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Research Article

Approximate Preservers on Banach Algebras and C^*-Algebras

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The aim of the present paper is to give approximate versions of Hua's theorem and other related results for Banach algebras and C^*-algebras. We also study linear maps approximately preserving the conorm between unital C^*-algebras.

1. Preliminaries

A well known formulation of the celebrated Hua's theorem [1] asserts that every bijective additive map $T : K \to K$ on a division ring K such that $T(1) = 1$ and $T(x^{-1}) = T(x)^{-1}$ for every invertible element x is either an automorphism or an antiautomorphism. This result was later moved to matrix algebras in [2] and finally extended to Banach algebras in [3] (see also [4]). In [3], the author called the previous relation *strongly preserving invertibility*.

Let A and B be unital Banach algebras. Recall that an additive map $T : A \to B$ is a *Jordan homomorphism* if $T(ab + ba) = T(a)T(b) + T(b)T(a)$ for every $a, b \in A$, or equivalently, $T(a^2) = T(a)^2$ for all $a \in A$. Obvious examples of Jordan homomorphisms are homomorphisms and anti-homomorphisms. It is well known that every unital (i.e., $T(1) = 1$) Jordan homomorphism strongly preserves invertibility. Reciprocally, one of the results in [4] proves that every additive map between Banach algebras strongly preserving invertibility is a multiple of a Jordan homomorphism. In particular, if the map is unital, the map itself is a Jordan homomorphism.

There exist also other versions of Hua's theorem involving some important kinds of generalized invertibility. Given a ring A and an element $a \in A$, a is said to be *Drazin invertible* if there exist $b \in A$ and a nonnegative integer k such that

$$bab = b, \qquad a^k ba = a^k, \qquad ab = ba. \tag{1}$$

Such b is unique whenever it exists. In this case, it is called the *Drazin inverse* of a and it is denoted by $b = a^D$. This notion was introduced by Drazin in [5] and it has proved useful in many fields of pure and applied mathematics (see for instance [6, 7]). If the previous identities are satisfied for some $b \in A$ and $k = 1$, b is called the *group inverse* of a and we will denote it by a^G. Let A^{-1}, A^D, and A^G denote the sets of all invertible, Drazin invertible, and group invertible elements in A, respectively. Clearly

$$A^{-1} \subset A^G \subset A^D. \tag{2}$$

Linear or additive maps between unital Banach algebras strongly preserving group and Drazin invertibility were introduced in [3, 4] (see also [8], where the author described additive maps between operator algebras of infinite-dimensional Hilbert spaces strongly preserving Drazin invertibility). If $T : A \to B$ is a Jordan homomorphism between Banach algebras, it was shown in [3, Theorem 2.1] that

(i) T strongly preserves group invertibility, that is, $T(a^G) = T(a)^G$ for every $a \in A^G$,

(ii) T strongly preserves Drazin invertibility, that is, $T(a^D) = T(a)^D$ for every $a \in A^D$.

Conversely, if $T : A \to B$ is an additive map strongly preserving invertibility, group invertibility, or Drazin invertibility, and $T(1) = 1$ (resp., $T(1)$ is invertible or $1 \in T(A)$), then T (resp., $T(1)T$) is a unital Jordan homomorphism

and $T(1)$ commutes with the image of T, [4, Theorem 4.2]. In [9], the authors showed that the same holds without any hypothesis on $T(1)$ and even when B is not necessarily unital.

Recall that an element $a \in A$ is called *regular* if there exists $b \in A$ (not necessarily unique) such that $a = aba$ and $b = bab$. Notice that the first equality $a = aba$ is a necessary and sufficient condition for a to be regular, and that, $p = ab$ and $q = ba$ are idempotents in A fulfilling $aA = pA$ and $Aa = Aq$. For an element $a \in A$, let us consider the left and right multiplication operators $L_a : x \mapsto ax$ and $R_a : x \mapsto xa$, respectively. If a is regular, then so are L_a and R_a, and thus its ranges $aA = L_a(A)$ and $Aa = R_a(A)$ are both closed.

Regular elements in a unital C^*-algebra A were studied by Harte and Mbekhta in [10, 11].

An element $a \in A$ has *Moore-Penrose inverse* b when $a = aba$, $b = bab$ and the associated idempotents ab and ba are self-adjoint. In the aforementioned papers by Harte and Mbekhta, it is shown that every regular element in a C^*-algebra has a Moore-Penrose inverse, and that it is unique. For a regular element a in a C^*-algebra A, $a^\dagger$ will denote its Moore-Penrose inverse. The set of all Moore-Penrose invertible elements in a C^*-algebra A will be denoted by $A^\dagger$.

Let A and B be unital C^*-algebras. A linear map $T : A \to B$ *strongly preserves Moore-Penrose invertibility* if $T(a^\dagger) = T(a)^\dagger$ for every $a \in A^\dagger$. Every Jordan $*$-homomorphism strongly preserves Moore-Penrose invertibility, and the question is whether or not the converse holds. Some partial positive answers are given by Mbekhta in [3], and more recently by the authors of the present paper in [12], when A has a rich structure of projections. The problem for general C^*-algebras remains open. However, we can consider an alternative approach.

Recall that the class of C^*-algebras is contained in a wider class of Banach spaces, the so called JB^*-triples, in which the concept of regularity extends the one given for C^*-algebras. A JB^*-*triple* is a complex Banach space E together with a continuous triple product $\{\cdot,\cdot,\cdot\} : E \times E \times E \to E$, which is conjugate linear in the middle variable, and symmetric and bilinear in the outer variables satisfying that

(a) $L(a,b)L(x,y) = L(x,y)L(a,b) + L(L(a,b)x,y) - L(x,L(b,a)y)$, where $L(a,b)$ is the operator on E given by $L(a,b)x = \{a,b,x\}$;

(b) $L(a,a)$ is an hermitian operator with nonnegative spectrum;

(c) $\|L(a,a)\| = \|a\|^2$.

For each x in a JB^*-triple E, $Q(x)$ will stand for the conjugate linear operator on E defined by $Q(x)(y) = \{x,y,x\}$.

An element a in a JB^*-triple E is called *von Neumann regular* if there exists (a unique) $b \in E$ such that $Q(a)(b) = a$, $Q(b)(a) = b$, and $Q(a)Q(b) = Q(b)Q(a)$. The element b is called the *generalized inverse* of a and it will be denoted by $a^\wedge$. We will also note by $E^\wedge$ the set of all elements in E with generalized inverse. We refer to [13–17] for basics results on von Neumann regularity in JB^*-triples.

Every C^*-algebra is a JB^*-triple via the triple product given by

$$\{x,y,z\} = \frac{1}{2}\left(xy^*z + zy^*x\right). \tag{3}$$

For a C^*-algebra A, it is well known that $A^\dagger = A^\wedge$ and, for every regular element a in A, we have $a^\wedge = (a^\dagger)^*$.

A linear map $T : E \to F$ between JB^*-triples is a *triple homomorphism* if

$$T(\{x,y,z\}) = \{T(x), T(y), T(z)\}, \tag{4}$$

for every $x,y,z \in E$. Every triple homomorphism $T : E \to F$ between JB^*-triples strongly preserves generalized invertibility; that is, $T(x^\wedge) = T(x)^\wedge$ for every $x \in E^\wedge$. In [9], the authors characterized the triple homomorphism between C^*-algebras as the linear maps strongly preserving generalized invertibility. As a consequence, it is proved that a self-adjoint linear map from a unital C^*-algebra A into a C^*-algebra B is a triple homomorphism if and only if it strongly preserves Moore-Penrose invertibility [9, Theorem 3.5].

The Hua-type descriptions belong to the framework of linear preserver problems. This has become an active research area in many topics of matrix theory, operator theory, and Banach algebras theory. Some of the most popular preserver problems are those dealing with determining the linear maps preserving properties related to invertibility. Every Jordan isomorphism $T : A \to B$ between unital Banach algebras is unital and preserves invertibility in both directions [18, Proposition 1.3], or equivalently, preserves the spectrum; that is, $\sigma(T(a)) = \sigma(a)$, for every $a \in A$. The celebrated Kaplansky's conjecture, [19], reformulated by Aupetit in [20], states that every unital surjective linear map T between unital semisimple Banach algebras preserving invertibility in both directions is a Jordan isomorphism. Many partial positive results are known so far [18, 20–24], but the general problem is still open even in the class of C^*-algebras. In the commutative setting, the classical Gleason-Kahane-Zelazko theorem (see [23]) states that a linear functional φ on a unital complex Banach algebra A is multiplicative if and only if $\varphi(a) \in \sigma(a)$, for all $a \in A$. On the other hand, Jafarian and Sourour proved in [22] that every spectrum preserving surjective linear map $T : \mathscr{B}(X) \to \mathscr{B}(Y)$ is either an isomorphism or an anti-isomorphism, where $\mathscr{B}(X)$ denotes the Banach algebra of all bounded linear operators on a Banach space X.

The authors of [25, 26] consider the problem of characterizing the approximately multiplicative linear functionals among all linear functionals on a commutative Banach algebra in terms of spectra. More recently, in [27], (see also [28]) Alaminos, Extremera, and Villena investigate approximate versions of Kaplansky's problem, by providing approximate formulations of [18, 22]. They considered linear maps that approximately preserve spectrum or spectral radius on operator algebras and established the relationship between approximately preserving spectrum (resp., spectral radius)

and approximately being a Jordan homomorphism (resp., weighted Jordan homomorphism).

Let A and B be Banach algebras and $T : A \to B$ be a bounded linear map. Following [27, 29], the *multiplicativity*, *antimultiplicativity*, and *Jordan-multiplicativity* of T can be measured by considering the following values:

$$\begin{aligned} &\operatorname{mult}(T) \\ &= \sup\{\|T(ab) - T(a)T(b)\| : a, b \in A, \|a\| = \|b\| = 1\}, \\ &\operatorname{amult}(T) \\ &= \sup\{\|T(ab) - T(b)T(a)\| : a, b \in A, \|a\| = \|b\| = 1\}, \\ &\operatorname{jmult}(T) = \sup\left\{\left\|T\left(a^2\right) - T(a)^2\right\| : a \in A, \|a\| = 1\right\}, \end{aligned} \tag{5}$$

respectively. Obviously, T is a homomorphism (antihomomorphism, Jordan homomorphism) if and only if $\operatorname{mult}(T) = 0$ (resp., $\operatorname{amult}(T) = 0$, $\operatorname{jmult}(T) = 0$).

For a bounded linear map $T : A \to B$ between C^*-algebras, we define the *triple multiplicativity* and the *self-adjointness* of T, respectively, as the following quantities:

$$\begin{aligned} &\operatorname{tmult}(T) \\ &\quad = \sup\{\|T(\{a,b,c\}) - \{T(a), T(b), T(c)\}\| : \\ &\qquad \|a\| = \|b\| = \|c\| = 1\}, \\ &\operatorname{sa}(T) = \sup\left\{\left\|T\left(a^*\right)^* - T(a)\right\| : \|a\| = 1\right\}. \end{aligned} \tag{6}$$

Clearly, $T : A \to B$ is a triple homomorphism if and only if $\operatorname{tmult}(T) = 0$, and T is self-adjoint if and only if $\operatorname{sa}(T) = 0$.

The aim of the present paper is to bring Hua type theorems into this framework. In order to make this possible, we have adapted some techniques from [27] involving ultraproducts of Banach algebras. Section 2 contains all the technical results about invertibility and coset representatives in ultraproducts of Banach algebras that we will need throughout the paper. Section 3 provides approximate versions of Hua's theorem for invertibility and group invertibility in Banach algebras. We translate the strongly invertibility preserving condition $T(a^{-1}) = T(a)^{-1}$ into

$$\sup_{\|a\|=1, a\in A^{-1}} \left\|T\left(a^{-1}\right) - T(a)^{-1}\right\| < \varepsilon, \tag{7}$$

and the condition $T(a^G) = T(a)^G$ into

$$\sup_{\|a\|=1, a\in A^{G}} \left\|T\left(a^{G}\right) - T(a)^{G}\right\| < \varepsilon, \tag{8}$$

for some $\varepsilon > 0$. We prove that for every unital Banach algebras A and B, if $\varepsilon \to 0$ in (7) or (8) then $\operatorname{jmult}(T(1)T) \to 0$, uniformly on any set of linear maps $T : A \to B$ with norms bounded above.

Section 4 includes an approximate formulation of [9, Theorem 3.5]. The condition $T(a^\dagger) = T(a)^\dagger$ for every $a \in A^\dagger$ is replaced by

$$\sup_{\|a\|=1, a\in A^{\dagger}} \left\|T\left(a^{\dagger}\right) - T(a)^{\dagger}\right\| < \varepsilon. \tag{9}$$

We show that for every unital C^*-algebras A and B if $\varepsilon \to 0$ in (9), then $\operatorname{tmult}(T) \to 0$, uniformly on any set of linear maps whose norms are bounded above.

In this section, we also study linear maps that approximately preserve the conorm. Recall that the *conorm* $c(a)$ of an element a in a unital Banach algebra is defined as the reduced minimum modulus of the left multiplication operator by a, $c(a) = \gamma(L_a)$ [11]. For a bounded linear operator T on a complex Banach space X, its *reduced minimum modulus* is given by

$$\gamma(T) = \inf\{\|Tx\| : \operatorname{dist}(x, \ker(T)) \geq 1\}. \tag{10}$$

It is well known that $\gamma(T) > 0$ if and only if T has closed range. In [10], it is shown that an element a in a unital C^*-algebra A is regular if and only if $c(a) > 0$. In this case,

$$c(a) = \left\|a^\dagger\right\|^{-1} \tag{11}$$

(see [11, Theorem 2]). In [30], the authors characterized the linear maps between unital C^*-algebras preserving the conorm. By [30, Theorem 3.1], if $T : A \to B$ is a unital linear map such that $c(T(a)) = c(a)$ for every $a \in A$, then T is an isometric Jordan $*$-homomorphism. Also, from [30, Theorem 3.2], if $T : A \to B$ is a surjective linear map such that $c(T(a)) = c(a)$ for every $a \in A$, then T is an isometric Jordan $*$-homomorphism multiplied by a unitary element. Hence, we replace the condition $c(T(a)) = c(a)$ by

$$\sup_{\|a\|=1} |c(T(a)) - c(a)| < \varepsilon, \tag{12}$$

in order to get approximate versions of [30, Theorems 3.1 and 3.2].

2. Ultraproducts of Banach Algebras: Basic Tools

Given a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and a sequence of Banach spaces $\{X_n\}_{n\in\mathbb{N}}$, the so called *ultraproduct* of the sequence is defined as follows:

$$(X_n)^{\mathcal{U}} := \frac{\ell^\infty(\mathbb{N}, X_n)}{\mathcal{N}_{\mathcal{U}}}, \tag{13}$$

where $\ell^\infty(\mathbb{N}, X_n)$ is the Banach space of all bounded sequences $\{x_n\}_{n\in\mathbb{N}}$, with $x_n \in X_n$ for all $n \in \mathbb{N}$, equipped with the ℓ_∞ norm and

$$\mathcal{N}_{\mathcal{U}} := \left\{\{x_n\}_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, X_n) : \lim_{\mathcal{U}} \|x_n\| = 0\right\}. \tag{14}$$

If the sequence $\{X_n\}_{n\in\mathbb{N}} = \{X\}$ is constant, $X^{\mathcal{U}} := \ell^\infty(\mathbb{N}, X)/\mathcal{N}_{\mathcal{U}}$, is called the *ultrapower* of X with respect to the ultrafilter $\mathcal{U}$. We will denote by $\mathbf{x} = [x_n]$ the equivalence class of the sequence $\{x_n\}_{n\in\mathbb{N}}$. The ultrapower of a Banach space is also a Banach space provided with the following norm:

$$\|\mathbf{x}\| := \lim_{\mathcal{U}} \|x_n\|. \tag{15}$$

Of course, the ultrapower $A^{\mathcal{U}}$ of a Banach algebra (resp., C^*-algebra) is also a Banach algebra (resp., C^*-algebra), with respect to the pointwise operations.

Finally, for every Banach spaces X and Y, the canonical linear isometry $\mathcal{B}(X,Y)^{\mathcal{U}} \to \mathcal{B}(X^{\mathcal{U}},Y^{\mathcal{U}})$ given by

$$\mathbf{T}(\mathbf{x}) = [T_n(x_n)], \tag{16}$$

for every $\mathbf{T} = [T_n] \in \mathcal{B}(X,Y)^{\mathcal{U}}$ and $\mathbf{x} = [x_n] \in X^{\mathcal{U}}$, allows us to consider $\mathcal{B}(X,Y)^{\mathcal{U}}$ as a closed subspace of $\mathcal{B}(X^{\mathcal{U}},Y^{\mathcal{U}})$. For $X = Y$, the canonical map gives an isometric unital homomorphism from $\mathcal{B}(X)^{\mathcal{U}}$ to $\mathcal{B}(X^{\mathcal{U}})$. The reader can see [31] in order to find basic results on ultraproducts.

Let A be a unital Banach algebra and $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$. The following proposition is devoted to the description of invertible elements in $A^{\mathcal{U}}$ through certain coset representatives. The result is probably well known, but the lack of an adequate reference moves us to include it here.

Proposition 1. *Let $\mathbf{a} \in A^{\mathcal{U}}$. The following assertions are equivalent.*

(1) $\mathbf{a}$ *is invertible.*

(2) $\mathbf{a}$ *has a coset representative $[u_n]$ such that $u_n \in A^{-1}$ for all $n \in \mathbb{N}$ and $\{u_n^{-1}\}_{n\in\mathbb{N}}$ is bounded.*

Proof. For (2) $\Rightarrow$ (1), just note that $[u_n^{-1}] \in A^{\mathcal{U}}$ is an inverse for $[u_n]$.

Reciprocally, assume that $\mathbf{a} = [a_n]$ is invertible. Then, there exists $\mathbf{b} = [b_n] \in A^{\mathcal{U}}$ such that $\mathbf{ab} = \mathbf{ba} = \mathbf{1}$. That is,

$$\begin{aligned} \lim_{\mathcal{U}} \|a_n b_n - \mathbf{1}\| &= 0, \\ \lim_{\mathcal{U}} \|b_n a_n - \mathbf{1}\| &= 0. \end{aligned} \tag{17}$$

Fix $0 < \delta < 1$. The above identities imply that

$$\begin{aligned} R &:= \{n \in \mathbb{N} : \|a_n b_n - \mathbf{1}\| < \delta\} \in \mathcal{U}, \\ L &:= \{n \in \mathbb{N} : \|b_n a_n - \mathbf{1}\| < \delta\} \in \mathcal{U}. \end{aligned} \tag{18}$$

In particular, a_n is right invertible for every $n \in R$ and a_n is left invertible for every $n \in L$. Thus, a_n is invertible for every $n \in I := R \cap L \in \mathcal{U}$. Moreover,

$$\begin{aligned} \|a_n^{-1}\| &= \|b_n(a_n b_n)^{-1}\| \le \|b_n\| \, \|(a_n b_n)^{-1}\| \\ &\le \frac{\|b_n\|}{1 - \|1 - a_n b_n\|} \le \frac{\|b_n\|}{1-\delta}, \end{aligned} \tag{19}$$

which shows that $\{a_n^{-1} : n \in I\}$ is bounded. Therefore, we can assume without loss of generality that $\{a_n\}_{n\in\mathbb{N}}$ consists of invertible elements and $\{a_n^{-1}\}_{n\in\mathbb{N}}$ is bounded. (Otherwise, we choose

$$a_n' = \begin{cases} a_n, & \text{if } n \in I, \\ 1, & \text{if } n \notin I. \end{cases} \tag{20}$$

Clearly, $[a_n] = [a_n']$.) □

Remark 2. It is clear that for $\mathbf{a} = [a_n] \in A^{\mathcal{U}}$ with $\|\mathbf{a}\| = 1$, we can choose a coset representative $\mathbf{a} = [b_n]$ such that $\|b_n\| = 1$ for all $n \in \mathbb{N}$ as follows:

$$b_n = \begin{cases} \dfrac{a_n}{\|a_n\|}, & \text{if } a_n \neq 0, \\ 1, & \text{if } a_n = 0. \end{cases} \tag{21}$$

Hence, for every invertible element $\mathbf{a}$ in $A^{\mathcal{U}}$, we can find a coset representative $\mathbf{a} = [a_n]$ fulfilling the conditions in Proposition 1 and satisfying $\|a_n\| = \|\mathbf{a}\|$ for all $n \in \mathbb{N}$. We will name this one a *normalized representative* for $\mathbf{a}$.

3. Approximate Preservers in Banach Algebras

Let A and B be unital Banach algebras. Recall that, Boudi and Mbekhta proved in [4, Theorem 2.2] that an additive map $T : A \to B$ strongly preserves invertibility if and only if $T(1)T$ is a unital Jordan homomorphism and $T(1)$ commutes with the range of T. Hence, for a bounded linear map $T : A \to B$ between unital Banach algebras, we consider the *unit-commutativity* of T, defined as

$$\operatorname{ucomm}(T) = \sup_{\|a\|=1} \|T(a)T(1) - T(1)T(a)\|, \tag{22}$$

in order to measure how close is our "approximately preserving invertibility" map to fulfilling that property.

Obviously, every bounded linear map satisfies ucomm $(T) \le 2\|T\|^2$. The next lemma shows the good behaviour of this concept with the ultraproduct of operators.

Some arguments in this section are inspired in [27].

Lemma 3. *Let $\{T_n\}_{n\in\mathbb{N}}$ be a bounded sequence of linear maps between Banach algebras A and B, where A is supposed to be unital. Consider $\mathbf{T} = [T_n] : A^{\mathcal{U}} \to B^{\mathcal{U}}$. Then,*

$$\lim_{\mathcal{U}} \operatorname{ucomm}(T_n) = \operatorname{ucomm}(\mathbf{T}). \tag{23}$$

Proof. Given $\mathbf{a} \in A^{\mathcal{U}}$ with $\|\mathbf{a}\| = 1$, we can choose $\mathbf{a} = [a_n]$ with $\|a_n\| = 1$ for every $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} &\|\mathbf{T}(\mathbf{a})\mathbf{T}(\mathbf{1}) - \mathbf{T}(\mathbf{1})\mathbf{T}(\mathbf{a})\| \\ &= \lim_{\mathcal{U}} \|T_n(a_n)T_n(1) - T_n(1)T_n(a_n)\| \le \lim_{\mathcal{U}} \operatorname{ucomm}(T_n), \end{aligned} \tag{24}$$

and hence,

$$\operatorname{ucomm}(\mathbf{T}) \le \lim_{\mathcal{U}} \operatorname{ucomm}(T_n). \tag{25}$$

Reciprocally, for each $n \in \mathbb{N}$, there exists $a_n \in A$ with $\|a_n\| = 1$ such that

$$\operatorname{ucomm}(T_n) - \frac{1}{n} < \|T_n(a_n)T_n(1) - T_n(1)T_n(a_n)\|. \tag{26}$$

Taking limit along $\mathcal{U}$ we obtain

$$\begin{aligned}&\lim_{\mathcal{U}} \operatorname{ucomm}(T_n)\\&\quad\le \lim_{\mathcal{U}} \left\|T_n(a_n)T_n(1)-T_n(1)T_n(a_n)\right\| \\&\quad= \left\|\mathbf{T}(\mathbf{a})\mathbf{T}(\mathbf{1})-\mathbf{T}(\mathbf{1})\mathbf{T}(\mathbf{a})\right\| \le \operatorname{ucomm}(\mathbf{T}).\end{aligned} \tag{27}$$

□

Our first main result provides an approximate version of Hua's theorem for Banach algebras [4, Theorem 2.2] in the above mentioned.

Theorem 4. *Let A and B be unital Banach algebras and $K, \varepsilon > 0$. Then, there exists $\delta > 0$ such that for every linear map $T : A \to B$ with $\|T\| < K$, the condition*

$$\sup_{\|a\|=1, a\in A^{-1}} \left\|T\left(a^{-1}\right)-T(a)^{-1}\right\| < \delta \tag{28}$$

implies that

$$\operatorname{jmult}(T(\mathbf{1})T) < \varepsilon, \qquad \operatorname{ucomm}(T) < \varepsilon. \tag{29}$$

Proof. Suppose that the assertion of the theorem is false. Then, we can find $K_0, \varepsilon_0 > 0$ and a sequence $\{T_n\}_{n\in\mathbb{N}}$ of linear maps from A to B such that, for every $n \in \mathbb{N}$,

(i) $\|T_n\| < K_0$,

(ii) $\sup_{\|a\|=1}\|T_n(a^{-1}) - T_n(a)^{-1}\| < 1/n$,

(iii) $\operatorname{jmult}(T_n(1)T_n) \ge \varepsilon_0$ or $\operatorname{ucomm}(T_n) \ge \varepsilon_0$.

Consider that $\mathbf{T} = [T_n] : A^{\mathcal{U}} \to B^{\mathcal{U}}$. We claim that $\mathbf{T}$ strongly preserves invertibility. Indeed, let $\mathbf{a} \in A^{\mathcal{U}}$ be an invertible element. We can suppose, without loss of generality, that $\|\mathbf{a}\| = 1$. Let $[a_n]$ be its normalized representative, with $\|a_n^{-1}\| < \alpha$, for some $\alpha > 0$ (see Proposition 1 and Remark 2). As

$$\begin{gathered}\left\|T_n\left(a_n^{-1}\right)\right\| \le \|T_n\|\left\|a_n^{-1}\right\| < K_0\alpha,\\ \left\|T_n\left(a_n^{-1}\right)-T_n(a_n)^{-1}\right\| < 1,\end{gathered} \tag{30}$$

we get

$$\left\|T_n(a_n)^{-1}\right\| < K_0\alpha + 1, \tag{31}$$

for all $n \in \mathbb{N}$. Hence, $\mathbf{T}(\mathbf{a})$ is invertible and $[T_n(a_n)^{-1}]$ is its inverse. This yields

$$\begin{aligned}&\left\|\mathbf{T}\left(\mathbf{a}^{-1}\right)-\mathbf{T}(\mathbf{a})^{-1}\right\|\\&\quad= \lim_{\mathcal{U}}\left\|T_n\left(a_n^{-1}\right)-T_n(a_n)^{-1}\right\| \le \lim_{\mathcal{U}}\frac{1}{n} = 0.\end{aligned} \tag{32}$$

Thus, $\mathbf{T}(\mathbf{a}^{-1}) = \mathbf{T}(\mathbf{a})^{-1}$ for every invertible element $\mathbf{a} \in A^{\mathcal{U}}$. By [4, Theorem 2.2], $\mathbf{T}(\mathbf{1})\mathbf{T}(\mathbf{a}^2) = (\mathbf{T}(\mathbf{1})\mathbf{T}(\mathbf{a}))^2$ and $\mathbf{T}(\mathbf{1})\mathbf{T}(\mathbf{a}) = \mathbf{T}(\mathbf{a})\mathbf{T}(\mathbf{1})$, for every $\mathbf{a} \in A^{\mathcal{U}}$. We apply [27, Lemmas 3.4] and Lemma 3 to obtain, respectively, the following:

$$\begin{gathered}0 = \operatorname{jmult}(\mathbf{T}(\mathbf{1})\mathbf{T}) = \lim_{\mathcal{U}}\operatorname{jmult}(T_n(1)T_n),\\ 0 = \operatorname{ucomm}(\mathbf{T}) = \lim_{\mathcal{U}}\operatorname{ucomm}(T_n).\end{gathered} \tag{33}$$

Consequently,

$$\begin{gathered}I = \{n \in \mathbb{N} : \operatorname{jmult}(T_n(1)T_n) < \varepsilon_0\} \in \mathcal{U},\\ J = \{n \in \mathbb{N} : \operatorname{ucomm}(T_n) < \varepsilon_0\} \in \mathcal{U}.\end{gathered} \tag{34}$$

Finally, $I \cap J \in \mathcal{U}$ gives us the desired contradiction. □

Our goal now is to achieve a group invertibility version for the previous theorem. Recall that given an additive map $T : A \to B$ from a unital Banach algebra A into a Banach algebra B, by [9, Theorem 2.4], if T strongly preserves group invertibility, then $T(1)T$ is a Jordan homomorphism and $T(1)$ commutes with the range of T. In order to take advantage of Proposition 1, our first step is to improve [9, Theorem 2.4] by showing that all the information required is located in A^{-1}.

Recall that the so-called Hua's identity asserts that, if a, b, and $a - b^{-1}$ are invertible elements in a ring, then

$$\left(a^{-1}-\left(a-b^{-1}\right)^{-1}\right)^{-1} = a - aba. \tag{35}$$

Theorem 5. *Let A and B be Banach algebras, A being unital, and $T : A \to B$ be an additive map such that $T(a^{-1}) = T(a)^G$ for all $a \in A^{-1}$. Then, $T(1)T$ is a Jordan homomorphism and $T(1)$ commutes with $T(A)$.*

Proof. A look to the arguments employed in [9, Lemma 2.1], allows us to show that T preserves the cubes of the invertible elements. Indeed, given $u \in A^{-1}$ and $\lambda \in \mathbb{Q}$ with $0 < |\lambda| < \|u^{-1}\|^{-2}$, as $\lambda^{-1}u$ and $u - \lambda u^{-1}$ are invertible elements, we can apply Hua' s identity to obtain

$$\left(u^{-1}-\left(u-\lambda u^{-1}\right)^{-1}\right)^{-1} = u - u\left(\lambda^{-1}u\right)u = u - \lambda^{-1}u^3. \tag{36}$$

Let us assume that $T(u) \neq 0$. Since $T(u) \in B^G$, it follows that $T(u)$ is invertible in the unital Banach algebra pBp for $p = T(u)T(u)^G$, with inverse $T(u)^G$. Identity (36) applied for $T(u)$ and $0 < |\lambda| < \| T(u)^G\|^{-2}$ gives

$$T(u) - \lambda^{-1}T(u)^3 = \left(T(u)^G - \left(T(u) - \lambda T(u)^G\right)^G\right)^G. \tag{37}$$

Hence, for every $\lambda \in \mathbb{Q}$ such that $0 < |\lambda| < \min\{\|u^{-1}\|^{-2}, \|T(u)^G\|^{-2}\}$, we get

$$\begin{aligned} &T(u) - \lambda^{-1}T(u)^3 \\ &= \left(T\left(u^{-1}\right) - T\left(u - \lambda u^{-1}\right)^G\right)^G \\ &= T\left(u^{-1} - \left(u - \lambda u^{-1}\right)^{-1}\right)^G = T\left(\left(u^{-1} - \left(u - \lambda u^{-1}\right)^{-1}\right)^{-1}\right) \\ &= T\left(u - \lambda^{-1}u^3\right) = T(u) - \lambda^{-1}T\left(u^3\right). \end{aligned} \tag{38}$$

Hence, $T(u^3) = T(u)^3$, as desired. From this last identity, reasoning as in [9, Proposition 2.3], we deduce that the following equalities hold for every $x \in A$:

$$\begin{aligned} 3T(x) &= T(\mathbf{1})^2T(x) + T(x)T(\mathbf{1})^2 + T(\mathbf{1})T(x)T(\mathbf{1}), \\ 3T\left(x^2\right) &= T(x)^2T(\mathbf{1}) + T(\mathbf{1})T(x)^2 + T(x)T(\mathbf{1})T(x). \end{aligned} \tag{39}$$

Finally, it only remains to repeat the arguments in (2) $\Rightarrow$ (3) in [9, Theorem 2.4] to conclude the proof. □

Now, we can state the following result.

Theorem 6. *Let A and B be Banach algebras where A is unital and $K, \varepsilon > 0$. Then, there exists $\delta > 0$ such that for every linear map $T : A \to B$ with $\|T\| < K$, the condition*

$$\sup_{\|a\|=1, a\in A^G} \left\|T\left(a^G\right) - T(a)^G\right\| < \delta \tag{40}$$

implies that

$$jmult\,(T(\mathbf{1})T) < \varepsilon, \qquad ucomm\,(T) < \varepsilon. \tag{41}$$

Proof. First, notice that if $\mathbf{b} \in A^{\mathcal{U}}$ has a coset representative $\mathbf{b} = [b_n]$ where b_n is group invertible for every $n \in \mathbb{N}$ and $\{b_n^G\}_{n\in\mathbb{N}}$ is bounded, then $\mathbf{b}$ is group invertible and $\mathbf{b}^G = [b_n^G]$. Hence, the same arguments used in Theorem 4 produce an operator $\mathbf{T} = [T_n] : A^{\mathcal{U}} \to B^{\mathcal{U}}$ satisfying $\mathbf{T}(\mathbf{a}^{-1}) = \mathbf{T}(\mathbf{a})^G$ for every invertible element $\mathbf{a} \in A^{\mathcal{U}}$. Now, Theorem 5 proves that $\mathbf{T}(\mathbf{1})\mathbf{T}$ is a Jordan homomorphism and $\mathbf{T}(\mathbf{1})$ commutes with $\mathbf{T}(A^{\mathcal{U}})$. Again, the final argument in Theorem 4 completes the proof. □

In [32, Proposition 2.5], the authors proved, in particular, that if an additive map $T : A \to B$ between unital Banach algebras satisfies

$$T(a)T\left(a^{-1}\right) = T(1)^2, \quad \text{for every } a \in A^{-1}, \tag{42}$$

and $T(1)$ is invertible, then $T(1)^{-1}T$ is a Jordan homomorphism and $T(1)$ commutes with $T(A)$. It is clear now that for a sequence of linear operators $T_n : A \to B$ satisfying that $\|T_n\|, \|T_n(1)^{-1}\| < K$ for all $n \in \mathbb{N}$, and

$$\left\|T_n(a)T_n\left(a^{-1}\right) - T_n(1)^2\right\| < \frac{1}{n} \quad \forall n \in \mathbb{N}, \tag{43}$$

its ultrapoduct $\mathbf{T} : A^{\mathcal{U}} \to B^{\mathcal{U}}$ fulfills $\mathbf{T}(\mathbf{a})\mathbf{T}(\mathbf{a}^{-1}) = \mathbf{T}(\mathbf{1})^2$ for every invertible $\mathbf{a} \in A^{\mathcal{U}}$. Therefore, $\mathbf{T}(\mathbf{1})^{-1}\mathbf{T}$ is a Jordan homomorphism and $\mathbf{T}(\mathbf{1})$ commutes with $\mathbf{T}(A^{\mathcal{U}})$. This leads us to the following approximate formulation of [32, Proposition 2.5].

Theorem 7. *Let A and B be unital Banach algebras and $K, \varepsilon > 0$. Then, there exists $\delta > 0$ such that for every linear map $T : A \to B$ with $\|T\|, \|T(1)^{-1}\| < K$, the following condition:*

$$\sup_{a\in A^{-1}} \left\|T(a)T\left(a^{-1}\right) - T(1)^2\right\| < \delta \tag{44}$$

implies that

$$jmult\left(T(1)^{-1}T\right) < \varepsilon, \quad ucomm\,(T) < \varepsilon. \tag{45}$$

4. Approximate Preservers in C^*-Algebras

The aim of this section is twofold. On the one hand, we prove that linear maps approximately preserving generalized invertibility in C^*-algebras are close to be triple homomorphisms. On the other hand, we study linear maps approximately preserving the conorm.

4.1. Approximate Preservers of the Moore-Penrose Inverse and the Generalized Inverse. Given a JB^*-triple E, the *triple cube* of an element $x \in E$ is defined as $x^{[3]} := \{x, x, x\}$. An element satisfying $e^{[3]} = e$ is called a *tripotent*. The following polarization identity allows us to write the triple product as linear combination of triple cubes:

$$\{x, y, z\} = \sum_{\alpha^2=1}\sum_{\beta^4=1} \alpha\beta(x + \alpha y + \beta z)^{[3]}, \quad \text{for } x, y, z \in E. \tag{46}$$

Hence, a linear map between JB^*-triples is a triple homomorphism if and only if it preserves triple cubes.

Each tripotent e in E gives rise to the so-called *Peirce decomposition* of E associated to e, that is,

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e), \tag{47}$$

where for $i = 0, 1, 2, E_i(e)$ is the $i/2$ eigenspace of $L(e, e)$. The peirce space $E_2(e)$ is a JB^*-algebra with product $x \cdot y := \{x, e, y\}$ and involution $x^\# := \{e, x, e\}$. Moreover, the triple product induced on $E_2(e)$ by this Jordan *-algebra structure coincides with its original triple product.

It is proved in [16, Lemma 3.2] (compare with [13, Theorem 3.4]) that for every regular element a in a JB^*-triple E, there exists a tripotent $e \in E$ such that a is a self-adjoint invertible element in the JB^*-algebra $E_2(e)$. If a is invertible, its inverse is denoted as usual by a^{-1}. Moreover, if a and b are invertible elements in the Jordan algebra $J = (J, \circ)$ such that $a - b^{-1}$ is also invertible, then $a^{-1} + (b^{-1} - a)^{-1}$ is invertible and the Hua identity

$$\left(a^{-1} + \left(b^{-1} - a\right)^{-1}\right)^{-1} = a - U_a(b) \tag{48}$$

holds, where $U_a(x) = 2a \circ (a \circ x) - a^2 \circ x$ (see [33], (11)).

Let A be a unital C^*-algebra, and $u \in A^\wedge \setminus \{0\}$. Then, there exists a unique partial isometry e, such that u is self-adjoint and invertible in the Jordan algebra $(E_A)_2(e) = ee^*Ae^*e$, with inverse $u^\wedge$. Hence, for every $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \|u^\wedge\|^{-2}$, the element $u - \lambda u^\wedge$ is invertible in ee^*Ae^*e. Reciprocally, the inverses of $u - \lambda u^\wedge$ and $u^\wedge - (u - \lambda u^\wedge)^\wedge$ in ee^*Ae^*e are their generalized inverses in A. By the Hua identity (48), we obtain

$$u - \lambda^{-1}u^{[3]} = \left(u^\wedge - \left(u - \lambda u^\wedge\right)^\wedge\right)^\wedge. \tag{49}$$

Theorem 8. *Let A and B be C^*-algebras, A being unital, and $T : A \to B$ a bounded linear map satisfying $T(x^{-1}) = T(x)^\wedge$ for every self-adjoint invertible element $x \in A$. Then, T is a triple homomorphism.*

Proof. Arguing as in [9, Lemma 3.1], pick a self-adjoint invertible element $u \in A$. We may assume that $T(u) \neq 0$. Then, given $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \min\{\|u^{-1}\|^{-2}, \|T(u)^\wedge\|^{-2}\}$, from identities (36) and (49), we get

$$\begin{aligned} &T(u) - \lambda^{-1}T(u)^{[3]} \\ &\quad = \left(T(u)^\wedge - \left(T(u) - \lambda T(u)^\wedge\right)^\wedge\right)^\wedge = T(u) - \lambda^{-1}T\left(u^3\right), \end{aligned} \tag{50}$$

which shows that $T(u^3) = T(u)^{[3]}$. Once we have proved that T preserves cubes of self-adjoint invertible elements, given a self-adjoint element $a \in A$, and $\alpha \in \mathbb{R}$ with $|\alpha| > \|a\|$, as the element $a + \alpha$ is self-adjoint and invertible, we get $T((a+\alpha)^3) = T(a+\alpha)^{[3]}$. Expanding this las equation we obtain

$$\begin{aligned} &T\left(a^3\right) + 3\alpha T\left(a^2\right) + 3\alpha^2 T(a) + \alpha^3 T(1) \\ &\quad = T(a)^{[3]} + \alpha^3 T(1) + 2\alpha\{T(a), T(a), T(1)\} \\ &\qquad + \alpha\{T(a), T(1), T(a)\} + 2\alpha^2\{T(1), T(1), T(a)\} \\ &\qquad + \alpha^2\{T(1), T(a), T(1)\}, \end{aligned} \tag{51}$$

for every $a \in A$, and $|\alpha| > \|a\|$. From this, we deduce that $T(a^3) = T(a)^{[3]}$ for every $a \in A$. That is, T preserves triple cubes of self-adjoint elements. By [34, Theorem 20], T is a triple homomorphism. □

Recall that we measure how close is a linear map $T : A \to B$ between C^*-algebras to being a triple homomorphism or self-adjoint by the triple multiplicativity and the self-adjointness of T, respectively as follows:

$$\begin{aligned} \operatorname{tmult}(T) &:= \sup\{\|T(\{a,b,c\}) - \{T(a), T(b), T(c)\}\| : \\ &\qquad \|a\| = \|b\| = \|c\| = 1\}, \\ \operatorname{sa}(T) &:= \sup_{\|a\|=1}\left\|T\left(a^*\right)^* - T(a)\right\|. \end{aligned} \tag{52}$$

Remark 9. Let A and B be C^*-algebras. It is clear that every Jordan *-homomorphism $T : A \to B$ is a triple homomorphism. We ask whether $\operatorname{jmult}(T)$ and $\operatorname{sa}(T)$ being small imply $\operatorname{tmult}(T)$ being small.

Let $T : A \to B$ be a bounded linear map. Define

$$\begin{aligned} T_J(a,b) &:= T(a \circ b) - T(a) \circ T(b), \\ T_T(a,b,c) &:= T(\{a,b,c\}) - \{T(a), T(b), T(c)\}, \\ T_*(a) &:= T\left(a^*\right)^* - T(a), \end{aligned} \tag{53}$$

for every $a, b, c \in A$. Then,

$$\begin{aligned} &T(\{a,b,c\}) \\ &\quad = T\left((a \circ b^*) \circ c\right) + T\left(a \circ \left(b^* \circ c\right)\right) - T\left((a \circ c) \circ b^*\right) \\ &\quad = T\left(a \circ b^*\right) \circ T(c) + T_J\left(a \circ b^*, c\right) \\ &\qquad + T(a) \circ T\left(b^* \circ c\right) + T_J\left(a, b^* \circ c\right) \\ &\qquad - T(a \circ c) \circ T\left(b^*\right) - T_J\left(a \circ c, b^*\right) \\ &\quad = \left(T(a) \circ T\left(b^*\right)\right) \circ T(c) + T_J\left(a, b^*\right) \circ T(c) \\ &\qquad + T_J\left(a \circ b^*, c\right) + T(a) \circ \left(T\left(b^*\right) \circ T(c)\right) \\ &\qquad + T(a) \circ T_J\left(b^*, c\right) + T_J\left(a, b^* \circ c\right) \\ &\qquad - (T(a) \circ T(c)) \circ T\left(b^*\right) - T_J(a,c) \circ T\left(b^*\right) \\ &\qquad - T_J\left(a \circ c, b^*\right) \\ &\quad = \left\{T(a), T\left(b^*\right)^*, T(c)\right\} + T_J\left(a, b^*\right) \circ T(c) \\ &\qquad + T_J\left(a \circ b^*, c\right) + T(a) \circ T_J\left(b^*, c\right) + T_J\left(a, b^* \circ c\right) \\ &\qquad - T_J(a,c) \circ T\left(b^*\right) - T_J\left(a \circ c, b^*\right). \end{aligned} \tag{54}$$

Therefore,

$$\begin{aligned} &T_T(a,b,c) \\ &\quad = \{T(a), T_*(b), T(c)\} + T_J\left(a, b^*\right) \circ T(c) + T_J\left(a \circ b^*, c\right) \\ &\qquad + T(a) \circ T_J\left(b^*, c\right) + T_J\left(a, b^* \circ c\right) \\ &\qquad - T_J(a,c) \circ T\left(b^*\right) - T_J\left(a \circ c, b^*\right). \end{aligned} \tag{55}$$

This implies that

$$\operatorname{tmult}(T) \leq \|T\|^2 \operatorname{sa}(T) + 3(\|T\| + 1)\operatorname{jmult}(T). \tag{56}$$

As in Lemma 3, it can be shown that, for an operator $\mathbf{T} = [T_n]$, the following holds:

$$\begin{aligned} \lim_{\mathcal{U}} \operatorname{tmult}\left(T_n\right) &= \operatorname{tmult}(\mathbf{T}), \\ \lim_{\mathcal{U}} \operatorname{sa}\left(T_n\right) &= \operatorname{sa}(\mathbf{T}). \end{aligned} \tag{57}$$

We will omit the proof of the next result in order to avoid repetition. The argument is analogous to the one used in the proofs of Theorems 4 and 6: assuming the contrary, we can construct a map $\mathbf{T} = [T_n] : A^{\mathcal{U}} \to B^{\mathcal{U}}$ between the ultrapowers fulfilling $\mathbf{T}(\mathbf{x}^{-1}) = \mathbf{T}(\mathbf{x})^{\wedge}$ for every self-adjoint invertible $\mathbf{x} \in A^{\mathcal{U}}$. By Theorem 8, $\mathbf{T}$ is a triple homomorphism.

Theorem 10. *Let A and B be C^*-algebras where A is unital and $K, \varepsilon > 0$. Then, there exists $\delta > 0$ such that for every linear map $T : A \to B$ with $\|T\| < K$, the condition*

$$\sup_{\|a\|=1, a=a^*} \left\| T\left(a^{\wedge}\right) - T(a)^{\wedge} \right\| < \delta \tag{58}$$

implies that

$$tmult\,(T) < \varepsilon. \tag{59}$$

Remark 11. Notice that in Theorem 8 it can be also obtained that $T(1)^*T$ is a Jordan *-homomorphism. Hence, the hypothesis of Theorem 10 also yields to $\mathrm{jmult}(T(1)^*T) < \varepsilon$ and $\mathrm{sa}(T(1)^*T) < \varepsilon$.

Corollary 12. *Let A and B be C^*-algebras where A is unital and $K, \varepsilon > 0$. Then, there exists $\delta > 0$ such that for every linear map $T : A \to B$ with $\|T\| < K$, the conditions*

$$\sup_{\|a\|=1, a\in A^{\dagger}} \left\| T\left(a^{\dagger}\right) - T(a)^{\dagger} \right\| < \delta, \qquad sa\,(T) < \delta \tag{60}$$

imply that

$$tmult\,(T) < \varepsilon. \tag{61}$$

Proof. Let us briefly sketch the proof: assuming the contrary, there exist $K_0, \varepsilon_0 > 0$ and a sequence $\{T_n\}_{n\in\mathbb{N}}$ of linear maps from A to B such that, for every $n \in \mathbb{N}$,

$$\begin{gathered} \|T_n\| < K_0, \qquad \sup_{\|a\|=1} \left\| T_n\left(a^{\dagger}\right) - T_n(a)^{\dagger} \right\| < \frac{1}{n}, \\ \mathrm{sa}\left(T_n\right) < \frac{1}{n}, \\ \mathrm{tmult}\left(T_n\right) \geq \varepsilon_0. \end{gathered} \tag{62}$$

Then, $\mathbf{T} = [T_n] : A^{\mathcal{U}} \to B^{\mathcal{U}}$ is a self-adjoint map such that $\mathbf{T}(\mathbf{a}^{-1}) = \mathbf{T}(\mathbf{a})^{\dagger}$, for every invertible element $\mathbf{a} \in A^{\mathcal{U}}$. In particular, $\mathbf{T}(\mathbf{a}^{-1}) = \mathbf{T}(\mathbf{a})^{\wedge}$, for every invertible self-adjoint element $\mathbf{a} \in A^{\mathcal{U}}$. From Theorem 8, $\mathbf{T}$ is a triple homomorphism (Contradiction). □

4.2. Maps Approximately Preserving the Conorm. Let A and B be unital C^*-algebras. Kadison proved in [35] that a surjective linear map $T : A \to B$ is an isometry if and only if T is a Jordan*-isomorphism multiplied by a unitary element in B. In [30], the authors address the question of characterizing surjective linear maps preserving some spectral quantities. Given an element a of a Banach algebra A, the *minimum modulus* and the *surjectivity modulus* of a are defined, respectively, by

$$\begin{gathered} m\,(a) := \inf\{\|ax\| : x \in A, \|x\| = 1\}, \\ q\,(a) := \inf\{\|xa\| : x \in A, \|x\| = 1\}. \end{gathered} \tag{63}$$

Obviously, $m(a) = 0$ (resp., $q(a) = 0$) if and only if a is a left (resp., right) topological divisor of zero. It is well known that for any invertible element $a \in A$,

$$m\,(a) = q\,(a) = \left\|a^{-1}\right\|^{-1}. \tag{64}$$

Moreover,

$$\begin{gathered} m\,(a)\,m\,(b) \leq m\,(ab) \leq \|a\|\,m\,(b), \\ |m\,(a) - m\,(b)| \leq \|a - b\|, \end{gathered} \tag{65}$$

for every $a, b \in A$. If A is a C^*-algebra, then $m(a) = q(a^*)$. In particular, $m(a) > 0$ (resp., $q(a) > 0$) if and only if a is left (resp., right) invertible.

Recall also from the introduction that the *conorm* of an element a in a Banach algebra A, is defined as

$$c\,(a) := \begin{cases} \inf\{\|ax\| : \mathrm{dist}\left(x, \ker\left(L_a\right)\right) \geq 1\}, & \text{if } a \neq 0, \\ \infty, & \text{if } a = 0. \end{cases} \tag{66}$$

For a regular element a in a C^*-algebra A,

$$c\,(a) = \left\|a^{\dagger}\right\|^{-1}. \tag{67}$$

Let A and B be unital C^*-algebras. By Theorems 3.1 and 3.2 in [30], if $T : A \to B$ is a linear map preserving any of these spectral quantities, then T is an isometric Jordan *-homomorphism whenever T is unital, and T is an isometric Jordan *-homomorphism multiplied by a unitary element, whenever T is surjective. In the next results, we show that the same holds if we just impose the preserving condition for invertible elements. Notice that we focus our attention on the conorm but identical results can be established for the minimum and surjective modulus.

Theorem 13. *Let A and B be unital C^*-algebras and $T : A \to B$ a unital linear map satisfying $c(T(x)) = c(x)$ for all $x \in A^{-1}$. Then, T is a Jordan *-homomorphism.*

Proof. First, let us prove that T is injective. Take $a_0 \in A$ such that $T(a_0) = 0$ and let $\alpha \in \mathbb{C}$ be sufficiently small so that $1 + \alpha a_0$ is invertible. Then,

$$1 = c\,(T\,(1)) = c\left(T\left(1 + \alpha a_0\right)\right) = c\left(1 + \alpha a_0\right). \tag{68}$$

In particular, we get

$$1 \leq c\left(1 + ita_0\right), \qquad 1 \leq c\left(1 - ta_0\right), \tag{69}$$

as $t \to 0$. Hence, by [30, Lemma 4.1], both ia_0 and a_0 are self-adjoint and, consequently, $a_0 = 0$.

We claim that T is positive. Indeed, given a self-adjoint element $a \in A$, we know that

$$1 + o(t) \leq c(1 + ita), \quad (\text{as } t \to 0). \tag{70}$$

Since $1 + ita$ is invertible for $t \in \mathbb{R}$ small enough, it follows that

$$1 + o(t) \leq c(\mathbf{1} + ita) = c(1 + itT(a)), \quad (\text{as } t \to 0). \tag{71}$$

This implies that $T(a)$ is self-adjoint.

Moreover, given $x \in A$ and $\lambda \notin \sigma(x)$, there exists a neighborhood U_λ of λ such that $U_\lambda \cap \sigma(x) = \emptyset$. If $\lambda \notin \sigma(x)$, then $\lambda \notin \sigma_K(x)$, where $\sigma_K(x)$ denotes the Kato spectrum of x as follows:

$$\sigma_K(a) := \left\{\lambda \in \mathbb{C} : \lim_{\mu \to \lambda} c(a - \mu) = 0\right\}. \tag{72}$$

As for $\mu \in U_\lambda$, the element $x - \mu$ is invertible; then, we have $c(T(x) - \mu) = c(x - \mu)$ for every $\mu \in U_\lambda$. Consequently,

$$\lim_{\mu \to \lambda} c(T(x) - \mu) > 0, \tag{73}$$

and $\lambda \notin \sigma_K(T(x))$. Since $\partial\sigma(a) \subseteq \sigma_K(a) \subseteq \sigma(a)$ for every $a \in A$ (see [36, Sections 12, 13]), we have just proved that

$$\partial\sigma(T(x)) \subseteq \sigma_K(T(x)) \subseteq \sigma(x), \tag{74}$$

for every $x \in A$. Being T self-adjoint, this implies that T is positive and hence, $\|T\| = 1$.

Arguing as in [30, Theorem 5.1], given a self-adjoint element $a \in A$ and t sufficiently small so that $u = e^{ita}$ is a unitary element with spectrum strictly contained in the unit circle $\mathbb{T}$, since

$$\begin{gathered} \partial\sigma(T(u)) \subsetneq \mathbb{T}, \quad \|T(u)\| \leq 1, \\ \left\|T(u)^{-1}\right\| = c(T(u))^{-1} = c(u)^{-1} = 1, \end{gathered} \tag{75}$$

the element $T(u)$ is unitary. From

$$\begin{aligned} 1 &= T(u)T(u)^* = T(u)T(u^*) \\ &= \left(\mathbf{1} + itT(a) - \frac{1}{2}t^2T(a^2) + \cdots\right) \\ &\quad \times \left(\mathbf{1} - itT(a) - \frac{1}{2}t^2T(a^2) + \cdots\right), \end{aligned} \tag{76}$$

we deduce that $T(a^2) = T(a)^2$ as desired. □

Theorem 14. *Let A and B be unital C^*-algebras and $T : A \to B$ a surjective linear map satisfying $c(T(x)) = c(x)$ for all $x \in A^{-1}$. Then, T is a Jordan *-homomorphism multiplied by a unitary element in B.*

Proof. First, let us prove that $b = T(1)$ is invertible. Since $c(T(1)) = c(1) = 1$, b is regular. Let $y = 1 - bb^\dagger$, and $x \in A$ such that $y = T(x)$. Notice that $b^*y = y^*b = 0$.

For $\alpha \in \mathbb{C}$ sufficiently small such that $1 + \alpha x \in A^{-1}$,

$$c(1 + \alpha x)^2 = c(b + \alpha y)^2 = c\left(bb^* + |\alpha|^2 yy^*\right). \tag{77}$$

Hence,

$$\lim_{|\alpha| \to 0} c\left(bb^* + |\alpha|^2 yy^*\right) = \lim_{|\alpha| \to 0} c(1 + \alpha x)^2 = 1 = c\left(bb^*\right). \tag{78}$$

Reasoning in a similar way to [30, Theorem 6.2], we get

$$c(1 + \alpha x)^2 = c\left(bb^* + |\alpha|\, yy^*\right) \geq 1 - |\alpha|^2\|y\|^2, \tag{79}$$

and therefore

$$\begin{gathered} c(1 + itx) \geq \left(1 - t^2\|y\|^2\right)^{1/2}, \\ c(1 - tx) \geq \left(1 - t^2\|y\|^2\right)^{1/2}, \end{gathered} \tag{80}$$

for small enough $t \in \mathbb{R}$. From these inequalities, we get, respectively, that x and ix are self-adjoint. This shows that $x = 0$ and thus $y = 0$. Consequently, $1 = bb^\dagger$, that is, b is right invertible. Similarly it can be proved that b is left invertible.

Note that, as in the previous theorem, T is injective. Therefore $S := b^{-1}T$ is a unital and bijective linear map satisfying

$$\begin{aligned} m(S(x)) &= m\left(b^{-1}T(x)\right) \leq \left\|b^{-1}\right\| m(T(x)) \\ &= m(T(x)) \leq c(T(x)) = c(x), \quad \forall x \in A^{-1}. \end{aligned} \tag{81}$$

Let y be a self-adjoint element in B and $t \in \mathbb{R}$ small such that $1 + itS^{-1}(y)$ is invertible. Taking $x = 1 + itS^{-1}(y)$ in the previous identity, we have

$$m(1 + ity) \leq c\left(1 + itS^{-1}(y)\right). \tag{82}$$

It follows that S^{-1} is self-adjoint and so is S.

We claim that S is positive. Note that for every $x \in A^\dagger$ and $u \in A^{-1}$, it is clear that $ux \in A^\dagger$, with

$$\begin{gathered} (ux)\left(x^\dagger u^{-1}\right)(ux) = ux, \\ \left(x^\dagger u^{-1}\right)(ux)\left(x^\dagger u^{-1}\right) = x^\dagger u^{-1}. \end{gathered} \tag{83}$$

This implies, by [11, Theorem 2], the following:

$$\begin{aligned} \frac{1}{\left\|x^\dagger u^{-1}\right\|} &\leq c(ux) \\ &\leq \frac{\left\|x^\dagger u^{-1}ux\right\| \left\|uxx^\dagger u^{-1}\right\|}{\left\|x^\dagger u^{-1}\right\|} \leq \frac{\|u\| \left\|u^{-1}\right\|}{\left\|x^\dagger u^{-1}\right\|}. \end{aligned} \tag{84}$$

Hence, for every $x \in A^{-1}$, we have

$$\begin{aligned} c(S(x)) &= c\left(b^{-1}T(x)\right) \geq \frac{1}{\left\|T(x)^\dagger b\right\|} \\ &\geq \frac{1}{\|b\| \left\|T(x)^\dagger\right\|} = \|b\|^{-1} c(T(x)), \\ c(S(x)) &= c\left(b^{-1}T(x)\right) \\ &\leq \frac{\left\|b^{-1}\right\| \|T(1)\|}{\left\|T(x)^\dagger b\right\|} \leq \|b\| \mid \left\|b^{-1}\right\|^2 c(T(x)). \end{aligned} \tag{85}$$

So, we have shown so far the following:

$$\|b\|^{-1}c(x) \le c(S(x)) \le \|b\| \left\|b^{-1}\right\|^2 c(x), \quad \forall x \in A^{-1}. \quad (86)$$

The first inequality can be used to show the following:

$$\partial\sigma(S(x)) \subset \sigma_K(S(x)) \subseteq \sigma(x), \quad (87)$$

in a similar way as in the previous theorem. As a consequence, S is positive. In order to conclude that S is an isometric Jordan *-isomorphism, it suffices to prove that S^{-1} is also positive (see for instance [35, Corollary 5]). So, let $h = S(a)$ be a positive element. As S^{-1} is self-adjoint, a is self-adjoint. We can therefore write $a = x - y$, where x and y are positive elements and $xy = yx = 0$. For every $\mu \in \mathbb{C}$, we have

$$\begin{aligned}\partial\sigma(S(x) + \mu S(y)) &\subset \sigma_K(S(x) + \mu S(y)) \\ &\subset \sigma(x + \mu y) \subset \mathbb{R} \cup \mu\mathbb{R}.\end{aligned} \quad (88)$$

(Recall that if $wz = zw = 0$, then $\sigma(w + z) \setminus \{0\} = (\sigma(w) \setminus \{0\}) \cup (\sigma(z) \setminus \{0\})$.) The previous spectral inclusion gives

$$\sigma(S(x) + \mu S(y)) \subset \mathbb{R} \cup \mu\mathbb{R}. \quad (89)$$

By Lemmas B and C in [37], we get $S(y) = 0$ and so $y = 0$. Consequently, a is positive as desired. We conclude the proof by showing that b is unitary. Indeed, since S is a Jordan $*$-isomorphism and $T(x) = bS(x)$, it is clear that, for every $x \in A^{-1}$, $T(x)$ is invertible with inverse $T(x)^{-1} = S(x^{-1})b^{-1}$. Moreover, $c(T(x)) = c(x)$; that is,

$$\left\|T(x)^{-1}\right\| = \left\|x^{-1}\right\|, \quad (90)$$

for every $x \in A^{-1}$. Since S is an isometry,

$$\left\|S\left(x^{-1}\right)\right\| = \left\|x^{-1}\right\| = \left\|S\left(x^{-1}\right)b^{-1}\right\|, \quad (91)$$

or equivalently,

$$\|y\| = \left\|yb^{-1}\right\|, \quad (92)$$

for every $y \in B^{-1}$. This yields that b is unitary. □

Note that, for every invertible element $\mathbf{a} \in A^{\mathcal{U}}$, where $\mathbf{a} = [a_n]$ is a normalized representative,

$$c(\mathbf{a}) = \left\|\mathbf{a}^{-1}\right\|^{-1} = \lim_{\mathcal{U}}\left\|a_n^{-1}\right\|^{-1} = \lim_{\mathcal{U}} c(a_n). \quad (93)$$

We are now in position to prove the main results in this section.

Theorem 15. *Let A and B be unital C^*-algebras and $K, \varepsilon > 0$. Then, there exists $\delta > 0$ such that for every linear map $T : A \to B$ with $\| T \| < K$, the following conditions:*

$$\sup_{\|a\|=1, a\in A^{-1}} |c(T(a)) - c(a)| < \delta, \qquad \|T(1) - 1\| < \delta \quad (94)$$

imply that

$$jmult(T) < \varepsilon, \qquad sa(T) < \varepsilon. \quad (95)$$

Proof. Note that if $\mathbf{b} \in B^{\mathcal{U}}$ has a coset representative $\mathbf{b} = [b_n]$ where b_n has Moore-Penrose inverse for every $n \in \mathbb{N}$ and $\{b_n^\dagger\}_{n\in\mathbb{N}}$ is bounded, then $\mathbf{b}$ is Moore-Penrose invertible and $\mathbf{b}^\dagger = [b_n^\dagger]$.

As we did above, suppose that the assertion is false, that is, we can find $K_0, \varepsilon_0 > 0$ and a sequence $\{T_n\}_{n\in\mathbb{N}}$ of linear maps from A to B satisfying

(i) $\|T_n\| \le K_0$,

(ii) $\sup_{\|a\|=1, a\in A^{-1}} |c(T_n(a)) - c(a)| < 1/n$, $\|T_n(1) - 1\| < 1/n$,

(iii) $\mathrm{jmult}(T_n) \ge \varepsilon_0$ or $\mathrm{sa}(T_n) \ge \varepsilon_0$,

for every $n \in \mathbb{N}$. Consider $\mathbf{T} = [T_n] : A^{\mathcal{U}} \to B^{\mathcal{U}}$. We claim that $\mathbf{T}$ is unital and preserves the conorm of invertible elements. On the one hand,

$$\|\mathbf{T}(\mathbf{1}) - \mathbf{1}\| = \lim_{\mathcal{U}}\left\|T_n(1) - 1\right\| \le \lim_{\mathcal{U}}\frac{1}{n} = 0, \quad (96)$$

so $\mathbf{T}(\mathbf{1}) = \mathbf{1}$. On the other hand, for an invertible element $\mathbf{a} \in A^{\mathcal{U}}$ with norm 1, let $[a_n]$ be its normalized representative: $\|a_n\| = 1$ and $\|a_n^{-1}\| < \alpha$ for some positive α. We know that

$$c(a_n) - \frac{1}{n} < c(T_n(a_n)), \quad \text{for every } n \in \mathbb{N}. \quad (97)$$

Hence $T_n(a_n) \in A^\dagger$, with $\|T_n(a_n)^\dagger\| < 2\alpha$, for $n > 2\alpha$. That is, $\{T_n(a_n)\}$ is Moore-Penrose invertible almost everywhere, with their respective Moore-Penrose inverses uniformly bounded in norm. This ensures that $\mathbf{T}(\mathbf{a})$ is Moore-Penrose invertible and $c(\mathbf{T}(\mathbf{a})) = \lim_{\mathcal{U}} c(T_n(a_n))$. Finally, Consider the following:

$$|c(\mathbf{T}(\mathbf{a})) - c(\mathbf{a})| = \lim_{\mathcal{U}}\left|c(T_n(a_n)) - c(a_n)\right| \le \lim_{\mathcal{U}}\frac{1}{n} = 0. \quad (98)$$

By Theorem 13, $\mathbf{T}$ is a Jordan *-homomorphism, which gives the contradiction. □

Let X and Y be complex Banach spaces, and let $\mathcal{B}(X, Y)$ be the algebra of all bounded linear operators from X to Y. Recall that the surjectivity modulus of T is given by $q(T) := \sup\{\varepsilon \ge 0 : \varepsilon B_Y \subseteq T(B_X)\}$, whereas usual B_X denotes the closed unit ball of X. Note that $q(T) > 0$ if and only if T is surjective, and $q(T) = \inf\{\|ST\| : S \in \mathcal{B}(X), \|S\| = 1\}$ (see [36, Theorem II.9.11]).

Theorem 16. *Let A and B be unital C^*-algebras and $K, \varepsilon > 0$. Then, there exists $\delta > 0$ such that for every linear map $T : A \to B$ with $\|T\| < K$ and $q(T) > K^{-1}$, the following condition:*

$$\sup_{\|a\|=1, a\in A^{-1}} |c(T(a)) - c(a)| < \delta \quad (99)$$

implies that

$$jmult(T(\mathbf{1})^* T) < \varepsilon, \qquad sa(T(\mathbf{1})^* T) < \varepsilon. \quad (100)$$

Proof. If we assume the contrary, hence, as above we find $K_0, \varepsilon_0 > 0$ and a sequence $\{T_n\}_{n\in\mathbb{N}}$ of linear maps from A to B satisfying

(i) $\|T_n\| \leq K_0$,

(ii) $\sup_{\|a\|=1, a\in A^{-1}} |c(T_n(a)) - c(a)| < 1/n$, $q(T_n) \geq K_0^{-1}$,

(iii) $\mathrm{jmult}(T_n(1)^* T_n) \geq \varepsilon_0$ or $\mathrm{sa}(T_n(1)^* T_n) \geq \varepsilon_0$,

for every, $n \in \mathbb{N}$.

As in the previous theorem, the map $\mathbf{T} = [T_n]$ preserves the conorm of all invertible elements. Moreover,

$$q(\mathbf{T}) = \lim_{\mathscr{U}} q(T_n) \geq K_0^{-1} > 0, \tag{101}$$

and thus $\mathbf{T}$ is surjective. By Theorem 14, $\mathbf{T}(\mathbf{1})$ is unitary and $\mathbf{T}(\mathbf{1})^* \mathbf{T}$ is a unital Jordan *-isomorphism. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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