{\lambda,\mu}^1)$. By Lemma 5,

$$I_{\lambda,\mu}(t_1^+ z_{\lambda,\mu}^1) < I_{\lambda,\mu}(\bar{t} z_{\lambda,\mu}^1) \leq I_{\lambda,\mu}(t_1^- z_{\lambda,\mu}^1) = I_{\lambda,\mu}(z_{\lambda,\mu}^1), \quad (43)$$

which is a contradiction. Hence, $z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^+$ and $I_{\lambda,\mu}(z_{\lambda,\mu}^1) = \theta_{\lambda,\mu} = \theta_{\lambda,\mu}^+$. $\square$

4. Existence of $k+1$ Solutions

Throughout this section, (H1)-(H2) will be assumed. First of all, we want to show that $I_{\lambda,\mu}$ satisfies the $(PS)_\gamma$ -condition in E for $\gamma \in (-\infty, (1/N)(S_{\alpha,\beta})^{N/2} - C_0(\lambda^{2/(2-q)} + \mu^{2/(2-q)}))$, where C_0 is defined in the following lemma.

Lemma 10. *If $\{z_n\} \subset E$ is a $(PS)_\gamma$ -sequence for $I_{\lambda,\mu}$ with $z_n \rightharpoonup z$ weakly in E , then $I'_{\lambda,\mu}(z) = 0$ and there exists a constant $C_0 = C_0(q, N, S, \gamma_\infty, |\Omega|) > 0$ such that $I_{\lambda,\mu}(z) \geq -C_0(\lambda^{2/(2-q)} + \mu^{2/(2-q)})$.*

Proof. Let $z_n = (u_n, v_n)$ and $z = (u, v)$. If $\{z_n\}$ is a $(PS)_\gamma$ -sequence for $I_{\lambda,\mu}$ with $z_n \rightharpoonup z$ weakly in E , it is easy to check that $I'_{\lambda,\mu}(z) = 0$ in E^{-1} . Then, we get $\langle I'_{\lambda,\mu}(z), z \rangle = 0$; that is, $\int_\Omega h(x)|u|^\alpha |v|^\beta dx = \|z\|_E^2 - K_{\lambda,\mu}(z)$. Thus, by (13), the Hölder, the Young, and the Sobolev inequalities, we have

$$\begin{aligned} I_{\lambda,\mu}(z) &\geq \frac{1}{N} \|z\|_E^2 - \frac{2^*-q}{2^*q} \gamma_\infty S^{-q/2} |\Omega|^{(2^*-q)/2^*} \\ &\quad \times \left(\lambda^{2/(2-q)} + \mu^{2/(2-q)} \right)^{(2-q)/2} \|z\|_E^q \\ &\geq \frac{1}{N} \|z\|_E^2 - \frac{1}{N} \|z\|_E^2 - C_0 \left(\lambda^{2/(2-q)} + \mu^{2/(2-q)} \right) \\ &= -C_0 \left(\lambda^{2/(2-q)} + \mu^{2/(2-q)} \right), \end{aligned} \quad (44)$$

where $C_0 = C_0(q, N, S, \gamma_\infty, |\Omega|) > 0$. $\square$

Lemma 11. *If $\{z_n\} \subset E$ is a $(PS)_\gamma$ -sequence for $I_{\lambda,\mu}$, then $\{z_n\}$ is bounded in E .*

Proof. See Hsu and Lin [8, Lemma 2.3]. $\square$

Recall that

$$S_{\alpha,\beta} = \inf_{u,v \in H_0^1(\Omega) \setminus \{0\}} \frac{\|(u, v)\|_E^2}{\left(\int_\Omega |u|^\alpha |v|^\beta dx \right)^{2/(\alpha+\beta)}}, \quad (45)$$

and let

$$c^* = \frac{1}{N} (S_{\alpha,\beta})^{N/2} - C_0 \left(\lambda^{2/(2-q)} + \mu^{2/(2-q)} \right), \quad (46)$$

where $C_0 > 0$ is given in Lemma 10.

Lemma 12. *$I_{\lambda,\mu}$ satisfies the $(PS)_\gamma$ -condition in E for $\gamma \in (-\infty, c^*)$.*

Proof. Let $\{z_n\} \subset E$ be a $(PS)_\gamma$ -sequence for $I_{\lambda,\mu}$ with $\gamma \in (-\infty, c^*)$. Write $z_n = (u_n, v_n)$. We know from Lemma 11 that $\{z_n\}$ is bounded in E , and then $z_n \rightharpoonup z = (u, v)$ weakly up to a subsequence; z is a critical point of $I_{\lambda,\mu}$. Furthermore, we may assume that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ weakly in $H_0^1(\Omega)$ and $u_n \rightarrow u$, $v_n \rightarrow v$ strongly in $L^s(\Omega)$ for all $1 \leq s < 2^*$, and $u_n \rightarrow u$, $v_n \rightarrow v$ a.e. on Ω . Hence, we have that $I'_{\lambda,\mu}(z) = 0$ and

$$K_{\lambda,\mu}(z_n) = K_{\lambda,\mu}(z) + o_n(1). \quad (47)$$

Let $\tilde{u}_n = u_n - u$, $\tilde{v}_n = v_n - v$ and $\tilde{z}_n = (\tilde{u}_n, \tilde{v}_n)$. Then, we obtain

$$\|\tilde{z}_n\|_E^2 = \|z_n\|_E^2 - \|z\|_E^2 + o_n(1), \quad (48)$$

and by an argument of Han [20, Lemma 2.1],

$$\begin{aligned} & \int_{\Omega} h(x) |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta dx \\ &= \int_{\Omega} h(x) |u_n|^\alpha |v_n|^\beta dx \\ & \quad - \int_{\Omega} h(x) |u|^\alpha |v|^\beta dx + o_n(1). \end{aligned} \quad (49)$$

Since $I_{\lambda,\mu}(z_n) = \gamma + o_n(1)$, $I'_{\lambda,\mu}(z_n) = o_n(1)$ in E^{-1} and (47)–(49), we deduce that

$$\frac{1}{2} \|\tilde{z}_n\|_E^2 - \frac{1}{2^*} \int_{\Omega} h(x) |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta dx = \gamma - I_{\lambda,\mu}(z) + o_n(1), \quad (50)$$

$$\|\tilde{z}_n\|_E^2 - \int_{\Omega} h(x) |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta dx = o_n(1). \quad (51)$$

Hence, we may assume that

$$\|\tilde{z}_n\|_E^2 \rightarrow l, \quad \int_{\Omega} h(x) |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta dx \rightarrow l. \quad (52)$$

Assume that $l \neq 0$; by the definition of $S_{\alpha,\beta}$, $|h|_{L^\infty(\Omega)} = 1$ and (52), we obtain

$$\begin{aligned} S_{\alpha,\beta} l^{2/2^*} &= S_{\alpha,\beta} \lim_{n \rightarrow \infty} \left(\int_{\Omega} h(x) |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta dx \right)^{2/2^*} \\ &\leq |h|_{L^\infty(\Omega)}^{2/2^*} \lim_{n \rightarrow \infty} \|\tilde{z}_n\|^2 = l, \end{aligned} \quad (53)$$

which implies that $l \geq (S_{\alpha,\beta})^{N/2}$. In addition, from Lemma 10, (50), and (52), we get

$$\begin{aligned} \gamma &= \left(\frac{1}{2} - \frac{1}{2^*} \right) l + I_{\lambda,\mu}(z) \\ &\geq \frac{1}{N} (S_{\alpha,\beta})^{N/2} - C_0 \left(\lambda^{2/(2-q)} + \mu^{2/(2-q)} \right), \end{aligned} \quad (54)$$

which is a contradiction. Hence, $l = 0$; that is, $z_n \rightarrow z$ strongly in E . $\square$

From assumption (H2), we can choose $r_0 \in (0, 1)$ such that

$$\overline{B_{r_0}(a_i)} \cap \overline{B_{r_0}(a_j)} = \emptyset \quad \text{for } i \neq j, \quad 1 \leq i, j \leq k, \quad (55)$$

and $\cup_{i=1}^k \overline{B_{r_0}(a_i)} \subset \Omega$, where $\overline{B_{r_0}(a_i)} = \{x \in \mathbb{R}^N \mid |x - a_i| \leq r_0\}$ and $h(a_i) = |h|_\infty = 1$ for $1 \leq i \leq k$.

Define

$$Q_i(z) = \frac{\int_{\Omega} \psi_i(x) (|\nabla u|^2 + |\nabla v|^2) dx}{\int_{\Omega} (|\nabla u|^2 + |\nabla v|^2) dx}, \quad z = (u, v) \in E \setminus \{0\}, \quad (56)$$

where $\psi_i(x) = \min\{1, |x - a_i|\}$, $i = 1, 2, \dots, k$.

Then, we have the following separation result.

Lemma 13. *If $Q_i(z) \leq r_0/3$ and $Q_j(z) \leq r_0/3$ for $z \in E \setminus \{0\}$, then $i = j$.*

Proof. For any $z \in E \setminus \{0\}$ satisfying $Q_i(z) \leq r_0/3$ ($1 \leq i \leq k$), we get

$$\begin{aligned} \frac{r_0}{3} \|z\|_E^2 &\geq \int_{\Omega} \psi_i(x) (|\nabla u|^2 + |\nabla v|^2) dx \\ &\geq \int_{\Omega \setminus B_{r_0}(a_i)} \psi_i(x) (|\nabla u|^2 + |\nabla v|^2) dx \\ &\geq r_0 \int_{\Omega \setminus B_{r_0}(a_i)} (|\nabla u|^2 + |\nabla v|^2) dx, \end{aligned} \quad (57)$$

which implies that

$$\|z\|_E^2 \geq 3 \int_{\Omega \setminus B_{r_0}(a_i)} (|\nabla u|^2 + |\nabla v|^2) dx, \quad 1 \leq i \leq k. \quad (58)$$

Hence, from (58), we obtain

$$\begin{aligned} 2\|z\|_E^2 &\geq 3 \left(\int_{\Omega \setminus B_{r_0}(a_i)} (|\nabla u|^2 + |\nabla v|^2) dx \right. \\ &\quad \left. + \int_{\Omega \setminus B_{r_0}(a_j)} (|\nabla u|^2 + |\nabla v|^2) dx \right) \\ &\geq 3\|z\|_E^2 \quad \text{if } i \neq j, \end{aligned} \quad (59)$$

which is a contradiction. $\square$

For $i = 1, 2, \dots, k$, we set

$$\begin{aligned} \mathcal{N}_{\lambda,\mu}^i &= \left\{ u \in \mathcal{N}_{\lambda,\mu}^- \mid Q_i(z) < \frac{r_0}{3} \right\}, \\ \partial \mathcal{N}_{\lambda,\mu}^i &= \left\{ u \in \mathcal{N}_{\lambda,\mu}^- \mid Q_i(z) = \frac{r_0}{3} \right\}, \end{aligned} \quad (60)$$

and define

$$\theta_{\lambda,\mu}^i = \inf_{\mathcal{N}_{\lambda,\mu}^i} I_{\lambda,\mu}(z), \quad \bar{\theta}_{\lambda,\mu}^i = \inf_{\partial \mathcal{N}_{\lambda,\mu}^i} I_{\lambda,\mu}(z). \quad (61)$$

Recall that the best Sobolev constant S is defined as

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx \right)^{2/2^*}}. \quad (62)$$

It is well known that

$$U(x) = \frac{[N(N-2)]^{(N-2)/4}}{[1+|x|^2]^{(N-2)/2}} \quad (63)$$

is a minimizer of S , and $|\nabla U|_{L^2(\mathbb{R}^N)}^2 = |U|_{L^{2^*}(\mathbb{R}^N)}^{2^*} = S^{N/2}$. Fix a maximum point a_i of h ($1 \leq i \leq k$). Let $\eta_i \in C_0^\infty(\Omega)$ be a cut-off function such that $0 \leq \eta_i \leq 1$, $|\nabla \eta_i| \leq C$, and $\eta_i(x) = 1$ for $|x - a_i| < r_0/2$, $\eta_i(x) = 0$ for $|x - a_i| > r_0$. We define

$$u_\varepsilon^i(x) = \varepsilon^{(2-N)/2} \eta_i(x) U\left(\frac{x - a_i}{\varepsilon}\right) = \frac{c_1 \varepsilon^{(N-2)/2} \eta_i(x)}{[\varepsilon^2 + |x - a_i|^2]^{(N-2)/2}}, \quad (64)$$

where $c_1 = [N(N-2)]^{(N-2)/4}$ and $\varepsilon > 0$.

From now on, we assume that $N/(N-2) < q < 2$ and $N > 4$.

Lemma 14. *There exist $\varepsilon_0 > 0$, $\Lambda_2 \in (0, (q/2)^{2/(2-q)} \Lambda_1)$, such that for $\varepsilon \in (0, \varepsilon_0)$ and $\lambda^{2/(2-q)} + \mu^{2/(2-q)} \in (0, \Lambda_2)$, one has*

$$\sup_{t \geq 0} I_{\lambda, \mu}(t\sqrt{\alpha}u_\varepsilon^i, t\sqrt{\beta}u_\varepsilon^i) < c^* \quad \text{uniformly in } i, \quad (65)$$

where c^* is the positive constant given in Lemma 12.

In particular, $0 < \theta_{\lambda, \mu}^- \leq \theta_{\lambda, \mu}^i < c^*$ for all $1 \leq i \leq k$.

Proof. It is well known that (or see Brézis and Nirenberg [21], Cheng and Ma [22, Lemma 3.2], Struwe [23], and Willem [24, Lemma 1.46]) as $\varepsilon \rightarrow 0^+$,

$$|u_\varepsilon^i|_{L^{2^*}(\Omega)}^2 = |U|_{L^{2^*}(\mathbb{R}^N)}^2 + O(\varepsilon^{N-2}), \quad (66)$$

$$|\nabla u_\varepsilon^i|_{L^2(\Omega)}^2 |\nabla U|_{L^2(\mathbb{R}^N)}^2 + O(\varepsilon^{N-2}). \quad (67)$$

For $N/(N-2) < q < 2$, $N > 4$ and $\varepsilon < r_0/2$,

$$\begin{aligned} |u_\varepsilon^i|_{L^q(\Omega)}^q &= \int_{B_{r_0/2}(a_i)} \left[\varepsilon^{(2-N)/2} U\left(\frac{x - a_i}{\varepsilon}\right) \right]^q dx + O(\varepsilon^{N-2}) \\ &\geq C\varepsilon^\theta + O(\varepsilon^{N-2}), \quad \text{where } \theta = N - \frac{(N-2)q}{2}. \end{aligned} \quad (68)$$

Set $\bar{z}_\varepsilon^i = (\sqrt{\alpha}u_\varepsilon^i, \sqrt{\beta}u_\varepsilon^i)$. By Lemma 5, there exists $t_\varepsilon^i > 0$ such that $z_\varepsilon^i = t_\varepsilon^i \bar{z}_\varepsilon^i \in \mathcal{N}_{\lambda, \mu}^-$ for $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_1$. Furthermore,

$$\begin{aligned} Q_i(z_\varepsilon^i) &= \frac{\int_\Omega \psi_i(x) |\nabla u_\varepsilon^i|^2 dx}{\int_\Omega |\nabla u_\varepsilon^i|^2 dx} \\ &= \frac{\int_{(\Omega - a_i)/\varepsilon} \psi_i(a_i + \varepsilon y) |\nabla (\eta_i(a_i + \varepsilon y) U(y))|^2 dy}{\int_{(\Omega - a_i)/\varepsilon} |\nabla (\eta_i(a_i + \varepsilon y) U(y))|^2 dy} \\ &\longrightarrow \psi_i(a_i) = 0 \quad \text{as } \varepsilon \longrightarrow 0. \end{aligned} \quad (69)$$

Hence, there exists $\bar{\varepsilon}_0 > 0$ for any

$$\varepsilon \in (0, \bar{\varepsilon}_0), \quad Q_i(z_\varepsilon^i) < \frac{r_0}{3}, \quad (70)$$

which implies

$$z_\varepsilon^i = t_\varepsilon^i \bar{z}_\varepsilon^i \in \mathcal{N}_{\lambda, \mu}^i \quad \text{for } \varepsilon \in (0, \bar{\varepsilon}_0), \quad (71)$$

and then

$$\theta_{\lambda, \mu}^- \leq \theta_{\lambda, \mu}^i \leq I_{\lambda, \mu}(z_\varepsilon^i) \leq \sup_{t \geq 0} I_{\lambda, \mu}(t t_\varepsilon^i \bar{z}_\varepsilon^i) = \sup_{t \geq 0} I_{\lambda, \mu}(t \bar{z}_\varepsilon^i). \quad (72)$$

First, we consider the functional $I_{0,0} : E \rightarrow \mathbb{R}$ defined by

$$I_{0,0}(u, v) = \frac{1}{2} \|(u, v)\|_E^2 - \frac{1}{2^*} \int_\Omega h(x) |u|^\alpha |v|^\beta dx. \quad (73)$$

Step I. Show that $\sup_{t \geq 0} I_{0,0}(\bar{z}_\varepsilon^i) \leq (1/N)(S_{\alpha, \beta})^{N/2} + O(\varepsilon^{N-2})$.

According to condition (H2), we conclude that

$$\begin{aligned} &\left| \int_\Omega h(x) |u_\varepsilon^i(x)|^{2^*} dx - \int_\Omega h(a_i) |u_\varepsilon^i(x)|^{2^*} dx \right| \\ &\leq \int_\Omega |h(x) - h(a_i)| |u_\varepsilon^i(x)|^{2^*} dx \\ &= O\left(\int_{B_{r_0}(a_i)} |x - a_i|^\sigma |u_\varepsilon^i(x)|^{2^*} dx\right) \\ &= O(\varepsilon^\sigma). \end{aligned} \quad (74)$$

From (66), (74), $h(a_i) = 1$, and $\sigma \geq N-2$, we can deduce that

$$\begin{aligned} &\left(\int_\Omega h(x) |u_\varepsilon^i(x)|^{2^*} dx \right)^{2/2^*} \\ &= \left(|u_\varepsilon^i|_{L^{2^*}(\Omega)}^{2^*} + O(\varepsilon^\sigma) \right)^{2/2^*} \\ &= |u_\varepsilon^i|_{L^{2^*}(\Omega)}^2 + O(\varepsilon^\sigma) \\ &= |U|_{L^{2^*}(\mathbb{R}^N)}^2 + O(\varepsilon^{N-2}). \end{aligned} \quad (75)$$

Using (67) and (75), then

$$\begin{aligned} &\frac{| \nabla u_\varepsilon^i |_{L^2(\Omega)}^2}{\left(\int_\Omega h(x) |u_\varepsilon^i(x)|^{2^*} dx \right)^{2/2^*}} \\ &= \frac{| \nabla U |_{L^2(\mathbb{R}^N)}^2 + O(\varepsilon^{N-2})}{|U|_{L^{2^*}(\mathbb{R}^N)}^2 + O(\varepsilon^{N-2})} \\ &= S + O(\varepsilon^{N-2}). \end{aligned} \quad (76)$$

Since

$$\begin{aligned} &\sup_{t \geq 0} \left(\frac{A}{2} t^2 - \frac{B}{2^*} t^{2^*} \right) \\ &= \frac{1}{N} \left(\frac{A}{B^{2/2^*}} \right)^{N/2}, \quad \text{for any } A > 0, B > 0, \end{aligned} \quad (77)$$

by (7) and (76), we conclude that

$$\begin{aligned} & \sup_{t \geq 0} I_{0,0}(t\bar{z}_\varepsilon^i) \\ &= \frac{1}{N} \left(\frac{(\alpha + \beta) |\nabla u_\varepsilon^i|^2_{L^2(\Omega)}}{(\alpha^{\alpha/2} \beta^{\beta/2} \int_\Omega h(x) |u_\varepsilon^i(x)|^{2^*} dx)^{2/2^*}} \right)^{N/2} \\ &\leq \frac{1}{N} (S_{\alpha,\beta})^{N/2} + O(\varepsilon^{N-2}). \end{aligned} \quad (78)$$

Step II. Let C_0 be the positive constant given in Lemma 10. We can choose $\delta_1 > 0$ such that for all $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \delta_1$, we have

$$c^* = \frac{1}{N} (S_{\alpha,\beta})^{N/2} - C_0 (\lambda^{2/(2-q)} + \mu^{2/(2-q)}) > 0. \quad (79)$$

Since $I_{\lambda,\mu}$ is continuous in E , $I_{\lambda,\mu}(0) = 0$, and $\{\bar{z}_\varepsilon^i\}$ is uniformly bounded in E for any $0 < \varepsilon < \min\{\bar{\varepsilon}_0, r_0/2\}$ (see (67)), then there exists $t_0 > 0$ (independent of ε) such that for any $0 < \varepsilon < \min\{\bar{\varepsilon}_0, r_0/2\}$,

$$\begin{aligned} & \sup_{0 \leq t \leq t_0} I_{\lambda,\mu}(t\bar{z}_\varepsilon^i) < c^*, \quad \text{uniformly in } i, \\ & \forall 0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \delta_1. \end{aligned} \quad (80)$$

According to condition (H1), $f_{\min} = \min_{x \in \bar{\Omega}} f(x) > 0$ and $g_{\min} = \min_{x \in \bar{\Omega}} g(x) > 0$. Applying the results of Step I and (68), we have that for $N/(N-2) < q < 2$ and $N > 4$,

$$\begin{aligned} & \sup_{t \geq t_0} I_{\lambda,\mu}(t\bar{z}_\varepsilon^i) \\ &= \sup_{t \geq t_0} \left(I_{0,0}(t\bar{z}_\varepsilon^i) - \frac{t^q}{q} K_{\lambda,\mu}(t\bar{z}_\varepsilon^i) \right) \\ &\leq \frac{1}{N} (S_{\alpha,\beta})^{N/2} + O(\varepsilon^{N-2}) - \frac{t_0^q}{q} m(\lambda + \mu) \\ &\quad \times \int_{B_{r_0/2}(a_i)} |u_\varepsilon^i|^q dx \\ &\leq \frac{1}{N} (S_{\alpha,\beta})^{N/2} + O(\varepsilon^{N-2}) - (\lambda + \mu) O_1(\varepsilon^\theta), \end{aligned} \quad (81)$$

where $m = \min\{\alpha^{q/2} f_{\min}, \beta^{q/2} g_{\min}\}$ and $\theta = N - ((N-2)q)/2$. Therefore, we can choose $\lambda = O_1(\varepsilon^{\tau_1})$ and $\mu = O_1(\varepsilon^{\tau_2})$ such that

$$\frac{2-q}{q} \theta < \tau_1, \quad \tau_2 < (N-2) - \theta. \quad (82)$$

This implies that

$$\begin{aligned} & \min\{\tau_1, \tau_2\} + \theta < \frac{2}{2-q} \min(\tau_1, \tau_2), \\ & \min\{\tau_1, \tau_2\} + \theta < N-2, \\ & (\lambda + \mu) O_1(\varepsilon^\theta) = O_1(\varepsilon^{\min\{\tau_1, \tau_2\} + \theta}), \\ & \lambda^{2/(2-q)} + \mu^{2/(2-q)} = O_1(\varepsilon^{2/(2-q) \min\{\tau_1, \tau_2\}}). \end{aligned} \quad (83)$$

There exist $\delta_2 > 0$, $\varepsilon_0 \in (0, \min\{\bar{\varepsilon}_0, r_0/2\})$ such that for all $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \delta_2$ and $0 < \varepsilon < \varepsilon_0$, we have

$$O(\varepsilon^{N-2}) - (\lambda + \mu) O_1(\varepsilon^\theta) < -C_0 (\lambda^{2/(2-q)} + \mu^{2/(2-q)}). \quad (84)$$

Thus, we can choose $\Lambda_2 = \min\{(q/2)^{2/(2-q)} \Lambda_1, \delta_1, \delta_2\} > 0$. Then, for all $\lambda^{2/(2-q)} + \mu^{2/(2-q)} \in (0, \Lambda_2)$, there holds

$$\sup_{t \geq 0} I_{\lambda,\mu}(t\bar{z}_\varepsilon^i) < c^* \quad \text{uniformly in } i. \quad (85)$$

Step III. For $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_2$ and $0 < \varepsilon < \varepsilon_0$, by Lemma 7, (72), and (85), we get

$$0 < \theta_{\lambda,\mu}^- \leq \theta_{\lambda,\mu}^i \leq I_{\lambda,\mu}(t\bar{z}_\varepsilon^i) < c^* \quad \forall 1 \leq i \leq k. \quad (86)$$

□

To proceed, we need to quote the concentration-compactness principle (see [24, 25]) about the case of systems.

Lemma 15. Let $\{u_n, v_n\} \subset H_0^1(\Omega) \times H_0^1(\Omega)$ be a sequence such that

$$\begin{aligned} & u_n \rightharpoonup u, \quad v_n \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega); \\ & u_n \rightarrow u, \quad v_n \rightarrow v \quad \text{a.e. on } \Omega, \\ & |\nabla(u_n - u)|^2 + |\nabla(v_n - v)|^2 \rightharpoonup \tilde{\mu} \\ & \quad \text{weakly in the sense of measures,} \\ & |u_n - u|^\alpha |v_n - v|^\beta \rightharpoonup \tilde{\nu} \\ & \quad \text{weakly in the sense of measures.} \end{aligned} \quad (87)$$

Then, it follows that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2) dx \\ &= \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx + \|\tilde{\mu}\|, \\ & \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta dx = \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx + \|\tilde{\nu}\|, \\ & \|\tilde{\nu}\|^{2/(\alpha+\beta)} \leq S_{\alpha,\beta}^{-1} \|\tilde{\mu}\|. \end{aligned} \quad (88)$$

Moreover, if $u \equiv v \equiv 0$ and $\|\tilde{\nu}\|^{2/(\alpha+\beta)} = S_{\alpha,\beta}^{-1} \|\tilde{\mu}\|$, then $\tilde{\mu}$ and $\tilde{\nu}$ concentrate at a single point.

Proof. See Han [20, Lemma 2.2]. $\square$

Lemma 16. For any $i \in \{1, 2, \dots, k\}$, there exist $\widetilde{\Lambda}_i > 0$ such that

$$\widetilde{\theta}_{\lambda, \mu}^i > \frac{1}{N} (S_{\alpha, \beta})^{N/2} \quad \forall 0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \widetilde{\Lambda}_i. \quad (89)$$

Proof. Fix $i \in \{1, 2, \dots, k\}$. Assume the contrary. There then exists a sequence $\{(\lambda_n, \mu_n)\}$ with $(\lambda_n, \mu_n) \rightarrow (0, 0)$ as $n \rightarrow \infty$ such that $\widetilde{\theta}_{\lambda_n, \mu_n}^i \rightarrow c \leq (1/N)(S_{\alpha, \beta})^{N/2}$ as $n \rightarrow \infty$. Consequently, there exists a sequence $\{z_n = (u_n, v_n)\} \subset \partial \mathcal{N}_{\lambda_n, \mu_n}^i$ such that as $n \rightarrow \infty$,

$$\begin{aligned} & \int_{\Omega} (|\nabla u_n|^2 + |\nabla v_n|^2) dx \\ &= \int_{\Omega} (\lambda_n f(x) |u_n|^q + \mu_n g(x) |v_n|^q) dx \\ &+ \int_{\Omega} h(x) |u_n|^\alpha |v_n|^\beta dx, \end{aligned} \quad (90)$$

$$I_{\lambda_n, \mu_n}(z_n) \rightarrow c \leq \frac{1}{N} (S_{\alpha, \beta})^{N/2} \quad \text{as } n \rightarrow \infty. \quad (91)$$

It then follows easily that $\{z_n\}$ is uniformly bounded in E , and since f and g are continuous on $\overline{\Omega}$, we obtain

$$\begin{aligned} K_{\lambda_n, \mu_n}(z_n) &= \int_{\Omega} (\lambda_n f(x) |u_n|^q + \mu_n g(x) |v_n|^q) dx \\ &= o_n(1) \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (92)$$

From (90), and by the Hölder and the Sobolev inequalities, we can fix $m_0 > 0$ such that

$$\begin{aligned} & \int_{\Omega} (|\nabla u_n|^2 + |\nabla v_n|^2) dx \geq m_0, \\ & \int_{\Omega} h(x) |u_n|^\alpha |v_n|^\beta dx \geq m_0. \end{aligned} \quad (93)$$

Thus, up to a subsequence, we infer that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\Omega} (|\nabla u_n|^2 + |\nabla v_n|^2) dx \\ &= \lim_{n \rightarrow \infty} \int_{\Omega} h(x) |u_n|^\alpha |v_n|^\beta dx = l > 0. \end{aligned} \quad (94)$$

Furthermore, by $|h|_{L^\infty(\Omega)} = 1$, we deduce

$$\begin{aligned} l &= \lim_{n \rightarrow \infty} \int_{\Omega} h(x) |u_n|^\alpha |v_n|^\beta dx \\ &\leq |h|_{L^\infty(\Omega)} \lim_{n \rightarrow \infty} \int_{\Omega} |u_n|^\alpha |v_n|^\beta dx \\ &\leq S_{\alpha, \beta}^{-2^*/2} \lim_{n \rightarrow \infty} \left(\int_{\Omega} (|\nabla u_n|^2 + |\nabla v_n|^2) dx \right)^{2^*/2} \\ &\leq S_{\alpha, \beta}^{-2^*/2} l^{2^*/2}, \end{aligned} \quad (95)$$

which implies

$$l \geq (S_{\alpha, \beta})^{N/2}. \quad (96)$$

On the other hand, we have, as $n \rightarrow \infty$,

$$\begin{aligned} \frac{1}{N} l &= \frac{1}{2} \|z_n\|^2 dx - \frac{1}{2^*} \int_{\Omega} h(x) |u_n|^\alpha |v_n|^\beta dx \\ &- \frac{1}{q} K_{\lambda_n, \mu_n}(z_n) + o_n(1) \\ &= I_{\lambda_n, \mu_n}(z_n) + o_n(1) \\ &\leq \frac{1}{N} (S_{\alpha, \beta})^{N/2}. \end{aligned} \quad (97)$$

Hence, together with (96), we get

$$l = (S_{\alpha, \beta})^{N/2}, \quad (98)$$

and then from (95), we also have

$$\lim_{n \rightarrow \infty} \int_{\Omega} h(x) |u_n|^\alpha |v_n|^\beta dx = \lim_{n \rightarrow \infty} \int_{\Omega} |u_n|^\alpha |v_n|^\beta dx = l. \quad (99)$$

Therefore,

$$\lim_{n \rightarrow \infty} \int_{\Omega} |u_n|^\alpha |v_n|^\beta dx = l. \quad (100)$$

Set $\widetilde{z}_n = (\widetilde{u}_n, \widetilde{v}_n) = z_n / \|z_n\|$; then, we have $\|\widetilde{z}_n\| = 1$. Moreover, by (94), (98), and (100), we get

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta dx = \lim_{n \rightarrow \infty} \frac{\int_{\Omega} |u_n|^\alpha |v_n|^\beta dx}{\|z_n\|^{2^*}} = S_{\alpha, \beta}^{-N/(N-2)}. \quad (101)$$

Thus, up to a subsequence, we may assume that

$$\begin{aligned} & \widetilde{u}_n \rightharpoonup u, \quad \widetilde{v}_n \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega); \\ & \widetilde{u}_n \rightarrow u, \quad \widetilde{v}_n \rightarrow v \quad \text{a.e. on } \Omega, \\ & |\nabla(\widetilde{u}_n - u)|^2 + |\nabla(\widetilde{v}_n - v)|^2 \rightharpoonup \widetilde{\mu} \\ & \text{weakly in the sense of measures,} \\ & |\widetilde{u}_n - u|^\alpha |\widetilde{v}_n - v|^\beta \rightharpoonup \widetilde{\nu} \\ & \text{weakly in the sense of measures.} \end{aligned} \quad (102)$$

Since Ω is bounded, from (101) and Lemma 15, we deduce that

$$1 = \int_{\Omega} (|\nabla u|^2 + |\nabla v|^2) dx + \|\widetilde{\mu}\|, \quad (103)$$

$$S_{\alpha, \beta}^{-N/(N-2)} = \int_{\Omega} |u|^\alpha |v|^\beta dx + \|\widetilde{\nu}\|, \quad (104)$$

$$\|\widetilde{\nu}\|^{2/(\alpha+\beta)} \leq S_{\alpha, \beta}^{-1} \|\widetilde{\mu}\|. \quad (105)$$

If $\int_{\Omega} (|\nabla u|^2 + |\nabla v|^2) dx \neq 0$ and $\|\tilde{\mu}\| \neq 0$, we deduce that

$$\begin{aligned}
 1 &= \left(\int_{\Omega} (|\nabla u|^2 + |\nabla v|^2) dx + \|\tilde{\mu}\| \right)^{(\alpha+\beta)/2} \\
 &> \left(\int_{\Omega} (|\nabla u|^2 + |\nabla v|^2) dx \right)^{(\alpha+\beta)/2} + \|\tilde{\mu}\|^{(\alpha+\beta)/2} \\
 &\geq S_{\alpha,\beta}^{(\alpha+\beta)/2} \int_{\Omega} |u|^{\alpha} |v|^{\beta} dx + S_{\alpha,\beta}^{(\alpha+\beta)/2} \|\tilde{v}\| \\
 &= S_{\alpha,\beta}^{(\alpha+\beta)/2} \cdot S_{\alpha,\beta}^{-N/(N-2)} \\
 &= 1,
 \end{aligned} \tag{106}$$

which is a contradiction.

Thus, $\int_{\Omega} (|\nabla u|^2 + |\nabla v|^2) dx = 0$ or $\|\tilde{\mu}\| = 0$. If $\|\tilde{\mu}\| = 0$, from (103)–(105), we get $\int_{\Omega} (|\nabla u|^2 + |\nabla v|^2) dx = 1$ and $\int_{\Omega} |u|^{\alpha} |v|^{\beta} dx = S_{\alpha,\beta}^{-N/(N-2)}$. Then,

$$\frac{\int_{\Omega} (|\nabla u|^2 + |\nabla v|^2) dx}{\left(\int_{\Omega} |u|^{\alpha} |v|^{\beta} dx \right)^{2/(\alpha+\beta)}} = S_{\alpha,\beta}, \tag{107}$$

which means that $S_{\alpha,\beta}$ is achieved by (u, v) . It is impossible since $S_{\alpha,\beta}$ cannot be achieved on any bounded domain Ω . Hence,

$$\int_{\Omega} (|\nabla u|^2 + |\nabla v|^2) dx = 0, \quad \|\tilde{\mu}\| = 1. \tag{108}$$

Then, $u \equiv v \equiv 0$ on Ω , and from (103), (104), we easily have $\|\tilde{v}\|^{2/(\alpha+\beta)} = S_{\alpha,\beta}^{-1} = S_{\alpha,\beta}^{-1} \|\tilde{\mu}\|$. By Lemma 15, we conclude that $x_0 \in \bar{\Omega}$ such that

$$\begin{aligned}
 &|\nabla \tilde{u}_n|^2 + |\nabla \tilde{v}_n|^2 \rightharpoonup \delta_{x_0} \\
 &\text{weakly in the sense of measures,} \\
 &|\tilde{u}_n|^{\alpha} |\tilde{v}_n|^{\beta} \rightharpoonup S_{\alpha,\beta}^{-N/(N-2)} \delta_{x_0} \\
 &\text{weakly in the sense of measures.}
 \end{aligned} \tag{109}$$

Observe that $Q_i(\tilde{z}_n) = Q_i(z_n) = r_0/3$;

$$\begin{aligned}
 \frac{r_0}{3} &= \lim_{n \rightarrow \infty} Q_i(z_n) \\
 &= \lim_{n \rightarrow \infty} \frac{\int_{\Omega} \psi_i(x) (|\nabla \tilde{u}_n|^2 + |\nabla \tilde{v}_n|^2) dx}{\int_{\Omega} (|\nabla \tilde{u}_n|^2 + |\nabla \tilde{v}_n|^2) dx} = \psi_i(x_0),
 \end{aligned} \tag{110}$$

which implies that $x_0 \neq a_i$ by the definition of $\psi_i(x)$. On the other hand, from (95) and (101), we get

$$\begin{aligned}
 S_{\alpha,\beta}^{-N/(N-2)} h(x_0) &= \lim_{n \rightarrow \infty} \int_{\Omega} h(x) |\tilde{u}_n|^{\alpha} |\tilde{v}_n|^{\beta} dx \\
 &= \lim_{n \rightarrow \infty} \frac{\int_{\Omega} h(x) |u_n|^{\alpha} |v_n|^{\beta} dx}{\|z_n\|^{2^*}} \\
 &= \lim_{n \rightarrow \infty} \frac{\int_{\Omega} |u_n|^{\alpha} |v_n|^{\beta} dx}{\|z_n\|^{2^*}} \\
 &= \lim_{n \rightarrow \infty} \int_{\Omega} |\tilde{u}_n|^{\alpha} |\tilde{v}_n|^{\beta} dx \\
 &= S_{\alpha,\beta}^{-N/(N-2)},
 \end{aligned} \tag{111}$$

which is impossible, because $h(x)$ is not a constant function by condition (H2). $\square$

Throughout this section, take $\Lambda^* = \min\{\Lambda_2, \min_{1 \leq i \leq k} \tilde{\Lambda}_i\}$; Λ_2 and $\tilde{\Lambda}_i$ are as in Lemmas 14 and 16. Using the idea of Tarantello [12], we have the following results. For $z = (u, v)$, $\varphi = (\varphi_1, \varphi_2) \in E$, we define

$$\begin{aligned}
 z - \varphi &= (u - \varphi_1, v - \varphi_2), \\
 \langle z, \varphi \rangle &= \int_{\Omega} (\nabla u \nabla \varphi_1 + \nabla v \nabla \varphi_2) dx,
 \end{aligned}$$

$$G_{\lambda,\mu}(z, \varphi) = \int_{\Omega} (\lambda f(x) |u|^{q-2} u \varphi_1 + \mu g(x) |v|^{q-2} v \varphi_2) dx,$$

$$\begin{aligned}
 H(z, \varphi) &= \frac{\alpha}{\alpha + \beta} \int_{\Omega} h(x) |u|^{\alpha-2} u |v|^{\beta} \varphi_1 dx \\
 &\quad + \frac{\beta}{\alpha + \beta} \int_{\Omega} h(x) |u|^{\alpha} |v|^{\beta-2} v \varphi_2 dx.
 \end{aligned} \tag{112}$$

Lemma 17. For each $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda^*$ and $z = (u, v) \in \mathcal{N}_{\lambda,\mu}^i$ ($1 \leq i \leq k$), there exist $\epsilon > 0$ and a differentiable function $\xi : B_{\epsilon}(0) \subset E \rightarrow \mathbb{R}^+$ such that $\xi(0) = 1$, $\xi(\phi)(z - \phi) \in \mathcal{N}_{\lambda,\mu}^i$ for all $\phi \in B_{\epsilon}(0)$ and

$$\langle \xi'(0), \varphi \rangle = \frac{2 \langle z, \varphi \rangle - q G_{\lambda,\mu}(z, \varphi) - 2^* H(z, \varphi)}{(2 - q) \|z\|_E^2 - (2^* - q) H(z, z)}, \tag{113}$$

for all $\varphi = (\varphi_1, \varphi_2) \in E$.

Proof. For $z \in \mathcal{N}_{\lambda,\mu}^i$, define a function $F_z : \mathbb{R} \times E \rightarrow \mathbb{R}$ by

$$\begin{aligned}
 F_z(\xi, \phi) &= \langle I'_{\lambda,\mu}(\xi(z - \phi)), \xi(z - \phi) \rangle \\
 &= \xi^2 \|z - \phi\|^2 - \xi^q G_{\lambda,\mu}(z - \phi, z - \phi) \\
 &\quad - \xi^{\alpha+\beta} H(z - \phi, z - \phi).
 \end{aligned} \tag{114}$$

Then, $F_u(1, 0) = \langle I'_{\lambda,\mu}(z), z \rangle = 0$ and

$$\begin{aligned}
 \frac{d}{d\xi} F_z(1, 0) &= 2 \|z\|_E^2 - q G_{\lambda,\mu}(z, z) - (\alpha + \beta) H(z, z) \\
 &= (2 - q) \|z\|_E^2 - (2^* - q) H(z, z) < 0.
 \end{aligned} \tag{115}$$

According to the implicit function theorem, there exist $\epsilon > 0$ and a differentiable function $\xi : B_\epsilon(0) \subset E \rightarrow \mathbb{R}$ such that $\xi(0) = 1$;

$$\langle \xi'(0), \varphi \rangle = \frac{2 \langle z, \varphi \rangle - q G_{\lambda, \mu}(z, \varphi) - 2^* H(z, \varphi)}{(2 - q) \|z\|_E^2 - (2^* - q) H(z, z)}, \quad (116)$$

$$F_z(\xi(\varphi), \varphi) = 0 \quad \forall \varphi \in B_\epsilon(0),$$

which is equivalent to

$$\langle I'_{\lambda, \mu}(\xi(\varphi)(z - \varphi)), \xi(\varphi)(z - \varphi) \rangle = 0 \quad \forall \varphi \in B(0; \epsilon); \quad (117)$$

that is, $\xi(\varphi)(z - \varphi) \in \mathcal{N}_{\lambda, \mu}$ for all $\varphi \in B_\epsilon(0)$. Furthermore, by the continuity of the functions ξ and Q_i , we have that

$$(2 - q) \|\xi(\varphi)(z - \varphi)\|^2 - (2^* - q) H(\xi(\varphi)(z - \varphi), \xi(\varphi)(z - \varphi)) < 0, \quad (118)$$

$$Q_i(\xi(\varphi)(z - \varphi)) < \frac{r_0}{3}$$

still holds if ϵ is sufficiently small. This implies that $\xi(\varphi)(z - \varphi) \in \mathcal{N}_{\lambda, \mu}^i$. $\square$

Proposition 18. *If $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda^*$, then there exists a $(PS)_{\theta_{\lambda, \mu}^i}$ -sequence $\{z_n^i\} \subset \mathcal{N}_{\lambda, \mu}^i$ in E for $I_{\lambda, \mu}$.*

Proof. If $\overline{\mathcal{N}_{\lambda, \mu}^i}$ denotes the closure of $\mathcal{N}_{\lambda, \mu}^i$, at first we note that $\overline{\mathcal{N}_{\lambda, \mu}^i} = \mathcal{N}_{\lambda, \mu}^i \cup \partial \mathcal{N}_{\lambda, \mu}^i$ for all $i = 1, 2, \dots, k$. It then follows from Lemmas 14 and 16, that

$$\theta_{\lambda, \mu}^i < \bar{\theta}_{\lambda, \mu}^i \quad \text{for } i = 1, 2, \dots, k, \quad 0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda^*. \quad (119)$$

Hence,

$$\theta_{\lambda, \mu}^i = \inf \{ I_{\lambda, \mu}(z) \mid z \in \overline{\mathcal{N}_{\lambda, \mu}^i} \} \quad \text{for } i = 1, 2, \dots, k. \quad (120)$$

Now, we fix $i \in \{1, 2, \dots, k\}$. Applying the Ekeland variational principle [26], there exists a minimizing sequence $\{z_n^i\} \subset \overline{\mathcal{N}_{\lambda, \mu}^i}$ such that

$$I_{\lambda, \mu}(z_n^i) < \theta_{\lambda, \mu}^i + \frac{1}{n},$$

$$I_{\lambda, \mu}(z_n^i) \leq I_{\lambda, \mu}(\varphi) + \frac{1}{n} \|\varphi - z_n^i\|_E \quad \text{for each } \varphi \in \overline{\mathcal{N}_{\lambda, \mu}^i}. \quad (121)$$

Using (119), we may assume that $z_n^i \in \mathcal{N}_{\lambda, \mu}^i$ for n sufficiently large. Applying Lemma 17 with $z = z_n^i$, we obtain the function $\xi_n : B_{\epsilon_n}(0) \rightarrow \mathbb{R}$ for some $\epsilon_n > 0$ such that $\xi_n(\varphi)(z_n^i - \varphi) \in \mathcal{N}_{\lambda, \mu}^i$ for all $\varphi \in B_{\epsilon_n}(0)$. Let $0 < \delta < \epsilon_n$ and $z \in E \setminus \{0\}$; we set

$$\varphi_\delta = \frac{\delta z}{\|z\|_E} \quad (122)$$

and $z_\delta = \xi_n(\varphi_\delta)(z_n^i - \varphi_\delta)$. Since $z_\delta \in \mathcal{N}_{\lambda, \mu}^i$, we deduce from (121) that

$$I_{\lambda, \mu}(z_\delta) - I_{\lambda, \mu}(z_n^i) \geq -\frac{1}{n} \|z_\delta - z_n^i\|_E. \quad (123)$$

By the mean-value theorem, we obtain

$$\langle I'_{\lambda, \mu}(z_n^i), (z_\delta - z_n^i) \rangle + o(\|z_\delta - z_n^i\|_E) \geq -\frac{1}{n} \|z_\delta - z_n^i\|_E. \quad (124)$$

Therefore,

$$\begin{aligned} & \langle I'_{\lambda, \mu}(z_n^i), -\varphi_\delta \rangle + (\xi_n(\varphi_\delta) - 1) \langle I'_{\lambda, \mu}(z_n^i), (z_n^i - \varphi_\delta) \rangle \\ & \geq -\frac{1}{n} \|z_\delta - z_n^i\|_E + o(\|z_\delta - z_n^i\|_E). \end{aligned} \quad (125)$$

Now, we observe that $\xi_n(\varphi_\delta)(z_n^i - \varphi_\delta) \in \mathcal{N}_{\lambda, \mu}^i$, and consequently we get from (125) that

$$\begin{aligned} & -\delta \left\langle I'_{\lambda, \mu}(z_n^i), \frac{z}{\|z\|_E} \right\rangle \\ & + \frac{(\xi_n(\varphi_\delta) - 1)}{\xi_n(\varphi_\delta)} \langle I'_{\lambda, \mu}(z_\delta), \xi_n(\varphi_\delta)(z_n^i - \varphi_\delta) \rangle \\ & + (\xi_n(\varphi_\delta) - 1) \langle I'_{\lambda, \mu}(z_n^i) - I'_{\lambda, \mu}(z_\delta), (z_n^i - \varphi_\delta) \rangle \\ & \geq -\frac{1}{n} \|z_\delta - z_n^i\|_E + o(\|z_\delta - z_n^i\|_E). \end{aligned} \quad (126)$$

Then, we write the pervious inequality in the following form:

$$\begin{aligned} & \left\langle I'_{\lambda, \mu}(z_n^i), \frac{z}{\|z\|_E} \right\rangle \\ & \leq \frac{\|z_\delta - z_n^i\|_E}{\delta n} + \frac{o(\|z_\delta - z_n^i\|_E)}{\delta} \\ & + \frac{(\xi_n(\varphi_\delta) - 1)}{\delta} \langle I'_{\lambda, \mu}(z_n^i) - I'_{\lambda, \mu}(z_\delta), (z_n^i - \varphi_\delta) \rangle. \end{aligned} \quad (127)$$

We can find a constant $C > 0$ independent of δ such that

$$\begin{aligned} & \|z_\delta - z_n^i\| \leq \delta + C(|\xi_n(\varphi_\delta) - 1|), \\ & \lim_{\delta \rightarrow 0} \frac{|\xi_n(\varphi_\delta) - 1|}{\delta} \leq \|\xi'_n(0)\| \leq C. \end{aligned} \quad (128)$$

For a fixed n , let $\delta \rightarrow 0$ in (127). Using the fact that

$$\lim_{\delta \rightarrow 0} \|z_\delta - z_n^i\|_E = 0, \quad (129)$$

we obtain

$$\left\langle I'_{\lambda, \mu}(z_n^i), \frac{z}{\|z\|_E} \right\rangle \leq \frac{C}{n}. \quad (130)$$

This implies

$$I_{\lambda,\mu}(z_n^i) = \theta_{\lambda,\mu}^i + o_n(1), \quad I'_{\lambda,\mu}(z_n^i) = o_n(1) \quad \text{in } E^{-1}. \quad (131)$$

□

Now, we complete the proof of Theorem 2. By Lemmas 12, 14 and Proposition 18, for all $\lambda^{2/(2-q)} + \mu^{2/(2-q)} \in (0, \Lambda^*)$, there exists a sequence $\{z_n^i\} \subset \mathcal{N}_{\lambda,\mu}^i$ and $z_0^i = (u_0^i, v_0^i) \in E$, $1 \leq i \leq k$, such that

$$\begin{aligned} I_{\lambda,\mu}(z_n^i) &= \theta_{\lambda,\mu}^i + o_n(1), \\ I'_{\lambda,\mu}(z_n^i) &= o_n(1) \quad \text{in } E^{-1}, \\ z_n^i &\longrightarrow z_0^i \quad \text{strongly in } E. \end{aligned} \quad (132)$$

Moreover, $\{z_n^i\} \subset \mathcal{N}_{\lambda,\mu}^-$, and by Lemma 7 (ii), we get $z_0^i \in \mathcal{N}_{\lambda,\mu}^-$, $u_0^i \not\equiv 0$, $v_0^i \not\equiv 0$ in Ω ,

$$\|z_0^i\|_E > \left(\frac{2-q}{2^*-q} \right)^{1/(2^*-2)} S^{N/4}, \quad (133)$$

$$\theta_{\lambda,\mu}^i \geq \theta_{\lambda,\mu}^- > 0 \quad \text{for } i = 1, 2, \dots, k.$$

Thus, z_0^i is a nontrivial solution of the problem $(P_{\lambda,\mu})$ and $I_{\lambda,\mu}(z_0^i) = \theta_{\lambda,\mu}^i$ for $i = 1, 2, \dots, k$. Set $u_+ = \max\{u, 0\}$ and $v_+ = \max\{v, 0\}$. Replace the terms $\int_{\Omega} h(x)|u|^{\alpha}|v|^{\beta}dx$ and $\int_{\Omega} (\lambda f(x)|u|^q + \mu g(x)|v|^q)dx$ of the functional $I_{\lambda,\mu}$ by $\int_{\Omega} h(x)u_+^{\alpha}v_+^{\beta}dx$ and $\int_{\Omega} (\lambda f(x)u_+^q + \mu g(x)v_+^q)dx$, respectively. It then follows that z_0^i is a nonnegative solution of the problem $(P_{\lambda,\mu})$. Applying the maximum principle [19], z_0^i is a positive solution of the problem $(P_{\lambda,\mu})$. Since $Q_i(z_0^i) < r_0/3$,

$$z_{\lambda,\mu}^1 \in \mathcal{N}_{\lambda,\mu}^+, \quad z_0^i \in \mathcal{N}_{\lambda,\mu}^i \subset \mathcal{N}_{\lambda,\mu}^- \quad \text{for } i = 1, 2, \dots, k, \quad (134)$$

where $z_{\lambda,\mu}^1$ is a positive solution of equation $(P_{\lambda,\mu})$ as in Theorem 1. From Lemma 13, we conclude that $\mathcal{N}_{\lambda,\mu}^i$ are disjoint for $i = 1, 2, \dots, k$. This implies that z_0^i ($1 \leq i \leq k$) and $z_{\lambda,\mu}^1$ are distinct positive solutions of the problem $(P_{\lambda,\mu})$.

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