Research Article

Neutral Slant Submanifolds of a Para-Kähler Manifold

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We define and study both neutral slant and semi-neutral slant submanifolds of an almost para-Hermitian manifold and, in particular, of a para-Kähler manifold. We give characterization theorems for neutral slant and semi-neutral slant submanifolds. We also investigate the integrability conditions for the distributions involved in the definition of a semi-neutral slant submanifold when the ambient manifold is a para-Kähler manifold.

1. Introduction

The geometry of slant submanifolds was initiated by Chen, as a generalization of both holomorphic and totally real submanifolds in complex geometry [1, 2]. Since then, many mathematicians have studied these submanifolds. Slant submanifolds have been studied by many geometers in various manifolds [3–5]. In particular, Papaghiuc [6] introduced semislant submanifolds. Lotta [7, 8] defined and studied slant submanifolds in contact geometry. Cabrerizo et al. studied slant, semislant, and bislant submanifolds in contact geometry [9, 10]. Recently, Arslan et al. [11] studied these submanifolds in the setting of neutral Kähler manifolds.

In this paper we define and study both neutral slant and semineutral slant submanifolds of an almost para-Hermitian manifold and, in particular, of a para-Kähler manifold. The paper is organized as follows. In Section 2, we review some formulas and definitions for an almost para-Hermitian manifold and their submanifolds. In Section 3, we define neutral slant submanifolds for an almost para-Hermitian manifold and give theorems for a neutral slant submanifold. In the last section, we define and study semineutral slant submanifolds of an almost para-Hermitian manifold. We give theorems for a semineutral slant submanifold. In the last part of Section 4, we obtain that the distributions are integrable and their leaves are totally geodesic in semineutral slant submanifold under the condition $\nabla f = 0$. Finally, the paper contains some examples.

2. Preliminaries

An almost para-Hermitian manifold $(\overline{M}, g, J)$ is a smooth manifold endowed with an almost paracomplex structure $J$ and a pseudo-Riemannian metric $g$ compatible in the sense that

$$J^2 = I, \quad g(JX,Y) + g(X, JY) = 0, \quad X,Y \in \Gamma(T\overline{M}),$$

(1)

where $\Gamma(TM)$ is the module of differentiable vector fields on $M$. It follows that the metric $g$ is neutral; that is, it has signature $(m,m)$, and the eigenbundles $T\overline{M}^{\pm}$ are totally isotropic with respect to $g$.

An almost para-Hermitian manifold $\overline{M}$ is called a para-Kähler manifold if

$$(\nabla_{X}J)Y = 0, \quad \forall X,Y \in \Gamma(T\overline{M}),$$

(2)

where $\nabla$ is the Levi-Civita connection on $\overline{M}$ [12, 13].

Let $M$ be an isometrically immersed submanifold of an almost para-Hermitian manifold $\overline{M}$. We denote the Levi-Civita connections on $M$ and $\overline{M}$ by $\nabla$ and $\overline{\nabla}$, respectively. Then, the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

$$\overline{\nabla}_X \mathcal{N} = -A_N X + \nabla^\perp_X \mathcal{N},$$

(3)
for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$, where $\nabla^\perp$ is the connection in the normal bundle $TM^\perp$, $h$ is the second fundamental form of $M$, and $A_N$ is the shape operator. The second fundamental form $h$ and the shape operator $A_N$ are related by

$$g(A_N X, Y) = g(h(X, Y), N),$$

where the induced pseudo-Riemannian metric on $M$ is denoted by the same symbol $g$.

Let us consider that $M$ is an immersed submanifold of an almost para-Hermitian manifold $\overline{M}$. For any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$, we put

$$JX = fX + \omega X,$$

$$JN = BN + CN,$$

where $fX$ (resp., $\omega X$) is tangential (resp., normal) part of $JX$ and $BN$ (resp., $CN$) is tangential (resp., normal) part of $JN$. From (1) and (5), we have

$$BN = -g(JX, N) = -g(fX + \omega X, N).$$

Similarly, for any $N \in \Gamma(TM^\perp)$, we have

$$N = \omega BN + C^2 N,$$

for any $X, Y \in \Gamma(TM^\perp)$.

Let $M$ be a submanifold of a para-Hermitian manifold $\overline{M}$. A tangent vector $X \in TM$ is said to be spacelike (resp., timelike) if $g(X, X) > 0$ (resp., $g(X, X) < 0$). If $X$ is a spacelike vector (resp., timelike), we have $\|X\| = \sqrt{g(X, X)}$ (resp., $\|X\| = \sqrt{-g(X, X)}$) [11].

### 3. Neutral Slant Submanifolds of Almost Para-Hermitian Manifolds

In this section, we study neutral slant immersions of an almost para-Hermitian manifold $\overline{M}$. First, we present definition of a neutral slant submanifold of an almost para-Hermitian manifold following Chen’s [1] definition for a Hermitian manifold. Let $M$ be a semi-Riemannian manifold isometrically immersed in an almost para-Hermitian manifold $\overline{M}$. For each nonzero spacelike vector $X$ tangent to $M$ at $x$, the angle $\theta(X)$, $0 \leq \theta(X) \leq \pi/2$ between $JX$ and $T_x M$ is called the Wirtinger angle of $X$. Then, $M$ is said to be neutral slant if the angle $\theta(X)$ is a constant, which is independent of the choice of $x \in M$ and $X \in \Gamma(TM)$. The angle $\theta$ of a neutral slant immersion is called the slant angle of the immersion. Thus, the invariant and anti-invariant immersions are neutral slant immersions with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A neutral slant immersion which is neither invariant nor anti-invariant is called a proper neutral slant immersion.

We note that our definition is quite different from Chen’s definition for slant submanifold [1], and the slant submanifold is given by Arslan et al. [11].

Next we give a useful characterization of neutral slant submanifolds in an almost para-Hermitian manifold.

**Theorem 1.** Let $M$ be a submanifold of a para-Hermitian manifold $\overline{M}$. Then,

(i) $M$ is neutral slant if and only if there exists a constant $\lambda \in [0, 1]$ such that $f^2 = \lambda I$. Furthermore, in this case, if $\theta$ is the slant angle of $M$, it satisfies $\lambda = \cos^2 \theta$;

(ii) $M$ is a neutral slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that $B^2 \omega = \lambda I$. Furthermore, in this case, if $\theta$ is the slant angle of $M$, it satisfies $\lambda = \sin^2 \theta$.

**Proof.** (i) Suppose that $M$ is a neutral slant submanifold. For any $X \in \Gamma(TM)$, we can write

$$\cos \theta(X) = \frac{\|fX\|}{\|JX\|},$$

where $\theta(X)$ is the slant angle. By using (7), (15), and (1), we get

$$g(f^2 X, X) = -g(fX, fX)$$

$$= -\cos^2 \theta(X) g(JX, JX)$$

$$= \cos^2 \theta(X) g(X, X),$$

for any $X \in \Gamma(TM)$. 

Let $M$ be a submanifold of a para-Hermitian manifold $\overline{M}$. A tangent vector $X \in TM$ is said to be spacelike (resp., timelike) if $g(X, X) > 0$ (resp., $g(X, X) < 0$). If $X$ is a spacelike vector (resp., timelike), we have $\|X\| = \sqrt{g(X, X)}$ (resp., $\|X\| = \sqrt{-g(X, X)}$) [11].
for all $X \in \Gamma(TM)$. Since $g$ is a neutral metric, from (16), we have
\begin{equation}
 f^2 X = \cos^2 \theta (X) X, \quad X \in \Gamma(TM).
\end{equation}

Let $\lambda = \cos^2 \theta$. Then it is obvious that $\lambda \in [0, 1]$. Conversely, let us assume that there exists a constant $\lambda \in [0, 1]$ such that $f^2 = \lambda I$ is satisfied. From (7), (17), and (1), we get
\begin{equation}
 \cos \theta (X) = \frac{g(JX, fX)}{\|JX\| fX} = \frac{g(X, f^2 X)}{\|X\| fX} = \lambda \frac{g(X, JX)}{\|JX\| fX},
\end{equation}
for all $X \in \Gamma(TM)$. Thus we have
\begin{equation}
 \cos \theta (X) = \frac{\lambda \|JX\|}{\|X\| fX}.
\end{equation}

Since $\cos \theta (X) = \|fX\|/\|X\|$, then by using the last equation we obtain $\cos^2 \theta (X) = \lambda$, which implies that $\theta (X)$ is a constant and so $M$ is a neutral slant.

(ii) From (8) and (i), we have (ii). \hfill \Box

**Corollary 2.** Let $M$ be a neutral slant submanifold of an almost para-Hermitian manifold $\overline{M}$ with slant angle $\theta$. Then, for any $X, Y \in \Gamma(TM)$, we have
\begin{align}
 &g(fX, fY) = -\cos^2 \theta g(X, Y), \\
 &g(\omega X, \omega Y) = -\sin^2 \theta g(X, Y).
\end{align}

**Proof.** From Theorem 1(i) and (7), we get
\begin{align}
 &g(fX, fY) = -g(f^2 X, Y), \\
 &g(fX, fY) = -\cos^2 \theta g(X, Y),
\end{align}
for any $X, Y \in \Gamma(TM)$. On the other hand, from (1), (5), and (20), we obtain
\begin{align}
 &g(JX, JY) = g(fX + \omega X, fY + \omega Y), \\
 &-g(X, Y) = g(fX, fY) + g(\omega X, \omega Y).
\end{align}

This completes the proof. \hfill \Box

Now, we give some examples of the neutral slant submanifolds in almost para-Hermitian manifolds inspired by Chen [1].

Note that given a semi-Euclidean space $R^{2n}_n$ with coordinates $(x_1, \ldots, x_{2n})$ on $R^{2n}_n$, we can naturally choose an almost paracomplex structure $J$ on $R^{2n}_n$ as follows:
\begin{align}
 &J\left(\frac{\partial}{\partial x_{2i}}\right) = \frac{\partial}{\partial x_{2i-1}}, \quad J\left(\frac{\partial}{\partial x_{2i-1}}\right) = \frac{\partial}{\partial x_{2i}},
\end{align}
where $i = 1, \ldots, n$. Let $R^{2n}_n$ be a semi-Euclidean space of signature $(+, -, +, -, \ldots)$ with respect to the canonical basis $(\partial/\partial x_1, \ldots, \partial/\partial x_{2n})$.

**Example 3.** Consider a submanifold $M$ in $R^{2n}_n$ given by
\begin{equation}
 \varphi (u, v) = (u \cos \alpha, v \sin \alpha, 0).
\end{equation}

It is easy to see that $M$ is a neutral slant submanifold with the slant angle $\alpha$.

**Example 4.** Consider a submanifold $M$ in $R^{2n}_n$ given by
\begin{equation}
 x(u, v) = (u \sin \alpha, v \cos \beta, u \cos \alpha, v \sin \beta),
\end{equation}
where $\alpha$ and $\beta$ are constant. Then $M$ is a neutral slant submanifold with the slant angle $\cos \theta = |\sin(\alpha + \beta)|$.

**Remark 5.** Consider $M_p^{2n}$ a neutral submanifold of an almost para-Hermitian manifold $(\overline{M}, g, J)$, in fact a neutral manifold $\overline{M}^n_2$, with
\begin{equation}
 |g(fX, JX)| \leq \|fX\|/\|JX\|,
\end{equation}
$M$ is called a neutral slant submanifold if the Wirtinger angle between $fX$ and $JX$ is constant, for all $x \in T_p M$ a spacelike vector field and all $x \in M$. It is well defined, because that angle can be measured as usual, it the same angle between $JX$ and $fX$ and they both are timelike vector fields.

In fact, if that conditions hold, it would be the same angle between $JY$ and $T_p M$ for $Y \in T_p M$ a timelike vector, both $JY$ and $fY$ would be spacelike vector fields. This condition is equivalent to
\begin{equation}
 |g(fX, JX)| \leq \|fX\|/\|JX\|,
\end{equation}
or $\|fX\| \leq \|JX\|$, in fact it is equivalent to Theorem 1 condition $f^2 X = \cos^2 \theta I$.

**4. Semineutral Slant Submanifolds of Almost Para-Hermitian Manifolds**

**Definition 6.** Let $(\overline{M}, g)$ be an almost para-Hermitian manifold with an almost paracomplex structure $J$. A differentiable distribution on $\overline{M}$ is called a neutral slant distribution if for each $p \in \overline{M}$ and each nonzero spacelike vector field $X \in \Gamma(D_p)$, the angle $\theta_p$ between $JX$ and $D_p$ is constant, that is, independent of the choice of $p \in \overline{M}$ and $X \in \Gamma(D_p)$. In this case, we call the constant angle $\theta_p$ the slant angle of the distribution $D_p$.

Let $M$ be an immersed submanifold of an almost para-Hermitian manifold $\overline{M}$ and $D$ a differentiable distribution on $M$. We denote the orthogonal distribution to $D$ on $M$ by $D^\perp$. Then, for all $X \in \Gamma(TM)$, we write
\begin{equation}
 JX = P_1 fX + P_2 fX + \omega X,
\end{equation}
where $P_1$ and $P_2$ are orthogonal projections on $D$ and $D^\perp$, respectively.

Next, we will give a sufficient and necessary condition for a distribution to be slant.

**Theorem 7.** Let $M$ be a submanifold of an almost para-Hermitian manifold $\overline{M}$ and $D$ a differentiable distribution on $\overline{M}$ with slant angle...
M. Then $D$ is a neutral slant distribution if and only if there exists a constant $\lambda \in [0, 1]$ such that

$$
(P_i f)^2 = \lambda I. \quad (30)
$$

Furthermore, in such case, if $\theta$ is the slant angle of $D$ then $\lambda = \cos^2 \theta$.

**Proof.** We suppose that $D$ is a neutral slant distribution on $M$. Then, from (29), we have

$$
\cos \theta(X) = \frac{g(JX, P_i fX)}{\|JX\| \|P_i fX\|} = -\frac{g(X, (P_i f)^2 X)}{\|JX\| \|P_i fX\|} = -\frac{\|P_i fX\|^2}{\|JX\|^2}, \quad \forall X \in \Gamma(D).
$$

which implies that

$$
\|P_i fX\| = \cos \theta(X) \|JX\|, \quad (32)
$$

for any $X \in \Gamma(D)$. By using (29), (32), and (1), we have

$$
g(X, (P_i f)^2 X) = -g(P_i fX, P_i fX) = -\cos^2 \theta(X) g(JX, JX) = \cos^2 \theta(X) g(X, X), \quad \forall X \in \Gamma(D).
$$

Since $g$ is a neutral metric, we obtain

$$
(P_i f)^2 X = \cos^2 \theta(X) X, \quad \forall X \in \Gamma(D). \quad (34)
$$

If we put $\lambda = \cos^2 \theta$, then we have (30). Conversely, let $\lambda \in [0, 1]$ be a constant such that (30) is satisfied. Then, from (1) we have

$$
\cos \theta(X) = \frac{g(JX, P_i fX)}{\|JX\| \|P_i fX\|} = -\lambda \frac{g(X, X)}{\|JX\| \|P_i fX\|}, \quad (35)
$$

for any $X \in \Gamma(D)$. Thus we get

$$
\cos \theta(X) = \frac{\lambda \|JX\|}{\|P_i fX\|}. \quad (36)
$$

On the other hand, since $\cos \theta(X) = \|P_i fX\| / \|JX\|$, then we obtain $\cos^2 \theta = \lambda$, which implies that $\theta$ is a constant and $D$ is a neutral slant distribution. This completes the proof. \hfill \square

**Definition 8.** $M$ is called a bineutral slant submanifold of an almost para-Hermitian manifold $\overline{M}$ if there exist two orthogonal distributions $D_1$ and $D_2$ on $M$ such that

(i) $TM$ admits the orthogonal direct decomposition $TM = D_1 \oplus D_2$;

(ii) $D_i$ is a neutral slant distribution with slant angle $\theta_i$ for $i = 1, 2$.

Given a bineutral slant submanifold $M$, we can write, for any $X \in \Gamma(TM)$,

$$
X = P_1 X + P_2 X, \quad (37)
$$

where $P_i$ denotes the component of $X$ in $D_i$ for any $i = 1, 2$. In particular, if $X \in \Gamma(D_1)$, then we obtain $X_i = P_i X$. If we define $f_i = P_i f$, then we have

$$
JX = f_1 X + f_2 X + \omega X, \quad (38)
$$

for any $X \in \Gamma(TM)$.

We note that semi-invariant submanifolds are particular cases of bineutral slant submanifolds with slant angles $\theta_1 = 0$ and $\theta_2 = \pi/2$.

**Theorem 9.** Let $M$ be a bineutral slant submanifold with angles $\theta_1 = \theta_2 = \theta$. If $g(JX, Y) = 0$, for any $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$, then $M$ is slant with angle $\theta$.

**Proof.** Since $g(JX, Y) = 0$, for any $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$, we have $g(f_1 X, Y) = 0$; that is, $f_1 X \in \Gamma(D_1)$. Similarly, for $Y \in \Gamma(D_2)$, we find. Then for any $X \in \Gamma(TM)$, $X$ can be written as follows: $X = X_1 + X_2$ such that $X_1 \in \Gamma(D_1)$ and $X_2 \in \Gamma(D_2)$ and $\cos^2 \theta_1 = \|f_1 X_1\|^2 / \|X_1\|^2$, $\cos^2 \theta_2 = \|f_2 X_2\|^2 / \|X_2\|^2$. Since $\theta_1 = \theta_2 = \theta$, we get

$$
\frac{g(f_1 X, f_1 X)}{g(JX, JX)} = \frac{g(f_1 X_1, f_1 X_1)}{g(X_1, X_1)} + \frac{g(f_2 X_2, f_2 X_2)}{g(X_2, X_2)} = \cos^2 \theta, \quad (39)
$$

which gives assertion of the theorem.

Now, as a generalization of semi-invariant submanifolds, we can define semineutral slant submanifolds of an almost para-Hermitian manifold. \hfill \square

**Definition 10.** $M$ is called a semineutral slant submanifold of an almost para-Hermitian manifold $\overline{M}$ if there exist two orthogonal distributions $D_1$ and $D_2$ on $M$ such that

(i) $TM$ admits the orthogonal direct sum $TM = D_1 \oplus D_2$;

(ii) the distribution $D_1$ is invariant; that is, $J(D_1) = D_1$;

(iii) the distribution $D_2$ is neutral slant with slant angle $\theta \neq 0$.

In this case, we call $\theta$ the slant angle of submanifold $M$.

It is obvious that the invariant and anti-invariant distributions of a semineutral slant submanifold are neutral slant distributions with the slant angles $\theta = 0$ and $\theta = \pi/2$, respectively.

Now, let $M$ be a semineutral slant submanifold of an almost para-Hermitian manifold $\overline{M}$. Let $M$ be a semisant submanifold with $d_1 \dim(D_1)$ and $d_2 \dim(D_2)$. Then we have the following particular cases.

(i) If $d_2 = 0$, then $M$ is an invariant submanifold.

(ii) If $d_1 = 0$ and $\theta = \pi/2$, then $M$ is an anti-invariant submanifold.

(iii) If $d_1 = 0$ and $\theta \neq \pi/2$, then $M$ is a proper neutral slant submanifold with slant angle $\theta$. 
(iv) If \( d_1 \cdot d_2 \neq 0 \) and \( \theta \neq \pi/2 \), then \( M \) is a proper semineutral slant submanifold.

We now give an example of bineutral slant submanifolds.

**Example 11.** Let \( x(u, v, t, s) = (u \sin \alpha, v, u \cos \alpha, 0, s \sin \beta, 0, t \cos \beta) \), where \( \alpha \) and \( \beta \) are constant. Then, \( M \) is a 4-dimensional submanifold of \( \overline{M} = R_4^8 \).

By defining

\[
D_1 = \left\{ \sin \alpha \frac{\partial}{\partial x_1} + \cos \alpha \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_2} \right\},
\]

\[
D_2 = \left\{ \frac{\partial}{\partial x_5}, \sin \beta \frac{\partial}{\partial x_6} + \cos \beta \frac{\partial}{\partial x_8} \right\},
\]

we have that \( TM = D_1 \oplus D_2 \) and \( D_1, D_2 \) are neutral slant with slant angles \( \cos^{-1}(|\sin \alpha|) \) and \( \cos^{-1}(|\sin \beta|) \), respectively. Thus \( M \) is a bineutral slant submanifold of \( \overline{M} \).

Now, let \( M \) be a semineutral slant submanifold of an almost para-Hermitian manifold \( \overline{M} \) and \( P_i \) (\( i = 1, 2 \)), denoting the orthogonal projections on \( D_i \) (\( i = 1, 2 \)). Then, for any \( X \in \Gamma(TM) \), applying \( J \) to (37), we have

\[
JX = fP_1X + fP_2X + \omega P_2X,
\]

where

\[
JP_1X = fP_1X, \quad \omega P_2X = 0.
\]

From (41) and (42), we have

\[
fX = JP_1X + fP_2X.
\]

By putting \( Y = P_1Y \) in (20) and \( Y = P_2Y \) in (21), we get

\[
g(fX, fP_1Y) = -\cos^3 \theta g(X, P_1Y), \quad X, Y \in \Gamma(TM),
\]

\[
g(\omega X, \omega P_2Y) = -\sin^3 \theta g(X, P_2Y),
\]

respectively.

We give a characterization for the semineutral slant submanifolds of an almost para-Hermitian manifold.

**Theorem 12.** Let \( M \) be an immersed submanifold of an almost para-Hermitian manifold \( \overline{M} \). Then \( M \) is a semineutral slant submanifold if and only if there exists a constant \( \lambda \in [0, 1] \) such that \( D \) = \{ \( X \in TM \mid f^2X = \lambda X \) \} is a distribution. Furthermore, in this case, \( \lambda = \cos^2 \theta \), where \( \theta \) denotes slant angle of \( M \).

**Proof.** Let \( M \) be a semineutral slant submanifold and \( TM = D_1 \oplus D_2 \), where \( D_1 \) is invariant and \( D_2 \) is neutral slant. We put \( \lambda = \cos^2 \theta \), where \( \theta \) denotes slant angle of \( M \). For any \( X \in \Gamma(D_1) \), if \( X \in \Gamma(D_2) \), then we have

\[
X = f^2X = fX = \lambda X,
\]

which implies that \( \lambda = 1 \). But this is a contradiction that \( \lambda \in [0, 1] \). Therefore we obtain \( D \subseteq D_2 \). On the other hand, since \( D_2 \) is a neutral slant distribution, it follows from Theorem 7 that \( f^2X = (fP_2)^2X = \lambda X \), which means that \( D_2 \subseteq D \). Thus \( D = D_2 \) is a distribution.

Conversely, we can consider the orthogonal direct decomposition \( TM = D \oplus D^\perp \). It is obvious that \( fD \subseteq D \), from which we have \( g(X, Y) = -g(X, JY) = -g(X, fY) = 0 \) for any \( X \in \Gamma(D^\perp) \) and \( Y \in \Gamma(D) \). Hence \( D^\perp \) is an invariant distribution. Finally, Theorem 7 implies that \( D \) is a neutral slant distribution, with slant angle \( \theta \) satisfying \( \lambda = \cos^2 \theta \).

We can easily present some examples of the above situations.

**Example 13.** \( x(u, v, t, r) = (u, 0, u \sin \theta, 0, v \cos \theta, t, s) \) defines a four-dimensional proper semineutral slant submanifold \( M \), with slant angle \( \cos^{-1}(|\sin \theta/\sqrt{2}|) \), in \( R_4^8 \).

Moreover, it is easy to see that

\[
X_1 = \frac{\partial}{\partial x_1}, \quad X_3 = \frac{\partial}{\partial x_3} , \quad X_4 = \sin \theta \frac{\partial}{\partial x_4} + \cos \theta \frac{\partial}{\partial x_6},
\]

from a local orthogonal frame of \( TM \). Then, we can define \( D_1 = \text{Span}[X_1, X_2] \) and \( D_2 = \text{Span}[X_3, X_4] \).

**Example 14.** \( x(u, v, t, s) = (u, v, t \sin \alpha, s \cos \beta, t \cos \alpha, s \sin \beta, 0, 0) \) defines a four-dimensional proper semineutral slant submanifold \( M \), with slant angle \( \cos \theta = |\sin(\alpha + \beta)| \), in \( R_4^8 \), where \( \alpha \) and \( \beta \) are constant.

Moreover it is easy to see that

\[
X_1 = \frac{\partial}{\partial x_1}, \quad X_3 = \sin \alpha \frac{\partial}{\partial x_3} + \cos \alpha \frac{\partial}{\partial x_5},
\]

\[
X_2 = \frac{\partial}{\partial x_2}, \quad X_4 = \cos \beta \frac{\partial}{\partial x_4} + \sin \beta \frac{\partial}{\partial x_6},
\]

from a local orthogonal frame of \( TM \). Then we can define \( D_1 = \text{Span}[X_1, X_2] \) and \( D_2 = \text{Span}[X_3, X_4] \).

Then, it is easy to show that all conditions of Theorem 12 are satisfied.

Next, we will give useful characterizations for integrable conditions of distributions.

**Theorem 15.** Let \( M \) be a semineutral slant submanifold of a para-Kähler manifold \( \overline{M} \). Then we have the following:

(a) the distribution \( D_1 \) is integrable if and only if

\[
h(X, fY) = h(fX, Y),
\]

for any \( X, Y \in \Gamma(D_1) \),

(b) the distribution \( D_2 \) is integrable if and only if

\[
P_i \left( \nabla_X fP_2Y - \nabla_Y fP_2X \right) = P_i \left( A_{\omega P_2X}X - A_{\omega P_2Y}X \right),
\]

for any \( X, Y \in \Gamma(D_2) \).
Proof. From (2), we get
\[ \nabla_X Y = f \nabla_X Y, \] (50)
for all \( X, Y \in \Gamma(TM) \).

(a) By using Gauss-Weingarten formulas, (5), and (6) in
(50), we have
\[ \nabla_X f Y + h (X, f Y) \\
= f \nabla_X Y + \omega \nabla_X Y + Bh (X, Y) + Ch (X, Y), \] (51)
for any \( X, Y \in \Gamma(D_1) \). From (41) and (51), we obtain
\[ \nabla_X f Y + h (X, f Y) = f P_1 \nabla_X Y + f P_2 \nabla_X Y + \omega P_3 \nabla_X Y \\
+ Bh (X, Y) + Ch (X, Y). \] (52)
By equating the normal part of the last equation, we have
\[ h (X, f Y) = \omega P_2 \nabla_X Y + Ch (X, Y). \] (53)
If we change the role of \( X \) and \( Y \) in (53), we write
\[ h (f X, Y) = \omega P_3 \nabla_X Y + Ch (Y, X). \] (54)
Since \( h \) is symmetric, from (53) and (54), we get
\[ h (X, f Y) - h (f X, Y) = \omega P_2 \nabla_X Y, \forall X, Y \in \Gamma(D_1). \] (55)
Assume that the distribution \( D_1 \) is integrable. Then, for
any \( X, Y \in \Gamma(D_1) \), we have \( [X, Y] \in \Gamma(D_1) \) which implies that
\( \omega P_2 \nabla_X Y = 0 \). Thus from (55) we obtain (48).
Conversely, if (48) is satisfied, then from (55), we have
\( \omega P_2 \nabla_X Y = 0 \), for any \( X, Y \in \Gamma(D_1) \), which implies that
\( P_2 \nabla_X Y = 0 \). Then we conclude that \( [X, Y] \in \Gamma(D_1) \).

(b) From (41) and Gauss-Weingarten formulae, we have
\[ \nabla_X f Y = \nabla_X P_1 Y + h (X, P_1 Y) + \nabla_X f P_2 Y + h (X, f P_2 Y) \\
- A_{\omega P_3} X + \nabla_\omega P_2 Y, \] (56)
for all \( X, Y \in \Gamma(TM) \). On the other hand, by using (5) and (6), we write
\[ f \nabla_X Y = f \nabla_X Y + \omega \nabla_X Y + Bh (X, Y) + Ch (X, Y). \] (57)
By using (56) and (57) in (50), we get
\[ \nabla_X f P_2 Y + h (X, f P_2 Y) - A_{\omega P_3} Y + \nabla_\omega P_2 Y \\
= f \nabla_X Y + \omega \nabla_X Y + Bh (X, Y) + Ch (X, Y), \] (58)
for any \( X, Y \in \Gamma(D_2) \). Since \( h \) is symmetric we obtain
\[ f [X, Y] = \nabla_X f P_2 Y - \nabla_Y f P_2 Y + A_{\omega P_3} X - A_{\omega P_3} Y \] (59)
which gives
\[ P_1 f [X, Y] = P_1 [\nabla_X f P_2 Y - \nabla_Y f P_2 Y] \\
- P_1 [A_{\omega P_3} X - A_{\omega P_3} Y]. \] (60)

Let the distribution \( D_2 \) be integrable. Then \( P_1 f [X, Y] = 0 \),
for all \( X, Y \in \Gamma(D_2) \), and hence from (60), the equation (49)
is obvious.

Conversely, if (49) is satisfied then \( P_1 f [X, Y] = 0 \); that is,
\( [X, Y] \in \Gamma(D_2) \) for any \( X, Y \in \Gamma(D_2) \). This completes the proof. \( \square \)

Definition 16. Let \( M \) be a semi-invariant submanifold of an
almost para-Hermitian manifold \( \overline{M} \). Then we say that
(i) \( M \) is \( D_1 \)-geodesic if
\[ h (X, Y) = 0, \forall X, Y \in \Gamma(D_1), \] (61)
(ii) \( M \) is \( D_2 \)-geodesic if
\[ h (X, Y) = 0, \forall X, Y \in \Gamma(D_2), \] (62)
(iii) \( M \) is mixed geodesic if
\[ h (X, Y) = 0, \forall X \in \Gamma(D_1), Y \in \Gamma(D_2). \] (63)

Lemma 17. Let \( M \) be a mixed-geodesic semineutral slant submanifold
of a para-K"ahler manifold \( \overline{M} \). Then the distribution \( D_1 \)
is integrable if and only if
\[ J A_N X = -A_N J X, \] (64)
for any \( X \in \Gamma(D_1) \) and \( N \in \Gamma(T^+M) \).

Proof. Since \( M \) is a mixed-geodesic submanifold, from (4)
we find that \( A_N X \) has no component on \( D_2 \). By using (4) and (i), we obtain
\[ g (J A_N X, Y) = -g (A_N J X, Y) = -g (h (X, Y), N), \] (65)
\[ g (A_N J X, Y) = g (h (X, Y), N). \]
Thus, we can write
\[ g (J A_N X + A_N J X, Y) = g (h (X, Y) - h (X, J Y), N), \] (66)
for all \( X, Y \in \Gamma(D_1) \). Taking into account Theorem 15(a) and
the last equation, the proof is completed. \( \square \)

Theorem 18. Let \( M \) be a semineutral slant submanifold of a
para-K"ahler manifold \( \overline{M} \). If \( \nabla \omega = 0 \), then \( M \) is a mixed-
geodesic submanifold. Furthermore,

(a) if \( X, Y \in \Gamma(D_1) \), then either \( M \) is a \( D_1 \)-geodesic
submanifold or \( h(X, Y) \) is an eigenvector of \( C^2 \) with
the eigenvalue 1,
(b) if \( X, Y \in \Gamma(D_2) \), then either \( M \) is a \( D_2 \)-geodesic
submanifold or \( h(X, Y) \) is an eigenvector of \( C^2 \) with
the eigenvalue \( \cos^2 \theta \).
Proof. If $\nabla \omega = 0$, then from (13) we get $Ch(X, Y) = h(X, fY)$, for all $X, Y \in \Gamma(TM)$. Since $D_1$ is an invariant and $D_2$ is a neutral slant distribution with the slant angle $\theta$, we obtain
\[
C^2 h(X, Y) = Ch(X, fY) = h(X, f^2 Y),
\]
\[
= h(X, \cos^2 \theta Y) = \cos^2 \theta h(X, Y),
\]
\[
C^2 h(X, Y) = C^2 h(Y, X) = Ch(Y, fX)
\]
\[
= h(Y, f^2 X) = h(Y, X) = h(X, Y),
\]
for any $X, Y \in \Gamma(D_1)$, $Y \in \Gamma(D_2)$. By using (67) we get
\[
\sin^2 \theta h(X, Y) = 0,
\]
which implies that $h(X, Y) = 0$, for any $X \in \Gamma(D_1)$, $Y \in \Gamma(D_2)$, that is, $M$ is mixed-geodesic. Similarly, we obtain
\[
C^2 h(X, Y) = h(X, Y),
\]
for all $X, Y \in \Gamma(D_1)$, and
\[
C^2 h(X, Y) = \cos^2 \theta h(X, Y),
\]
for all $X, Y \in \Gamma(D_2)$. This completes the proof.

Proposition 20. Let $M$ be a semineutral slant submanifold of a para-Kähler manifold $\overline{M}$. Then $\nabla \omega = 0$ if and only if
\[
A_{CN} Z = -A_N f Z,
\]
for all $Z \in \Gamma(TM)$, $N \in \Gamma(T^1 M)$.

Proof. From (13) and (1), we get
\[
g((\nabla_X \omega) Z, N) = g(Ch(X, Z) - h(X, fZ), N)
\]
\[
= -g(h(X, Z), CN) - g(h(X, fZ), N),
\]
for any $X, Z \in \Gamma(TM)$, $N \in \Gamma(T^1 M)$. Taking into account (4), we get
\[
g((\nabla_X \omega) Z, N) = -g(A_{CN} Z + A_N f Z, X),
\]
which completes the proof.

Proposition 21. Let $M$ be a semineutral slant submanifold of a para-Kähler manifold $\overline{M}$. If $\nabla f = 0$ then the distributions are integrable and their leaves are totally geodesic in $M$.

Proof. Since $\nabla f = 0$, then from (12) we obtain $Bh(X, Y) = 0$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D_1)$. By using (1) and (5), we have
\[
0 = (Bh(X, Y), Z) = g(Jh(X, Y), Z) = -g(h(X, Y), JZ),
\]
where $X, Z \in \Gamma(TM)$ and $Y \in \Gamma(D_1)$. Thus one can easily see that
\[
g(h(X, Y), \omega P_2 Z) = 0,
\]
\[
g(Jh(X, Y), \omega P_2 Z) = 0.
\]

Since $\overline{M}$ is a para-Kähler manifold, taking into account (78), we get
\[
0 = g(Jh(X, Y), \omega P_2 \nabla_X Y)
\]
\[
0 = -g(\omega P_2 \nabla_X Y, \omega P_2 \nabla_X Y)
\]
\[
0 = \sin^2 \theta g(P_2 \nabla_X Y, P_2 \nabla_X Y),
\]
which gives $P_2 \nabla_X Y = 0$; that is, $\nabla_X Y \in \Gamma(D_1)$. Now, let $Y \in \Gamma(D_1)$ and $V \in \Gamma(D_2)$. Since $D_1$ is orthogonal to $D_2$, the induced metric on $M$ is the neutral metric, and it is easy to see that $\nabla_X V \in \Gamma(D_2)$. Hence the proof is complete.

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