Research Article

On a Kind of Dirichlet Character Sums

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Let $p \geq 3$ be a prime and let χ denote the Dirichlet character modulo p . For any prime q with $q < p$, define the set $E(q, p) = \{a \mid 1 \leq a, \bar{a} \leq p, a\bar{a} \equiv 1 \pmod{p} \text{ and } a \equiv \bar{a} \pmod{q}\}$. In this paper, we study a kind of mean value of Dirichlet character sums $\sum_{a \leq p} \sum_{a \in E(q, p)} \chi(a)$, and use the properties of the Dirichlet L -functions and generalized Kloosterman sums to obtain an interesting estimate.

1. Introduction

Let $k \geq 3$ be an integer and let χ denote the Dirichlet character modulo k , for any real number $x \geq 1$, many scholars have studied the following sums:

$$\sum_{n \leq x} \chi(n), \quad (1)$$

where n are positive integers.

Perhaps one of the most famous results is Pólya's inequality [1]. That is, when χ is the primitive character modulo k , we have

$$\sum_{n \leq x} \chi(n) < k^{1/2} \log k. \quad (2)$$

In fact, the result can be extended to the nonprincipal character χ modulo k [2]. Further details about the estimates of character sums can be found in the literature, for example, [3, 4].

For any fixed integer $H > 0$ and any positive integer $k \geq 3$, define the following set:

$$L(H, k) = \{a \mid 1 \leq a, \bar{a} \leq k-1, (a, k) = 1, a\bar{a} \equiv 1 \pmod{k}, |a - \bar{a}| \leq H\}, \quad (3)$$

let χ denote the Dirichlet character modulo k , define the sums as follows:

$$\sum_{\substack{n \leq k \\ n \in L(H, k)}} \chi(n). \quad (4)$$

Xi and Yi [5] studied the problem for χ the nonprincipal Dirichlet character modulo k , and got

$$\sum_{\substack{n \leq k \\ n \in L(H, k)}} \chi(n) \ll k^{1/2} d(k) \log H, \quad (5)$$

where $0 < H \leq q$ was a constant and $d(k)$ was the divisor function. Before this, Wenpeng [6] got an asymptotic formula for the case that χ was the principal Dirichlet character modulo k .

On the other hand, for each integer a with $1 \leq a \leq k$ and $(a, k) = 1$, we know that there exists one and only one b with $1 \leq b \leq k$ such that $ab \equiv 1 \pmod{k}$. Let $r_2(k)$ be the number of solutions of the congruent equation $ab \equiv 1 \pmod{k}$ for $1 \leq a, b \leq k$ in which a and b are of opposite parity, this can be expressed as follows:

$$r_2(k) = \sum_{\substack{a=1 \\ a\bar{a} \equiv 1 \pmod{k} \\ 2 \nmid (a+\bar{a})}}^k 1. \quad (6)$$

Richard [7] asks us to find $r_2(k)$ or at least to say something nontrivial about it. About this problem, a lot of scholars have studied it [8–12]. Now we let m be another integer with $m < k$ and let $r_m(k)$ denote the number of all pairs of integers a, b satisfying $ab \equiv 1 \pmod{k}$, $1 \leq a, b \leq k$,

and $m\ddagger(a+b)$. Lu and Yi [13] have obtained the asymptotic formula of generalized D. H. Lehmer problem as follows:

$$r_m(k) = \sum_{\substack{a=1 \\ a\bar{a}\equiv 1 \pmod{k} \\ m\ddagger(a+\bar{a})}}^k 1 = \left(1 - \frac{1}{m}\right) \phi(k) + O(k^{1/2} \log^2 k), \quad (7)$$

where the O constant only depends on m .

In this paper, let p be an odd prime and let q be a fixed prime with $q < p$, define the set $E(q, p)$ for $a(1 \leq a \leq p)$ such that $a\bar{a} \equiv 1 \pmod{p}$ and $a \equiv \bar{a} \pmod{q}$, that is,

$$E(q, p) = \{a \mid 1 \leq a, \bar{a} \leq p-1, \\ a\bar{a} \equiv 1 \pmod{p}, a \equiv \bar{a} \pmod{q}\}. \quad (8)$$

As another case of (7), we will consider the mean value of Dirichlet character sums as follows:

$$\sum_{\substack{a \leq p-1 \\ a \in E(q, p)}} \chi(a), \quad (9)$$

and get an interesting estimate. That is, we will prove the following theorem.

Theorem 1. Let p be an odd prime and let q be a fixed prime with $q < p$, and let χ denote the Dirichlet character modulo p . Let $E(q, p)$ denote the following set:

$$E(q, p) = \{a \mid 1 \leq a \leq p-1, \\ a\bar{a} \equiv 1 \pmod{p}, a \equiv \bar{a} \pmod{q}\}, \quad (10)$$

then, for any nonprincipal Dirichlet character $\chi \pmod{p}$, we have the following estimate:

$$\sum_{\substack{a \leq p-1 \\ a \in E(q, p)}} \chi(a) = O(p^{1/2+\epsilon}), \quad (11)$$

where the O constant only depends on q .

From this Theorem we can get

$$\sum_{\substack{a=1 \\ a\bar{a}\equiv 1 \pmod{p} \\ q\ddagger(a-\bar{a})}}^{p-1} \chi(a) = \sum_{\substack{a=1 \\ a\bar{a}\equiv 1 \pmod{p}}}^{p-1} \chi(a) - \sum_{\substack{a \leq p-1 \\ a \in E(q, p)}} \chi(a) \\ = O(p^{1/2+\epsilon}). \quad (12)$$

For any integer k and fixed integer m such that $(m, k) = 1$, whether or not there exists an estimate for

$$\sum_{\substack{a \leq k \\ a \in E(m, k)}} \chi(a) \quad (13)$$

is still an open problem.

2. Some Lemmas

In this section, we will give several lemmas which are necessary in the proof of the theorem.

Lemma 2. Let Q be an integer, and let χ be a primitive character modulo Q . Then, for any real number u and v with $u < v$, we have

$$\sum_{uQ < n \leq vQ} \chi(n) = \tau(\chi) \sum_{0 < |h| \leq H} \bar{\chi}(h) \frac{e(-hu) - e(-hv)}{2\pi i h} \\ + O\left(1 + \frac{Q \log Q}{H}\right), \quad (14)$$

where $e(x) = e^{2\pi i x}$ and $\tau(\chi) = \sum_{a=1}^Q \chi(a)e(a/Q)$ are Gauss sums.

Epecially, let $u = 0$, we have a slight modification

$$\sum_{0 < n \leq vQ} \chi(n) = \begin{cases} \frac{\tau(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n) \sin(2\pi n v)}{n} + O(1), & \text{if } \chi(-1) = 1, \\ \frac{\tau(\chi)}{\pi i} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n) (1 - \cos(2\pi n v))}{n} + O(1), & \text{if } \chi(-1) = -1. \end{cases} \quad (15)$$

Proof. (See [1]). □

Lemma 3. Let q be a prime, let Q be an integer with $Q > q$, and let χ be a primitive character modulo Q , then, we have

$$\sum_{n \leq Q/q} \chi(n) = \begin{cases} \frac{\tau(\chi)}{(q-1)\pi} \sum_{l=1}^{q-1} \sum_{\chi_2 \pmod{q}} \left(\chi_2(l) \sin \frac{2\pi l}{q} \right) \\ \times L(1, \chi \chi_2) + O(1), & \chi(-1) = 1, \\ \frac{\tau(\chi)}{(q-1)\pi i} \sum_{l=1}^{q-1} \sum_{\chi_2 \pmod{q}} \chi_2(l) \left(1 - \cos \frac{2\pi l}{q} \right) \\ \times L(1, \chi \chi_2) + O(1), & \chi(-1) = -1, \end{cases} \quad (16)$$

where $L(1, \chi)$ are the Dirichlet L -functions corresponding to χ .

Proof. From Lemma 2, we take $u = 0$, and $v = 1/q$ and get

$$\sum_{n \leq Q/q} \chi(n) = \tau(\chi) \sum_{0 < h \leq H} \bar{\chi}(h) \frac{1 - e(-h/q)}{2\pi i h} \\ + \tau(\chi) \sum_{-H < h \leq 0} \bar{\chi}(h) \frac{1 - e(-h/q)}{2\pi i h} \\ + O\left(1 + \frac{Q \log Q}{H}\right). \quad (17)$$

When $\chi(-1) = 1$, we have

$$\begin{aligned} \sum_{n \leq Q/q} \chi(n) &= \tau(\chi) \sum_{0 < h \leq H} \bar{\chi}(h) \frac{\sin(2\pi h/q)}{\pi h} \\ &\quad + O\left(1 + \frac{Q \log Q}{H}\right) \\ &= \frac{\tau(\chi)}{\pi} \sum_{l=1}^{q-1} \left(\sin \frac{2\pi l}{q}\right) \sum_{\substack{0 < h \leq H \\ h \equiv l \pmod{q}}} \frac{\bar{\chi}(h)}{h} \\ &\quad + O\left(1 + \frac{Q \log Q}{H}\right) \\ &= \frac{\tau(\chi)}{(q-1)\pi} \sum_{l=1}^{q-1} \left(\sin \frac{2\pi l}{q}\right) \sum_{0 < h \leq H} \frac{\bar{\chi}(h)}{h} \\ &\quad \times \sum_{\chi_2 \bmod q} \bar{\chi}_2(h) \chi_2(l) + O\left(1 + \frac{Q \log Q}{H}\right) \\ &= \frac{\tau(\chi)}{(q-1)\pi} \sum_{l=1}^{q-1} \sum_{\chi_2 \bmod q} \left(\chi_2(l) \sin \frac{2\pi l}{q}\right) \\ &\quad \times \sum_{0 < h \leq H} \frac{\bar{\chi}\bar{\chi}_2(h)}{h} + O\left(1 + \frac{Q \log Q}{H}\right). \end{aligned} \quad (18)$$

Let $H \rightarrow \infty$, then, we have

$$\begin{aligned} \sum_{n \leq Q/q} \chi(n) &= \frac{\tau(\chi)}{(q-1)\pi} \sum_{l=1}^{q-1} \sum_{\chi_2 \bmod q} \left(\chi_2(l) \sin \frac{2\pi l}{q}\right) L(1, \bar{\chi}\bar{\chi}_2) + O(1). \end{aligned} \quad (19)$$

The case of $\chi(-1) = -1$ can be treated in the same way. This proves Lemma 3. $\square$

Lemma 4. Let p, q be odd primes and let χ_1, χ_2 be the Dirichlet characters modulo p and q , respectively, such that $(p, q) = 1$, denote $\chi = \chi_1\chi_2$, $k = pq$, and χ is the Dirichlet character modulo pq , the famous Gauss sums are defined as follows:

$$G(n, \chi) = \sum_{a=1}^k \chi(a) e\left(\frac{na}{k}\right), \quad (20)$$

where $e(y) = e^{2\pi i y}$. Hence, we have

$$G(n, \chi) = \chi_1(q) \chi_2(p) G(n, \chi_1) G(n, \chi_2). \quad (21)$$

When $n = 1$, we denote $\tau(\chi) = G(1, \chi)$; therefore, we have

$$\tau(\chi) = \chi_1(q) \chi_2(p) \tau(\chi_1) \tau(\chi_2). \quad (22)$$

Proof. (See [14]). $\square$

Lemma 5. Let m , and n be integers and let $q \geq 3$ be prime, let χ denote the Dirichlet character modulo q , the generalized Kloosterman sums are defined by

$$S_\chi(m, n; q) = \sum_{a \bmod q} \chi(a) e\left(\frac{m\bar{a} + na}{q}\right), \quad (23)$$

where $a\bar{a} \equiv 1 \pmod{q}$ and $e(y) = e^{2\pi i y}$.

Then, we have the following estimate:

$$S_\chi(m, n; q) \ll q^{1/2+\epsilon}(m, n, q)^{1/2}, \quad (24)$$

where (m, n, q) denotes the gcd of m, n , and q .

Proof. (See [15]). $\square$

Lemma 6. Let $p \geq 3$ be an odd prime, let χ, χ_1 be a Dirichlet character modulo p , and $\chi\chi_1 \neq \chi_0^p$. For any odd prime q with $q < p$, let χ_2, χ_3, χ_4 be any Dirichlet characters with $\chi_2 \bmod q$, $\chi_3 \bmod q$ and $\chi_4 \bmod q$, respectively, then, no matter χ is odd character or even character modulo p , we have

$$\sum_{\substack{\chi_1 \neq \chi_0^p \\ \chi_1\chi_2(-1)=1}} \chi\chi_1(q) \chi_1(q) \tau(\chi\chi_1) \tau(\chi_1) \quad (25)$$

$$\times L(1, \bar{\chi}\bar{\chi}_1\bar{\chi}_2\bar{\chi}_3) L(1, \bar{\chi}_1\bar{\chi}_2\bar{\chi}_4) \ll p^{3/2+\epsilon};$$

$$\sum_{\substack{\chi_1 \neq \chi_0^p \\ \chi_1\chi_2(-1)=-1}} \chi\chi_1(q) \chi_1(q) \tau(\chi\chi_1) \tau(\chi_1) \quad (26)$$

$$\times L(1, \bar{\chi}\bar{\chi}_1\bar{\chi}_2\bar{\chi}_3) L(1, \bar{\chi}_1\bar{\chi}_2\bar{\chi}_4) \ll p^{3/2+\epsilon};$$

$$\sum_{\substack{\chi_1 \neq \chi_0^p \\ \chi(-1)=1}} \chi\chi_1(q) \chi_1(q) \tau(\chi\chi_1) \tau(\chi_1) \quad (27)$$

$$\times L(1, \bar{\chi}\bar{\chi}_1\bar{\chi}_3) L(1, \bar{\chi}_1\bar{\chi}_4) \ll p^{3/2+\epsilon};$$

$$\sum_{\substack{\chi_1 \neq \chi_0^p \\ \chi(-1)=-1}} \chi\chi_1(q) \chi_1(q) \tau(\chi\chi_1) \tau(\chi_1) \quad (28)$$

$$\times L(1, \bar{\chi}\bar{\chi}_1\bar{\chi}_3) L(1, \bar{\chi}_1\bar{\chi}_4) \ll p^{3/2+\epsilon},$$

where the $\ll$ constant only depends on q .

Proof. For any integer n with $(n, k) = 1$ ($k \geq 3$ is any positive integer), we have

$$\begin{aligned} \sum_{\substack{\chi \bmod k \\ \chi(-1)=1}} \chi(n) &= \begin{cases} \frac{1}{2} \phi(k), & n \equiv \pm 1 \pmod{k}, \\ 0, & \text{otherwise.} \end{cases} \\ \sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}} \chi(n) &= \begin{cases} \frac{1}{2} \phi(k), & n \equiv 1 \pmod{k}, \\ -\frac{1}{2} \phi(k), & n \equiv -1 \pmod{k}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (29)$$

Now let $y > p$ and let $A(y, \chi) = \sum_{p < n \leq y} \chi(n)$. Then, from the Pólya-Vinogradov inequality, we obtain

$$A(y, \chi) \ll \sqrt{p} \log p. \quad (30)$$

Hence, from Abel's identity, for any $\operatorname{Re}(s) \geq 1$, we can easily get

$$\begin{aligned} L(s, \chi) &= \sum_{n \leq p} \frac{\chi(n)}{n^s} + \int_p^\infty \frac{A(y, \chi)}{y^{s+1}} dy \\ &= \sum_{n \leq p} \frac{\chi(n)}{n^s} + O\left(\frac{\log p}{p^{s-1/2}}\right) \\ &= \sum_{n \leq p} \frac{\chi(n)}{n^s} + O\left(\frac{\log p}{p^{1/2}}\right). \end{aligned} \quad (31)$$

We will take (25), for example, to prove this lemma. For $(q, p) = 1$, from the definition of $\tau(\chi)$ and (31), since χ_1 is not the principle character modulo p and $\chi\chi_1$ is not the principle character modulo p , we have

$$\begin{aligned} &\sum_{\substack{\chi_1 \neq \chi_p^0 \\ \chi_1 \chi_2(-1)=1}} \chi\chi_1(q) \chi_1(q) \tau(\chi\chi_1) \tau(\chi_1) \\ &\quad \times L(1, \overline{\chi\chi_1\chi_2\chi_3}) L(1, \overline{\chi_1\chi_2\chi_4}) \\ &= \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \chi_2(-1)=1}} \chi\chi_1(q) \chi_1(q) \tau(\chi\chi_1) \tau(\chi_1) \\ &\quad \times \left(\sum_{k \leq p} \frac{\overline{\chi\chi_1\chi_2\chi_3}(k)}{k} + O\left(\frac{\log pq}{(pq)^{1/2}}\right) \right) \\ &\quad \times \left(\sum_{n \leq p} \frac{\overline{\chi_1\chi_2\chi_4}(n)}{n} + O\left(\frac{\log pq}{(pq)^{1/2}}\right) \right) \\ &\quad - \chi(q) \tau(\chi) \tau(\chi_0^p) L(1, \overline{\chi\chi_2\chi_3}) L(1, \chi_2\chi_4) \\ &= \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \chi_2(-1)=1}} \chi\chi_1(q) \chi_1(q) \tau(\chi\chi_1) \tau(\chi_1) \\ &\quad \times \sum_{k \leq p} \frac{\overline{\chi\chi_1\chi_2\chi_3}(k)}{k} \sum_{n \leq p} \frac{\overline{\chi_1\chi_2\chi_4}(n)}{n} + O(p^{3/2} \log^2 p) \\ &= \sum_{k \leq p} \sum_{n \leq p} \frac{\overline{\chi\chi_1\chi_3}(k) \chi_2\chi_4(n) \chi(qa)}{kn} \\ &\quad \times \sum_{a=1}^p \sum_{b=1}^p \chi(a) e\left(\frac{a+b}{p}\right) \\ &\quad \times \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \chi_2(-1)=1}} \chi_1(q^2 ab) \overline{\chi_1}(kn) + O(p^{3/2} \log^2 p) \end{aligned}$$

$$\begin{aligned} &= \frac{\phi(p)}{2} \sum_{k \leq q} \sum_{n \leq p} \frac{\overline{\chi\chi_1\chi_3}(k) \chi_2\chi_4(n) \chi(qa)}{kn} \\ &\quad \times \left(\sum_{a=1}^p \sum_{b=1}^p \chi(a) e\left(\frac{a+b}{p}\right) \right. \\ &\quad \left. \pm \sum_{a=1}^p \sum_{b=1}^p \chi(a) e\left(\frac{a+b}{p}\right) \right) \\ &\quad + O(p^{3/2} \log^2 p) \\ &= \frac{\phi(p)}{2} \sum_{k \leq p} \sum_{n \leq p} \frac{\overline{\chi\chi_1\chi_3}(k) \chi_2\chi_4(n) \chi(qa)}{kn} \\ &\quad \times (S_\chi(1, \overline{q^2 kn}; p) \pm S_\chi(1, -\overline{q^2 kn}; p)) \\ &\quad + O(p^{3/2} \log^2 p), \end{aligned} \quad (32)$$

where $abq^2 \overline{kn} \equiv 1 \pmod{p}$. From Lemma 5, we can easily obtain

$$\begin{aligned} &\sum_{\substack{\chi_1 \neq \chi_p^0 \\ \chi_1 \chi_2(-1)=1}} \chi\chi_1(q) \chi_1(q) \tau(\chi\chi_1) \tau(\chi_1) L(1, \overline{\chi\chi_1\chi_2\chi_3}) \\ &\quad \times L(1, \overline{\chi_1\chi_2\chi_4}) \ll p^{3/2+\epsilon_1} \log^2 p \ll p^{3/2+\epsilon}, \end{aligned} \quad (33)$$

where the $\ll$ constant only depends on q . Therefore, this completes the proof of Lemma 6. $\square$

Lemma 7. Let q be a fixed odd prime and let p be a prime with $p > q$, let χ denote the nonprincipal Dirichlet character modulo p , and χ_1 denote the Dirichlet character modulo p , then, we have

$$\begin{aligned} &\sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \chi\chi_1(q) \chi_1(q) \\ &\quad \times \sum_{a \leq (p-1)/q} \chi\chi_1(a) \sum_{b \leq (p-1)/q} \chi_1(b) \ll p^{3/2+\epsilon}, \end{aligned} \quad (34)$$

where the $\ll$ constant only depends on q .

Proof. For primes p and q , χ_1 and $\chi\chi_1$ are nonprincipal and primitive characters modulo p , hence, from Lemmas 3 and 6, when $\chi(-1) = 1$, we can get

$$\begin{aligned}
& \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \chi\chi_1(q) \chi_1(q) \sum_{a \leq (p-1)/q} \chi\chi_1(a) \sum_{b \leq (p-1)/q} \chi_1(b) \\
&= \sum_{\substack{\chi_1(-1)=1 \\ \chi_1 \neq \chi_p^0}} \chi\chi_1(q) \chi_1(q) \\
&\quad \times \left(\frac{\tau(\chi\chi_1)}{(q-1)\pi} \sum_{u=1}^{q-1} \sum_{\chi_3 \bmod q} \left(\chi_3(u) \sin \frac{2\pi u}{q} \right) \right. \\
&\quad \times L(1, \overline{\chi\chi_1\chi_3}) + O(1) \Big) \\
&\quad \times \left(\frac{\tau(\chi_1)}{(q-1)\pi} \sum_{v=1}^{q-1} \sum_{\chi_4 \bmod q} \left(\chi_4(v) \sin \frac{2\pi v}{q} \right) \right. \\
&\quad \times L(1, \overline{\chi_1\chi_4}) + O(1) \Big) \\
&+ \sum_{\substack{\chi_1(-1)=-1 \\ \chi_1 \neq \chi_p^0}} \chi\chi_1(q) \chi_1(q) \\
&\quad \times \left(\frac{\tau(\chi\chi_1)}{(q-1)\pi i} \sum_{u=1}^{q-1} \sum_{\chi_3 \bmod q} \left(\chi_3(u) \left(1 - \cos \frac{2\pi u}{q} \right) \right) \right. \\
&\quad \times L(1, \overline{\chi\chi_1\chi_3}) + O(1) \Big) \\
&\quad \times \left(\frac{\tau(\chi_1)}{(q-1)\pi i} \sum_{v=1}^{q-1} \sum_{\chi_4 \bmod q} \left(\chi_4(v) \left(1 - \cos \frac{2\pi v}{q} \right) \right) \right. \\
&\quad \times L(1, \overline{\chi_1\chi_4}) + O(1) \Big) \\
&= \frac{1}{(q-1)^2 \pi^2} \\
&\quad \times \sum_{u=1}^{q-1} \sum_{v=1}^{q-1} \sum_{\chi_3 \bmod q} \sum_{\chi_4 \bmod q} \chi_3(u) \chi_4(v) \sin \frac{2\pi u}{q} \sin \frac{2\pi v}{q} \\
&\quad \times \sum_{\substack{\chi_1 \neq \chi_p^0 \\ \chi_1(-1)=1}} \chi\chi_1(q) \chi_1(m) \tau(\chi\chi_1) \tau(\chi_1) \\
&\quad \times L(1, \overline{\chi\chi_1\chi_3}) L(1, \overline{\chi_1\chi_4}) \\
&\quad + \frac{-1}{(q-1)^2 \pi^2}
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{u=1}^{q-1} \sum_{v=1}^{q-1} \sum_{\chi_3 \bmod q} \sum_{\chi_4 \bmod m} \chi_3(u) \chi_4(v) \\
&\quad \times \left(1 - \cos \frac{2\pi u}{q} \right) \left(1 - \cos \frac{2\pi v}{q} \right) \\
&\quad \times \sum_{\substack{\chi_1 \neq \chi_p^0 \\ \chi_1(-1)=-1}} \chi\chi_1(q) \chi_1(q) \tau(\chi\chi_1) \tau(\chi_1) \\
&\quad \times L(1, \overline{\chi\chi_1\chi_3}) L(1, \overline{\chi_1\chi_4}) + O(p^{1/2+\epsilon}) \\
&\ll p^{3/2+\epsilon},
\end{aligned} \tag{35}$$

where the $\ll$ constant is only concerned with q .

When $\chi(-1) = -1$, by the similar method, we can also obtain

$$\begin{aligned}
& \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \chi\chi_1(q) \chi_1(q) \sum_{a \leq (p-1)/q} \chi\chi_1(a) \sum_{b \leq (p-1)/q} \chi_1(b) \\
&\ll p^{3/2+\epsilon},
\end{aligned} \tag{36}$$

where the $\ll$ constant is only concerned with q . Combining (35) and (36), we can obtain Lemma 7. This completes the proof of Lemma 7. $\square$

Lemma 8. Let q be a fixed odd prime and let p be a prime with $p > q$, let χ denote the nonprincipal Dirichlet character modulo p , let χ_1, χ_2 denote the Dirichlet character modulo p, q respectively, then, we have

$$\sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\substack{a \leq p-1 \\ q \nmid a}} \sum_{\substack{b \leq p-1 \\ q \nmid b}} \chi\chi_1\chi_2(a) \chi_1\bar{\chi}_2(b) \ll p^{3/2+\epsilon}, \tag{37}$$

where the $\ll$ constant is only concerned with q .

Proof. According to the properties of Dirichlet character, we can get

$$\begin{aligned}
& \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\substack{a \leq p-1 \\ q \nmid a}} \sum_{\substack{b \leq p-1 \\ q \nmid b}} \chi\chi_1\chi_2(a) \chi_1\bar{\chi}_2(b) \\
&= \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \left(\sum_{a \leq p-1} \chi\chi_1\chi_2(a) - \sum_{\substack{a \leq p-1 \\ q \mid a}} \chi\chi_1\chi_2(a) \right) \\
&\quad \times \left(\sum_{b \leq p-1} \chi_1\bar{\chi}_2(b) - \sum_{\substack{b \leq p-1 \\ q \mid b}} \chi_1\bar{\chi}_2(b) \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \left(\sum_{a \leq p-1} \chi \chi_1 \chi_2 (a) - \chi \chi_1 \chi_2 (q) \right. \\
&\quad \left. \times \sum_{a \leq (p-1)/q} \chi \chi_1 \chi_2 (a) \right) \\
&\quad \times \left(\sum_{b \leq p-1} \chi_1 \bar{\chi}_2 (b) - \chi_1 \bar{\chi}_2 (q) \sum_{b \leq (p-1)/q} \chi_1 \bar{\chi}_2 (b) \right) \\
&= \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{a \leq p-1} \chi \chi_1 \chi_2 (a) \sum_{b \leq p-1} \chi_1 \bar{\chi}_2 (b).
\end{aligned} \tag{38}$$

For primes p and q , let $\chi' = \chi \chi_1$ be a nonprincipal and primitive character modulo p , χ_2 is also a primitive character modulo q , so $\chi \chi_1 \chi_2$ is a primitive character modulo pq ; therefore, from Lemmas 3, 4, and 6 and from (38), it is clear that when $\chi(-1) = 1$, we can obtain

$$\begin{aligned}
&\sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{a \leq p-1} \sum_{b \leq p-1} \chi \chi_1 \chi_2 (a) \chi_1 \bar{\chi}_2 (b) \\
&= \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\substack{a \leq p-1 \\ \chi_1 \chi_2(-1)=1}} \chi \chi_1 \chi_2 (a) \chi_1 \bar{\chi}_2 (b) \\
&\quad \times \left(\frac{\tau(\chi \chi_1 \chi_2)}{(q-1)\pi} \sum_{u=1}^{q-1} \sum_{\chi_3 \bmod q} \left(\chi_3 (u) \sin \frac{2\pi u}{q} \right) \right. \\
&\quad \left. \times L(1, \overline{\chi \chi_1 \chi_2 \chi_3}) + O(1) \right) \\
&\quad \times \left(\frac{\tau(\chi_1 \bar{\chi}_2)}{(q-1)\pi} \sum_{v=1}^{q-1} \sum_{\chi_4 \bmod q} \left(\chi_4 (v) \sin \frac{2\pi v}{q} \right) \right. \\
&\quad \left. \times L(1, \overline{\chi_1 \chi_2 \chi_4}) + O(1) \right) \\
&\quad + \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\substack{a \leq p-1 \\ \chi_1 \chi_2(-1)=-1}} \chi \chi_1 \chi_2 (a) \chi_1 \bar{\chi}_2 (b) \\
&\quad \times \left(\frac{\tau(\chi \chi_1 \chi_2)}{(q-1)\pi i} \sum_{u=1}^{q-1} \sum_{\chi_3 \bmod q} \left(\chi_3 (u) \left(1 - \cos \frac{2\pi u}{q} \right) \right) \right. \\
&\quad \left. \times L(1, \overline{\chi \chi_1 \chi_2 \chi_3}) + O(1) \right)
\end{aligned}$$

$$\begin{aligned}
&\times \left(\frac{\tau(\chi_1 \bar{\chi}_2)}{(q-1)\pi i} \sum_{v=1}^{q-1} \sum_{\chi_4 \bmod q} \left(\chi_4 (v) \left(1 - \cos \frac{2\pi v}{q} \right) \right) \right. \\
&\quad \left. \times L(1, \overline{\chi_1 \chi_2 \chi_4}) + O(1) \right) \\
&= \frac{1}{(q-1)^2 \pi^2} \sum_{u=1}^{q-1} \sum_{v=1}^{q-1} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\chi_3 \bmod q} \sum_{\chi_4 \bmod q} \\
&\quad \times \chi_2 (p) \tau(\chi_2) \bar{\chi}_2 (p) \tau(\bar{\chi}_2) \chi_3 (u) \chi_4 (v) \sin \frac{2\pi u}{q} \\
&\quad \times \sin \frac{2\pi v}{q} \sum_{\substack{\chi_1 \neq \chi_p^0 \\ \chi_1 \chi_2(-1)=1}} \chi \chi_1 (q) \chi_1 (q) \tau(\chi \chi_1) \tau(\chi_1) \\
&\quad \times L(1, \overline{\chi \chi_1 \chi_2 \chi_3}) L(1, \overline{\chi_1 \chi_2 \chi_4}) \\
&\quad + \frac{-1}{(q-1)^2 \pi^2} \sum_{u=1}^{q-1} \sum_{v=1}^{q-1} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\chi_3 \bmod q} \sum_{\chi_4 \bmod q} \\
&\quad \times \chi_2 (p) \tau(\chi_2) \bar{\chi}_2 (p) \tau(\bar{\chi}_2) \chi_3 (u) \chi_4 (v) \\
&\quad \times \left(1 - \cos \frac{2\pi u}{q} \right) \left(1 - \cos \frac{2\pi v}{q} \right) \\
&\quad \times \sum_{\substack{\chi_1 \neq \chi_p^0 \\ \chi_1 \chi_2(-1)=-1}} \chi \chi_1 (q) \chi_1 (q) \tau(\chi \chi_1) \tau(\chi_1) \\
&\quad \times L(1, \overline{\chi \chi_1 \chi_2 \chi_3}) L(1, \overline{\chi_1 \chi_2 \chi_4}) \\
&\ll p^{3/2+\epsilon},
\end{aligned} \tag{39}$$

where the $\ll$ constant is only concerned with q .

When $\chi(-1) = -1$, in the similar way, we can also obtain

$$\sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{a \leq p-1} \sum_{b \leq p-1} \chi \chi_1 \chi_2 (a) \chi_1 \bar{\chi}_2 (b) \ll p^{3/2+\epsilon}, \tag{40}$$

where the $\ll$ constant is only concerned with q .

Therefore, from (39) and (40), we can easily get Lemma 8. This completes the proof of Lemma 8. $\square$

3. Proof of Theorem

In this section, we will complete the proof of the theorem. According to the orthogonality relation for character sums, we have

$$\begin{aligned}
\sum_{\substack{a \leq p \\ a \in E(q,p)}} \chi(a) &= \sum_{b=1}^{p-1} \sum_{a=1}^{p-1} \chi(a) \\
&\quad \begin{matrix} a \equiv b \pmod{q} \\ ab \equiv 1 \pmod{p} \end{matrix} \\
&= \frac{1}{p-1} \sum_{\chi_1 \bmod p} \sum_{b=1}^{p-1} \sum_{a=1}^{p-1} \chi \chi_1 (a) \chi_1 (b) \\
&\quad \begin{matrix} b \equiv 1 \pmod{q} \\ a \equiv b \pmod{q} \end{matrix}
\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{p-1} \sum_{a=1}^{p-1} \sum_{\substack{b=1 \\ a \equiv b \pmod{q}}}^{p-1} \chi(a) \\
 &\quad + \frac{1}{p-1} \sum_{\substack{\chi_1 \pmod{p} \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{a=1 \\ a \equiv b \pmod{q}}}^{p-1} \sum_{b=1}^{p-1} \chi \chi_1(a) \chi_1(b) \\
 &= \frac{1}{p-1} \sum_{a=1}^{p-1} \chi(a) \left(\sum_{\substack{b=1 \\ b \equiv a \pmod{q}}}^{p-1} 1 \right) \\
 &\quad + \frac{1}{p-1} \sum_{\substack{\chi_1 \pmod{p} \\ \chi_1 \neq \chi_p^0}} \sum_{a=1}^{p-1} \sum_{\substack{b=1 \\ q|a, q|b}}^{p-1} \chi \chi_1(a) \chi_1(b) \\
 &\quad + \frac{1}{p-1} \sum_{\substack{\chi_1 \pmod{p} \\ \chi_1 \neq \chi_p^0}} \sum_{a=1}^{p-1} \sum_{\substack{b=1 \\ q \nmid a, q \nmid b}}^{p-1} \chi \chi_1(a) \chi_1(b) \\
 &= \frac{1}{p-1} \sum_{a=1}^{p-1} \chi(a) \left(\frac{p}{q} + O(1) \right) \\
 &\quad + \frac{1}{p-1} \sum_{\substack{\chi_1 \pmod{p} \\ \chi_1 \neq \chi_p^0}} \chi \chi_1(q) \chi_1(q) \\
 &\quad \times \sum_{a \leq (p-1)/q} \chi \chi_1(a) \sum_{b \leq (p-1)/q} \chi_1(b) \\
 &\quad + \frac{1}{(p-1)\phi(q)} \\
 &\quad \times \sum_{\substack{\chi_1 \pmod{p} \\ \chi_1 \neq \chi_p^0}} \sum_{\chi_2 \pmod{q}} \sum_{a \leq p-1} \sum_{\substack{b \leq p-1 \\ q \nmid a, q \nmid b}} \chi \chi_1(a) \chi_1(b) \\
 &\quad \times \chi_2(a) \bar{\chi}_2(b) \\
 &= \frac{p}{(p-1)q} \sum_{a=1}^{p-1} \chi(a) + O(1) \\
 &\quad + \frac{1}{p-1} \sum_{\substack{\chi_1 \pmod{p} \\ \chi_1 \neq \chi_p^0}} \chi \chi_1(q) \chi_1(q) \\
 &\quad \times \sum_{a \leq (p-1)/q} \chi \chi_1(a) \sum_{b \leq (p-1)/q} \chi_1(b) \\
 &\quad + \frac{1}{(p-1)\phi(q)} \\
 &\quad \times \sum_{\substack{\chi_1 \pmod{p} \\ \chi_1 \neq \chi_p^0}} \sum_{a \leq p-1} \sum_{\substack{b \leq p-1 \\ m \nmid a, q \nmid b}} \chi \chi_1(a) \chi_1(b) \\
 &\quad + \frac{1}{(p-1)\phi(q)}
 \end{aligned}$$

$$\begin{aligned}
 &\times \sum_{\substack{\chi_1 \pmod{p} \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \pmod{q} \\ \chi_2 \neq \chi_q^0}} \sum_{a \leq p-1} \sum_{\substack{b \leq p-1 \\ q \nmid a, q \nmid b}} \chi \chi_1(a) \chi_1(b) \\
 &\quad \times \chi_2(a) \bar{\chi}_2(b) \\
 &= O(1) + \frac{1}{p-1} \sum_{\substack{\chi_1 \pmod{p} \\ \chi_1 \neq \chi_p^0}} \chi \chi_1(q) \chi_1(q) \\
 &\quad \times \sum_{a \leq (p-1)/q} \chi \chi_1(a) \sum_{b \leq (p-1)/q} \chi_1(b) \\
 &\quad + \frac{1}{(p-1)\phi(q)} \\
 &\quad \times \sum_{\substack{\chi_1 \pmod{p} \\ \chi_1 \neq \chi_p^0}} \sum_{a \leq p-1} \sum_{\substack{b \leq p-1 \\ q \nmid a, q \nmid b}} \chi \chi_1(a) \chi_1(b) \\
 &\quad + \frac{1}{(p-1)\phi(q)} \\
 &\quad \times \sum_{\substack{\chi_1 \pmod{p} \\ \chi_1 \neq \chi_p^0}} \sum_{\chi_2 \pmod{q}} \sum_{a \leq p-1} \sum_{\substack{b \leq p-1 \\ q \nmid a, q \nmid b}} \chi \chi_1 \chi_2(a) \chi_1 \bar{\chi}_2(b).
 \end{aligned} \tag{41}$$

Note that for any nonprincipal Dirichlet character χ modulo k ($k \geq 3$ is a positive integer), we have $\sum_{n=1}^k \chi(n) = 0$, hence, we obtain

$$\begin{aligned}
 &\sum_{\substack{a \leq p-1 \\ q \nmid a}} \sum_{\substack{b \leq p-1 \\ q \nmid b}} \chi \chi_1(a) \chi_1(b) \\
 &= \left(\sum_{a \leq p-1} \chi \chi_1(a) - \sum_{\substack{a \leq p-1 \\ q|a}} \chi \chi_1(a) \right) \\
 &\quad \times \left(\sum_{b \leq p-1} \chi_1(b) - \sum_{\substack{b \leq p-1 \\ q|b}} \chi_1(b) \right) \\
 &= \sum_{\substack{a \leq p-1 \\ q|a}} \chi \chi_1(a) \sum_{\substack{b \leq p-1 \\ q|b}} \chi_1(b) \\
 &= \chi \chi_1(q) \chi_1(q) \sum_{a \leq (p-1)/q} \chi \chi_1(a) \sum_{b \leq (p-1)/q} \chi_1(b).
 \end{aligned} \tag{42}$$

From (41) and (42) and Lemmas 7 and 8, we get

$$\begin{aligned}
 &\sum_{\substack{a \leq p \\ a \in E(q,p)}} \chi(a) \\
 &= \frac{1}{p-1} \left(1 + \frac{1}{\phi(q)} \right)
 \end{aligned}$$

$$\begin{aligned}
& \times \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \chi \chi_1(q) \chi_1(q) \sum_{a \leq (p-1)/q} \chi \chi_1(a) \sum_{b \leq (p-1)/q} \chi_1(b) \\
& + \frac{1}{(p-1)\phi(q)} \\
& \times \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \neq \chi_p^0}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \neq \chi_q^0}} \sum_{\substack{a \leq p-1 \\ q \nmid a}} \sum_{\substack{b \leq p-1 \\ q \nmid b}} \chi \chi_1 \chi_2(a) \chi_1 \bar{\chi}_2(b) + O(1) \\
& \ll p^{1/2+\epsilon},
\end{aligned} \tag{43}$$

where the $\ll$ constant only depends on q . This completes the proof of Theorem.

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