Research Article

Explicit Multistep Mixed Finite Element Method for RLW Equation

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1. Introduction

In this paper, we consider the following initial boundary problem of RLW equation:

\[ u_t + \gamma u_x + \delta u_{xx} - \beta u_{xxt} = 0, \quad (x, t) \in I \times J, \]

\[ u(a, t) = u(b, t) = 0, \quad t \in J, \]

\[ u(x, 0) = u_0(x), \quad x \in I, \]

where \( I = (a, b) \) is a bounded open interval, \( J = (0, T] \) with \( 0 < T < \infty \). The initial value \( u_0(x) \) is given function, and the coefficients \( \beta, \gamma, \delta \) are all positive constants.

In recent years, large nonlinear phenomena are found in many research fields, for example, physics, biology, fluid dynamics, and so forth. These phenomena can be described by the mathematical model of some nonlinear evolution equations. In particular, some attention has also been paid to nonlinear RLW equations [1, 2] which play a very important role in the study of nonlinear dispersive waves. Solitary waves are wave packet or pulses, which propagate in nonlinear dispersive media. Due to dynamical balance between the nonlinear and dispersive effects, these waves retain an unchanged waveform. A soliton is a very special type of solitary wave, which also keeps its waveform after collision with other solitons. The regularized long wave (RLW) equation is an alternative description of nonlinear dispersive waves to the more usual Korteweg-de Vries (KdV) equation. Mathematical theories and numerical methods for (1) were considered in [1–24]. The existence and uniqueness of the solution of RLW equation are discussed in [5]. Their analytical solutions were found under restricted initial and boundary conditions, and therefore they got interest from a numerical point of view. Several numerical methods for the solution of the RLW equation have been introduced in the literature. These include a variety of difference methods [5–8, 17], finite element methods based on Galerkin and collocation principles [9–13], mixed finite element methods [14–16], meshfree method [18], adomian decomposition method [19], and so on.

In [25], Chatzipantelidis studied the explicit multistep methods for some nonlinear partial differential equations and discussed some mathematical theories. Akrivis et al. [26] studied the multistep method for some nonlinear evolution equations. Mei and Chen [20] presented the explicit multistep method based on Galerkin method for regularized long wave (RLW) equation. In this paper, our purpose is to propose and study an explicit multistep mixed method, which combines a mixed Galerkin method in the spatial direction and the explicit multistep method in the time direction, for RLW equation. We derive optimal error estimates in \( L^2 \) and \( H^1 \) norms for the scalar unknown \( u \) and its flux \( q = u_x \)
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for the fully discrete explicit multistep mixed scheme and compare our method's accuracy with some other numerical schemes. Compared to the numerical methods in [20, 25, 26], we not only obtain the approximation solution for \( u \), but also get the approximation solution for \( q = u_x \).

The layout of the paper is as follows. In Section 2, an explicit multistep mixed scheme and numerical process are given. The optimal error estimates in \( L^2 \) and \( H^1 \) norms for the scalar unknown \( u \) and its flux \( q = u_x \) for the fully discrete explicit multistep mixed scheme are proved in Section 3. In Section 4, some numerical results are shown to confirm our theoretical results. Finally, some concluding remarks are given in Section 5. Throughout this paper, \( C \) will denote a generic positive constant which does not depend on the spatial mesh parameter \( h \) or time discretization parameter \( \Delta t \).

### 2. The Mixed Numerical Scheme

With the auxiliary variable \( q = u_x \), we reformulate (1) as the following first-order coupled system:

\[
\begin{align*}
    u_x &= q, \\
    u_t + \gamma q + \delta u q - \beta q &= 0.
\end{align*}
\]

(2)

We consider the following mixed weak formulation of (2). Find \( (u, q) : [0, T] \to H^1_0 \times H^1 \) satisfying:

\[
\begin{align*}
    (u_t, v_h) &= (q, v_h), \quad \forall v_h \in H^1_0, \quad (u_x, w_h) = \gamma (q, w_h) + \delta (u q, w_h), \quad \forall w_h \in H^1. \quad (3) \\
    (q, w) + \beta (q x_t, w) &= \gamma (q, w x_t) + \delta (u q, w) \quad (4)
\end{align*}
\]

Noting the Dirichlet boundary conditions \( u_t(a, t) = u_t(b, t) = 0 \) and \( q_t = u_x \), we can get \( (u_t, -w_x) = (u_t, w) = (q_t, w) \) easily and then get the scheme (4).

Let \( V_h \) and \( W_h \) be finite dimensional subspaces of \( H^1_0 \) and \( H^1 \), respectively, defined by

\[
\begin{align*}
    V_h &= \left\{ v_h | v_h \in C^0(\bar{T}), v_h|_{I_j} \in P_k(I_j), \forall I_j \in T_h, v_h(a) = v_h(b) = 0 \right\} \subset H^1_0, \\
    W_h &= \left\{ w_h | w_h \in C^0(\bar{T}), w_h|_{I_j} \in P_k(I_j), \forall I_j \in T_h \right\} \subset H^1.
\end{align*}
\]

(5)

The semidiscrete mixed finite element method for (3) and (4) consists in determining \( \{ u^h, q^h \} : [0, T] \to V_h \times W_h \) such that

\[
\begin{align*}
    (u^h_t, v^h) &= (q^h, v^h), \quad \forall v^h \in V_h, \\
    (q^h, w^h) + \beta (q^h x_t, w^h) &= \gamma (q^h, w^h_x) + \delta (u^h q^h, w^h), \quad \forall w^h \in W_h. \quad (6)
\end{align*}
\]

In the following discussion, we will give an explicit multistep mixed scheme. We take linear finite element spaces \( V_h = \text{span}\{\phi_0, \phi_1, \ldots, \phi_N\} \) and \( W_h = \text{span}\{\varphi_0, \varphi_1, \ldots, \varphi_N\} \), and then \( u_h \) and \( q_h \) can be expressed as the following formulation:

\[
\begin{align*}
    u_h(x, t) &= \sum_{i=0}^N u_i(t) \varphi_i(x), \quad (x, t) \in \Omega \times J, \\
    q_h(x, t) &= \sum_{i=0}^N q_i(t) \varphi_i(x), \quad (x, t) \in \Omega \times J. \quad (8)
\end{align*}
\]

Substitute (8) into (6) and (7), and take \( v^h = \varphi_j \) and \( w^h = \varphi_j \) in (6) and (7), respectively, to obtain

\[
\begin{align*}
    \sum_{i=0}^N \left[ \left( \int_a^b \varphi_i \cdot \varphi_j dx + \beta \int_a^b \varphi_{i,x} \cdot \varphi_{j,x} dx \right) \frac{\partial q_i}{\partial t} \right. \\
    &\left. \quad - \gamma \int_a^b \varphi_i \cdot \varphi_j dx + \delta \int_a^b \left( \sum_{i=0}^N u_i \varphi_i \right) \varphi_j dx \right] q_i = 0, \quad (9)
\end{align*}
\]

where \( j = 0, 1, 2, \ldots, N \).

We subdivide the space variable domain \([a, b]\) into uniform subintervals with \( N + 1 \) grid points \( x_k, k = 0, \ldots, N \), such that \( a = x_0 < x_1 < \cdots < x_N = b, h = x_{k+1} - x_k = (b - a)/N \). Using the local coordinate transformation \( x = x_k + \mu h, 0 \leq \mu \leq 1 \), we transform a subinterval \([x_k, x_{k+1}]\) into a standard interval \([0, 1] \). Furthermore, we have

\[
\begin{align*}
    \sum_{i=k}^{k+1} \left[ \left( \int_0^1 \varphi_i \cdot \varphi_j dx + \frac{\beta}{h^2} \int_0^1 \varphi_{i,x} \cdot \varphi_{j,x} dx \right) \frac{\partial q_i}{\partial t} \right. \\
    &\left. \quad - \frac{1}{h} \left( \gamma \int_0^1 \varphi_i \cdot \varphi_j dx + \delta \int_0^1 \left( \sum_{i=0}^N u_i \varphi_i \right) \varphi_j dx \right) q_i \right] = 0, \quad (10)
\end{align*}
\]
Table 1: Solitary wave Amp. 0.3 and the errors in $L^2$ and $L^\infty$ norms for $u, Q_1, Q_2,$ and $Q_3$ at $t = 20, h = 0.125, \Delta t = 0.1,$ and $-40 \leq x \leq 60.$

<table>
<thead>
<tr>
<th>Method</th>
<th>Time</th>
<th>$Q_1$</th>
<th>$Q_2$</th>
<th>$Q_3$</th>
<th>$L^2$ for $u$</th>
<th>$L^\infty$ for $u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Our method</td>
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<td>3.9797</td>
<td>0.8104</td>
<td>2.5787</td>
<td>0</td>
<td>0</td>
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<td></td>
<td>4</td>
<td>3.9797</td>
<td>0.8104</td>
<td>2.5786</td>
<td>3.6304e-004</td>
<td>5.2892e-005</td>
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<tr>
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<td>0.8104</td>
<td>2.5786</td>
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<td>0.8104</td>
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<td>6.3283e-005</td>
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<tr>
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<td>16</td>
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<td>0.8104</td>
<td>2.5787</td>
<td>1.4186e-003</td>
<td>6.8010e-005</td>
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<tr>
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<td>0.8104</td>
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<td>1.7569e-003</td>
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<td>4.6880e-003</td>
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<tr>
<td>[22]</td>
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<td>0.8235</td>
<td>2.6740</td>
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<td>—</td>
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<tr>
<td>[23]</td>
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<tr>
<td>[24]</td>
<td>20</td>
<td>3.9821</td>
<td>0.8112</td>
<td>2.5813</td>
<td>0.5110e-003</td>
<td>0.1980e-003</td>
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</tbody>
</table>

Table 2: Convergence order and error in $L^2$ norm for $u$ of time with $h = 0.125$ and $c = 0.1.$

<table>
<thead>
<tr>
<th>Time</th>
<th>$\Delta t = 0.4$</th>
<th>$\Delta t = 0.2$</th>
<th>$\Delta t = 0.1$</th>
<th>Order (0.2/0.4)</th>
<th>Order (0.1/0.2)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>5.3805e-003</td>
<td>1.4267e-003</td>
<td>3.6304e-004</td>
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<tr>
<td>8</td>
<td>1.6888e-002</td>
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</tr>
<tr>
<td>12</td>
<td>1.7830e-002</td>
<td>4.3997e-003</td>
<td>1.0817e-003</td>
<td>2.0188</td>
<td>2.0241</td>
</tr>
<tr>
<td>16</td>
<td>2.3751e-002</td>
<td>5.7916e-003</td>
<td>1.4186e-003</td>
<td>2.0360</td>
<td>2.0295</td>
</tr>
<tr>
<td>20</td>
<td>2.9434e-002</td>
<td>7.1148e-003</td>
<td>1.7396e-003</td>
<td>2.0486</td>
<td>2.0321</td>
</tr>
</tbody>
</table>

We take linear basis functions defined as follows:

$$L_1 = 1 - \mu, \quad L_2 = \mu,$$

and then the variables $u$ and $q$ over the element $[x_k, x_{k+1}]$ are written as

$$u^e = \sum_{j=1}^{2} L_j u_j, \quad q^e = \sum_{j=1}^{2} L_j q_j.$$  \hspace{1cm} (12)

Then, we get the following equations:

$$\sum_{i=1}^{2} \left[ \left( \int_{0}^{1} L_i \cdot L_j d\mu + \frac{\beta}{h^2} \int_{0}^{1} L_i \mu \cdot L_j d\mu \right) \frac{\partial q_i}{\partial t} - \frac{1}{h} \left( \int_{0}^{1} L_i \cdot L_j d\mu + \delta \int_{0}^{b} \left( \sum_{j=1}^{2} u_j L_i \right) L_i \cdot L_j d\mu \right) q_i \right] = 0,$$

$$\sum_{i=1}^{2} \left[ \frac{1}{h^2} \left( \int_{0}^{1} L_i \cdot L_j d\mu \right) u_i - \frac{1}{h} \left( \int_{0}^{b} L_i \cdot L_j d\mu \right) q_i \right] = 0.$$

Then, the system (13) has the following matrix form:

$$(A_{ij}^e + \beta B_{ij}^e) \frac{\partial q_i}{\partial t} - (\gamma C_{ij}^e + \delta D_{ij}^e (u^e)) q_i = 0,$$

$$B_{ij}^e u^e - C_{ij}^e q_i = 0,$$

with the following element matrices:

$$u^e = (u_1, u_2)^T, \quad q^e = (q_1, q_2)^T,$$

$$A_{ij}^e = \int_{0}^{1} L_i \cdot L_j d\mu, \quad B_{ij}^e = \int_{0}^{1} \frac{1}{h^2} L_i \cdot L_j d\mu + \int_{0}^{b} \left( \sum_{j=1}^{2} u_j L_i \right) L_i \cdot L_j d\mu,$$

$$C_{ij}^e = \frac{1}{h} \int_{0}^{1} L_i \cdot L_j d\mu,$$

$$D_{ij}^e = \frac{\delta}{h} \int_{0}^{1} \left( \sum_{j=1}^{2} u_j L_i \right) L_i \cdot L_j d\mu.$$

Assembling contributions from all elements, we obtain the following coupled system of nonlinear matrix equations:

$$(A + \beta B) \frac{\partial q_i}{\partial t} - (\gamma C + \delta D (u^e)) q_i = 0,$$

$$B u^e - C q_i = 0.$$

To formulate a fully discrete scheme, we consider a uniform partition of $\tilde{J} = [0, T]$ with time step length $\Delta t = T/N, \quad N \in \mathbb{Z}^+$, and time levels $r^n = n \Delta t, \quad n = 0, \ldots, N.$ We now discuss a fully discrete scheme based on a linear explicit multistep method. We now define $U^n \in V_h$ and $Z^n \in W_h$ as approximations to $u(t^n)$ and $q(t^n)$, respectively, and formulate the following fully discrete linear explicit multistep mixed scheme:

$$(A + \beta B) \sum_{i=0}^{p} a_i Z^{n+i} = \Delta t \sum_{i=0}^{p-1} \sigma_i \left[ (\gamma C + \delta D (U^{n+i})) Z^{n+i} \right],$$

$$B U^{n+p} = C Z^{n+p},$$

(17)
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Table 3: Convergence order and error in $L^\infty$ norm for $u$ of time with $h = 0.125$ and $c = 0.1$.

<table>
<thead>
<tr>
<th>Time</th>
<th>$\Delta t = 0.4$</th>
<th>$\Delta t = 0.2$</th>
<th>$\Delta t = 0.1$</th>
<th>Order (0.2/0.4)</th>
<th>Order (0.1/0.2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>8.074e-004</td>
<td>2.029e-004</td>
<td>5.289e-005</td>
<td>1.9919</td>
<td>1.9403</td>
</tr>
<tr>
<td>8</td>
<td>9.177e-004</td>
<td>2.290e-004</td>
<td>5.866e-005</td>
<td>2.0023</td>
<td>1.9653</td>
</tr>
<tr>
<td>12</td>
<td>9.986e-004</td>
<td>2.496e-004</td>
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<tr>
<td>16</td>
<td>1.017e-003</td>
<td>2.685e-004</td>
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<td>2.0059</td>
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<tr>
<td>20</td>
<td>1.127e-003</td>
<td>2.860e-004</td>
<td>7.715e-005</td>
<td>1.9788</td>
<td>1.8907</td>
</tr>
</tbody>
</table>

Table 4: Convergence order and error in $L^2$ norm for $q$ of time with $h = 0.125$ and $c = 0.1$.

<table>
<thead>
<tr>
<th>Time</th>
<th>$\Delta t = 0.4$</th>
<th>$\Delta t = 0.2$</th>
<th>$\Delta t = 0.1$</th>
<th>Order (0.2/0.4)</th>
<th>Order (0.1/0.2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
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<td>6.4427e-004</td>
<td>1.6260e-004</td>
<td>1.9229</td>
<td>1.9863</td>
</tr>
<tr>
<td>8</td>
<td>5.1823e-003</td>
<td>1.2934e-003</td>
<td>3.1976e-004</td>
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<tr>
<td>12</td>
<td>7.6616e-003</td>
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</tr>
<tr>
<td>16</td>
<td>9.8719e-003</td>
<td>2.3787e-003</td>
<td>5.8232e-004</td>
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<td>2.0303</td>
</tr>
<tr>
<td>20</td>
<td>1.1843e-002</td>
<td>2.8251e-003</td>
<td>6.8991e-004</td>
<td>2.0677</td>
<td>2.0338</td>
</tr>
</tbody>
</table>

with given the initial approximations $U^0, \ldots, U^{p-1}$ and $Z^0, \ldots, Z^{p-1}$. In the explicit multistep mixed system (17), the parameter variable $\alpha_i$ and $\sigma_i$ is described by the coefficients of the term $\chi^i$, for the following polynomials $\alpha(\chi)$ and $\sigma(\chi)$, respectively:

$$\alpha(\chi) := \sum_{j=1}^{p} \chi^{p-j}(\chi-1)^j,$$

$$\sigma(\chi) := \chi^p - (\chi-1)^p.$$  (18)

In this paper, we consider the explicit 2-step mixed method for the RLW equation. For $p = 2$, we obtained easily

$$\alpha_0 = \frac{3}{2}, \quad \alpha_1 = -2, \quad \alpha_2 = \frac{1}{2}, \quad \sigma_0 = -1, \quad \sigma_1 = 2.$$  (19)

Substituting (19) into (17), we obtain the following 2-step mixed scheme:

$$(A + \beta B) \left( \frac{3}{2} Z^{n+2} - 2 Z^{n+1} + \frac{1}{2} Z^n \right) = \Delta t \left[ \gamma C (2Z^{n+1} - Z^n) + \delta (2D(U^{n+1})Z^{n+1} - D(U^n)Z^n) \right],$$

$$BU^{n+2} = CZ^{n+2}.$$  (20)

**Remark 1.** There have been many numerical schemes for the RLW equation, but we have not seen the related research on explicit multistep mixed element method for RLW equation in the literature. From the viewpoint of numerical theory, we propose a mixed element scheme (6) and (7), which is different from some other mixed finite element methods in [14–16], for the RLW equation and derive some a priori error estimates based on the explicit multistep mixed element method. From the perspective of numerical calculation, our method is efficient for RLW equation.

3. Two-Step Mixed Scheme and Optimal Error Estimates

3.1. Two-Step Mixed Scheme and Some Lemmas. In this section, we will discuss some a priori error estimates based on explicit 2-step mixed finite element method for the RLW equation. For the fully discrete procedure, let $0 = t_0 < t_1 < \cdots < t_N = T$ be a given partition of the time interval $[0, T]$ with step length $\Delta t = T/N$, for some positive integer $N$. For a smooth function $\phi$ on $[0, T]$, define $\phi^n = \phi(t_n)$.

The system (3) and (4) has the following formulation at $t = t_{n+1}$:

$$(u^{n+1}_x, u_x) = (q^{n+1}, v_x), \quad \forall v \in H^1_0,$$

$$(q^{n+1}, w) + \beta(q^{n+1}_x, w_x) = \gamma(q^{n+1}, w_x) + \delta(u^{n+1}q^{n+1}, w_x), \quad \forall w \in H^1.$$  (21)

Based on system (17), we get an equivalent formulation for system (21) as

$$(u^{n+1}_x, v_x) = (q^{n+1}, v_x), \quad \forall v \in H^1_0,$$  (22)

$$\frac{3q^{n+1} - 4q^n + q^{n-1}}{2\Delta t} + \beta \left( \frac{3q^{n+1} - 4q^n + q^{n-1}}{2\Delta t}, w_x \right) = -\left( \frac{q^{n+1}}{\Delta t}, \frac{q^{n+1}_x}{\Delta t} \right) + \gamma \left( 2q^n - q^{n-1}, w_x \right) + \delta \left( 2u^n q^n - u^{n-1} q^{n-1}, w_x \right) + \gamma \left( R^{n+1}_1, w_x \right) + \delta \left( R^{n+1}_2, w_x \right), \quad \forall w \in H^1.$$  (23)
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Table 5: Convergence order and error in $L^\infty$ norm for $q$ of time with $h = 0.125$ and $c = 0.1$.

<table>
<thead>
<tr>
<th>Time</th>
<th>$\Delta t = 0.4$</th>
<th>$\Delta t = 0.2$</th>
<th>$\Delta t = 0.1$</th>
<th>Order (0.2/0.4)</th>
<th>Order (0.1/0.2)</th>
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</thead>
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</tr>
<tr>
<td>16</td>
<td>2.9012e-004</td>
<td>6.8641e-005</td>
<td>1.5594e-005</td>
<td>2.0795</td>
<td>2.1381</td>
</tr>
<tr>
<td>20</td>
<td>3.1768e-004</td>
<td>7.5864e-005</td>
<td>1.7195e-005</td>
<td>2.0661</td>
<td>2.1414</td>
</tr>
</tbody>
</table>

Table 6: Convergence order and error in $L^2$ norm for $u$ of space with $\Delta t = 0.01$ and $c = 0.1$.

<table>
<thead>
<tr>
<th>Time</th>
<th>$h = 0.8$</th>
<th>$h = 0.4$</th>
<th>$h = 0.2$</th>
<th>Order (0.4/0.8)</th>
<th>Order (0.2/0.4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.7109e-003</td>
<td>4.2677e-004</td>
<td>1.1449e-004</td>
<td>1.9940</td>
<td>1.8982</td>
</tr>
<tr>
<td>8</td>
<td>1.8945e-003</td>
<td>4.6862e-004</td>
<td>1.1924e-004</td>
<td>2.0195</td>
<td>1.9746</td>
</tr>
<tr>
<td>12</td>
<td>2.1232e-003</td>
<td>5.2130e-004</td>
<td>1.3025e-004</td>
<td>2.0102</td>
<td>2.0008</td>
</tr>
<tr>
<td>16</td>
<td>2.3559e-003</td>
<td>5.7598e-004</td>
<td>1.4412e-004</td>
<td>2.0589</td>
<td>1.9978</td>
</tr>
<tr>
<td>20</td>
<td>2.5778e-003</td>
<td>6.3407e-004</td>
<td>1.7877e-004</td>
<td>2.0358</td>
<td>1.8265</td>
</tr>
</tbody>
</table>

where

$$r^{n+1} = q^{n+1}_x - \frac{3q^{n+1}_x - 4q^n_x + q^{n-1}_x}{2\Delta t},$$

$$\kappa_1^{n+1} = q \Delta t (t_{n+1}) - \frac{3q^{n+1}_x - 4q^n_x + q^{n-1}_x}{2\Delta t},$$

$$R_1^{n+1} = q^{n+1}_x - \left(2q^n_x - q^{n-1}_x\right),$$

$$R_2^{n+1} = u_x^{n+1} q^{n+1}_x - \left(2u_x^n q^n_x - u_x^{n-1} q^{n-1}_x\right).$$

We now find a pair $(U^{n+1}, Z^{n+1})$ in $V_h \times W_h$ satisfying

$$\left(U_x^{n+1}, v_h\right) = (Z_x^{n+1}, v_h), \quad \forall v_h \in V_h,$$

$$(3Z_x^{n+1} - 4Z_x^n + Z_x^{n-1})/2\Delta t, w_h\right) + \beta \left(3Z_x^{n+1} - 4Z_x^n + Z_x^{n-1}, w_h\right) = \gamma \left(2Z^n_x - Z_x^{n-1}, w_h\right) + \delta \left(2U^n Z^n_x - U^{n-1} Z^{n-1}_x, w_h\right),$$

$$\forall w_h \in W_h.$$

For the theoretical analysis of a priori error estimates, we define the following projections.

**Lemma 2** (see [15, 27, 28]). One defines the elliptic projection $\overline{u}_h \in V_h$ by

$$(u_x - \overline{u}_h, v_h)_x = 0, \quad \forall v_h \in V_h.$$  (27)

With $\eta = u - \overline{u}$, the following estimates are well known for $j = 0, 1$:

$$\|\eta\| \leq C h^{j+1-\eta} \|u_{k+1}\|.$$  (28)

**Lemma 3** (see [15, 27, 28]). Furthermore, one also defines a Ritz projection $q_h \in W_h$ of $q$ as the solution of

$$A(q_h, w_h) = 0, \quad w_h \in W_h,$$  (29)

where $A(q, w) = (q_x, w_x) + \lambda(q, w)$, and $\lambda$ is taken appropriately so that

$$A(w, w) \geq \mu_0 \|w\|^2_1, \quad w \in H^1,$$  (30)

where $\mu_0$ is a positive constant. Moreover, it is easy to verify that $A(\cdot, \cdot)$ is bounded.

With $\rho = q - q_h$, the following estimates hold:

$$\left\|\frac{\partial \rho}{\partial t}\right\| \leq Ch^{r+1-j} \left\|\frac{\partial q}{\partial t}\right\|, \quad i = 0, 1, 2, 3; \quad j = 0, 1.$$  (31)

For fully discrete error estimates, we now write the errors as

$$u(t_n) - U^n = (u(t_n) - \overline{u}_h(t_n)) + (\overline{u}_h(t_n) - U^n) = \eta^n + \zeta^n,$$

$$q(t_n) - Z^n = q(t_n) - q_h(t_n) + (q_h(t_n) - Z^n) = \xi^n + \xi_h^n.$$  (32)

Combine (27), (29), (22), (23), (25), and (36) at $t = t_{n+1}$ to get the following error equations:

$$\left(3\xi_x^{n+1} + \xi_h^{n+1}, v_h\right) = \left(3\xi_x^n + 3\xi_h^n, v_h\right),$$

$$\beta \left(3\xi_x^{n+1} - 4\xi_h^{n+1} + \xi_h^{n-1}, v_h\right) + \delta \left(2u_x^n Z^n_x - U^{n-1} Z^{n-1}_x, v_h\right),$$

$$\forall w_h \in W_h.$$  (33)

$$\left(3\xi_x^{n+1} - 4\xi_h^{n+1} + \xi_h^{n-1}, v_h\right) = -(1 - \beta \lambda) \frac{3p^{n+1}_x - 4p^n_x + p^{n-1}_x}{2\Delta t} + v^{n+1}_x, w_h.$$  (34)
where
\[ \kappa_2^{n+1} = \frac{\partial^3}{\partial t^3} (t_{n+1}) - \frac{3 \partial^3 u^{n+1}}{\partial x^3} - 4 \partial^3 q^n + \frac{\partial^3 h_n}{\partial x^3}. \] (35)

Lemma 4. For \( r^{n+1}, \kappa_2^{n+1}, R_1^{n+1}, \) and \( R_2^{n+1}, \) the following estimates hold:
\[ \| r^{n+1} \| \leq C \Delta t^2 \| q_{tt}\|_{L^\infty(L^2)}, \]
\[ \| \kappa_2^{n+1} \| \leq C \Delta t^2 \left( \| q_{ttt}\|_{L^\infty(L^2)} + \| q_{xtt}\|_{L^\infty(L^2)} \right), \]
\[ \| R_1^{n+1} \| \leq C \Delta t^2 \| q_t\|_{L^\infty(L^2)}, \]
\[ \| R_2^{n+1} \| \leq C \Delta t^2 \left( \| u \|_{L^\infty(L^2)} + \| t u_t \|_{L^\infty(L^2)} + \| t u_{tt} \|_{L^\infty(L^2)} \right). \] (36)

Proof. Using the Taylor expansion, we have
\[ q(t_n) = q(t_{n+1}) + t q_t(t_{n+1})(t_n - t_{n+1}) \]
\[ + \frac{\partial}{\partial t} (t_{n+1})^2, \quad t_n < t_{n+1} \] \( \leq t_{n+1} < t_{n+1} \)
\[ q(t_{n-1}) = q(t_{n+1}) + t q_t(t_{n+1})(t_{n-1} - t_{n+1}) \]
\[ + \frac{\partial}{\partial t} (t_{n+1})^2, \quad t_{n-1} < t_{n+1}. \] (37)
(38)

Combining (37) and (38) and noting that \(-2 \Delta t = 2(t_n - t_{n+1}) = t_{n-1} - t_{n+1},\) we obtain
\[ q(t_{n+1}) = 2q(t_n) - q(t_{n-1}) + (q_{tt}(t_{n+1}) - 2q_t(t_{n+1}) \Delta t^2). \] (39)

From (39), we have
\[ \| q(t_{n+1}) - (2q(t_n) - q(t_{n-1})) \| \leq C \Delta t^2 \| q_{tt}\|_{L^\infty(L^2)}. \] (40)

Using a similar estimate as the one for \( \| R_1^{n+1} \|, \) we have
\[ u(t_{n+1}) q(t_{n+1}) - u(t_{n-1}) q(t_{n-1}) \]
\[ + (u q_{tt})(t_{n+1}) - 2 (u q_{tt})(t_{n+1}) \Delta t^2, \] \( \quad \leq (u q_{tt})(t_{n+1}) - 2 (u q_{tt})(t_{n+1}) \Delta t^2, \] (41)

where \( t_n < t_{n+1} < t_{n+1} \).

From (41), we have
\[ \| u(t_{n+1}) q(t_{n+1}) - (2u(t_n) q(t_n) - u(t_{n-1}) q(t_{n-1})) \| \]
\[ \leq C \Delta t^2 \left( \| u \|_{L^\infty(L^2)} + \| u_t \|_{L^\infty(L^2)} + \| u_{tt} \|_{L^\infty(L^2)} \right). \] (42)

Using the Taylor expansion and noting that \(-2 \Delta t = 2(t_n - t_{n+1}) = t_{n-1} - t_{n+1},\) we have
\[ 4q(t_n) = 4q(t_{n+1}) - 4q(t_{n+1}) \Delta t + 2q_t(t_{n+1}) \Delta t^2 \]
\[ - \frac{2q_{tt}(t_{n+1})}{3} \Delta t^3, \quad t_n < t_{n+1}, \]
\[ q(t_{n+1}) = 2q(t_n) - q(t_{n-1}) + q_{tt}(t_{n+1}) \Delta t^2 \]
\[ - \frac{4q_{tt}(t_{n+1})}{3} \Delta t^3, \quad t_n < t_{n+1}. \] (43)

Using (43), we obtain
\[ q_{tt}^{n+1} = \frac{3q^{n+1} - 4q^n + q^{n-1}}{2 \Delta t} - \left( \frac{2q_{tt}(t_{n+1})}{3} - \frac{4q_{tt}(t_{n+1})}{3} \right) \Delta t^2. \] (44)

By (44), we have
\[ \| r^{n+1} \| \leq C \Delta t^2 \| q_{tt} \|_{L^\infty(L^2)}. \] (45)
Using the similar method to the estimate for \( \| r^{n+1} \| \) and (31), we obtain
\[
\| \mathbf{k}_2^{n+1} \| \leq C \Delta t^2 \| q_{att}^h \|_{L^\infty(L^2)} \\
\leq C \Delta t^2 \left( \| \rho_{att} \|_{L^\infty(L^2)} + \| q_{att}^h \|_{L^\infty(L^2)} \right) \\
\leq C \Delta t^2 \left( h' \| q_{att}^h \|_{L^\infty(L^2)} + \| q_{att}^h \|_{L^\infty(L^2)} \right). 
\]

3.2. Optimal Error Estimates. In this subsection, we derive the fully discrete optimal error estimates and obtain the following theorem.

**Theorem 5.** Assuming that \( U^0, U^1 \in V_h \) and \( Z^0, Z^1 \in W_h \) are given, then for \( 1 \leq j \leq M, j = 0, 1 \), one has
\[
\| U^{j+1} - U^{j+1} \| \leq C \left( h^{\min(j+1-k, j+1)} + \Delta t^2 \right), \\
\| q^{j+1} - Z^{j+1} \| + 2 \left( \| q^{j+1} - Z^{j+1} \| - \| q^j - Z^j \| \right) \\
\leq C \left( h^{\min(j+1-k, j+1)} + \Delta t^2 \right). 
\]

**Proof.** Taking \( v^h = \chi^{n+1} \) in (33) and using Cauchy-Schwarz’s inequality and Poincaré’s inequality, we get
\[
\| \chi^{n+1} \| \leq C \left( \| \rho^{n+1} \| + \| \chi^{n+1} \| \right). 
\]

Set \( u^h = \chi^{n+1} \) in (34) to obtain
\[
\left( \frac{3 \xi^{n+1} - 4 \xi^n + \chi^{n+1}}{2 \Delta t}, \chi^{n+1} \right) + \beta \left( \frac{3 \xi^{n+1} - 4 \xi^n + \chi^{n+1}}{2 \Delta t}, \xi^{n+1} \right) \\
= - \left( 1 - \beta \right) \left( \frac{3 \xi^{n+1} - 4 \xi^n + \chi^{n+1}}{2 \Delta t}, \xi^{n+1} \right) \\
- \beta \left( \frac{3 \xi^{n+1} - 4 \chi^n + \Gamma^{n+1}}{2 \Delta t}, \xi^{n+1} \right) \\
+ \chi^n \left( \frac{3 \xi^{n+1} - 4 \chi^n + \Gamma^{n+1}}{2 \Delta t}, \xi^{n+1} \right) \\
+ \beta \left( \frac{3 \xi^{n+1} - 4 \chi^n + \Gamma^{n+1}}{2 \Delta t}, \xi^{n+1} \right) \\
= \left( 1 - \beta \right) \left( \frac{3 \xi^{n+1} - 4 \chi^n + \Gamma^{n+1}}{2 \Delta t}, \chi^{n+1} \right) \\
- \beta \left( \frac{3 \xi^{n+1} - 4 \chi^n + \Gamma^{n+1}}{2 \Delta t}, \xi^{n+1} \right) \\
+ \chi^n \left( \frac{3 \xi^{n+1} - 4 \chi^n + \Gamma^{n+1}}{2 \Delta t}, \chi^{n+1} \right) \\
+ \beta \left( \frac{3 \xi^{n+1} - 4 \chi^n + \Gamma^{n+1}}{2 \Delta t}, \xi^{n+1} \right). 
\]
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\[
\left\| \left(2u(t_n)q(t_n) - U^n Z^n\right) \right. \\
- \left. \left( u(t_{n-1})q(t_{n-1}) - U^{n-1} Z^{n-1}\right), \xi^{n+1}_x \right\| \\
\leq \left| \left(2u(t_n)q(t_n) - Z^n\right) - \left( u(t_{n-1})q(t_{n-1}) - Z^{n-1}\right), \xi^{n+1}_x \right| \\
+ \left| \left( \left( u(t_n) - u(t_{n-1})\right) \left( q(t_{n-1}) - Z^{n-1}\right), \xi^{n+1}_x \right) \right| \\
+ \left| \left( \left( u(t_n) - U^n\right) Z^n - \left( u(t_{n-1}) - U^{n-1}\right) Z^{n-1}\right), \xi^{n+1}_x \right| \\
\leq T_1 + T_2 + T_3 + T_4. \tag{52}
\]

We now estimate \( T_1, T_2, T_3, \) and \( T_4 \) as

\[
T_1 \leq \left| \left( u(t_n) \left( 2\xi^n - \xi^{n-1}\right), \xi^{n+1}_x \right) \right| \\
+ \left| \left( u(t_n) \left( 2\rho^n - \rho^{n-1}\right), \xi^{n+1}_x \right) \right| \\
\leq \| u(t_n) \|_\infty \left\| \left( 2\xi^n - \xi^{n-1}\right) \right\|_\infty \| \xi^{n+1}_x \|_\infty \\
+ \| u(t_n) \|_\infty \left\| \left( 2\rho^n - \rho^{n-1}\right) \right\|_\infty \| \xi^{n+1}_x \|_\infty , \tag{53}
\]

\[
T_2 = \left( \int_{t_n}^{t_{n+1}} u ds \left( \rho^n - \rho^{n-1} + \xi^{n-1} + \xi^{n+1}_x \right) \right) \\
\leq \Delta t \| u \|_\infty \left\| \left( \rho^n - \rho^{n-1} + \xi^{n-1} \right) \right\|_\infty \| \xi^{n+1}_x \|_\infty .
\]

\[
T_3 \leq \left| \left( Z^n \left( 2\xi^n - \xi^{n-1}\right), \xi^{n+1}_x \right) \right| \\
+ \left| \left( Z^n \left( 2\eta^n - \eta^{n-1}\right), \xi^{n+1}_x \right) \right| \\
\leq \| Z^n \|_\infty \left\| \left( 2\xi^n - \xi^{n-1}\right) \right\|_\infty \| \xi^{n+1}_x \|_\infty \\
+ \| Z^n \|_\infty \left\| \left( 2\eta^n - \eta^{n-1}\right) \right\|_\infty \| \xi^{n+1}_x \|_\infty .
\]

\[
T_4 \leq \left( \int_{t_n}^{t_{n+1}} Z_n ds \left( \eta^{n-1} + \xi^{n-1} + \xi^{n+1}_x \right) \right) \\
\leq \Delta t \| Z_n \|_\infty \left\| \left( \eta^{n-1} + \xi^{n-1} \right) \right\|_\infty \| \xi^{n+1}_x \|_\infty .
\]

Substituting (36), (51), and (54) into (50), using (48), and summing from \( n = 1, 2, \ldots, J \), the resulting inequality becomes

\[
(1 - C \Delta t) \left( \| \xi^{J+1}_x \|_\infty^2 + \| \xi^J_x \|_\infty^2 + \| \xi^{J+1}_x \|_\infty^2 \right) \\
+ \| \xi^J_x \|_\infty^2 \right) \\
\leq C \Delta t \sum_{n=1}^{J} \left( \| \xi^{n-1}_x \|_\infty^2 + \| \rho^{n-1} \|_\infty^2 \right) + C \int_0^{t_{J+1}} \| \rho_t \|_2^2 ds \\
+ C \Delta t \sum_{n=1}^{J} \left( \| \xi^{n-1}_x \|_\infty^2 + \| \rho^{n-1} \|_\infty^2 \right) + \| \xi^J_x \|_\infty^2 \right) \\
\leq C \Delta t \sum_{n=1}^{J} \left( \| \xi^{n-1}_x \|_\infty^2 + \| \rho^{n-1} \|_\infty^2 \right) + C \int_0^{t_{J+1}} \| \rho_t \|_2^2 ds \\
+ C \Delta t \sum_{n=1}^{J} \left( \| \xi^{n-1}_x \|_\infty^2 + \| \rho^{n-1} \|_\infty^2 \right) + \| \xi^J_x \|_\infty^2 \right) .
\]

Choose \( \Delta t_0 \) in such a way that for \( 0 < \Delta t \leq \Delta t_0, (1 - C \Delta t) > 0 \). Then, as an application of Gronwall’s lemma, we obtain

\[
\| \xi^{J+1}_x \|_\infty^2 + \| \xi^J_x \|_\infty^2 + \| \xi^{J+1}_x \|_\infty^2 \right) \\
\leq C \Delta t \sum_{n=1}^{J} \left( \| \xi^{n-1}_x \|_\infty^2 + \| \rho^{n-1} \|_\infty^2 \right) + \int_0^{t_{J+1}} \| \rho_t \|_2^2 ds \\
+ \left( \| \xi^{n-1}_x \|_\infty^2 + \| \rho^{n-1} \|_\infty^2 \right) + \int_0^{t_{J+1}} \| \rho_t \|_2^2 ds \\
+ \left( \| \xi^{n-1}_x \|_\infty^2 + \| \rho^{n-1} \|_\infty^2 \right) + \int_0^{t_{J+1}} \| \rho_t \|_2^2 ds \\
+ \left( \| \xi^{n-1}_x \|_\infty^2 + \| \rho^{n-1} \|_\infty^2 \right) .
\]

Combine (28), (31), and (56) with the triangle inequality to complete the \( L^2 \) and \( H^1 \) error estimates for \( \eta \). Furthermore, use (48) and the triangle inequality to complete the optimal error estimates for \( \| u(t_n) - U^n \| \) and \( \| u(t_n) - U^n \|_1 \).

Remark 6. Compared to a variety of difference methods in [17], our method is studied based on mixed element scheme (6) and (7).

Remark 7. Although some convergence proofs of multistep methods for RLW/BBM are provided in [20, 25, 29], our convergence results of multistep methods are proved based on a mixed finite element scheme. Based on the current discussion, we have to provide the detailed proofs for multistep mixed finite element methods in this paper.
4. Numerical Results

In order to test the viability of the proposed method, we consider a test problem. We write conservation laws as [3, 4]

\[
Q_1 = \int_a^b u \, dx = h \sum_{j=1}^N u_j^n,
\]

\[
Q_2 = \int_a^b \left( u^2 + \mu(u_x)^2 \right) \, dx = h \sum_{j=1}^N \left[ (u_j^n)^2 + \mu (u_x_j^n)^2 \right],
\]

\[
Q_3 = \int_a^b \left( u^3 + 3u^2 \right) \, dx = h \sum_{j=1}^N \left[ (u_j^n)^3 + 3 (u_j^n)^2 \right],
\]

where \( Q_1, Q_2, \) and \( Q_3 \) are usually called mass, momentum, and energy, respectively, which are observed to check the conservation of the numerical scheme.

We consider RLW equation (1) and let, in (1), \( \delta = \gamma = \beta = 1. \) Then, the solitary wave solution of (1) is

\[
u (x, t) = 3 \csech^2 \left( k \left[ x - x_0 - V t \right] \right),
\]

where

\[
k = \frac{1}{2} \sqrt{\frac{c}{1 + c}}, \quad \nu = 1 + c.
\]

We consider the motion of a single solitary wave and take as initial condition, with \( c = 0.1 \) and \( x_0 = 0, \)

\[
u (x, 0) = 0.3 \csech^2 \left( \frac{x}{2 \sqrt{11}} \right).
\]

The corresponding exact solution with initial condition (60) is

\[
u (x, t) = 0.3 \csech^2 \left( \frac{x - 1.1 t}{2 \sqrt{11}} \right).
\]

In this procedure, we take space-time domain as \(-40 \leq x \leq 60 \) and \( 0 \leq t \leq 20. \)
In Table 1, we take spatial mesh parameter \( h = 0.125 \) and time discretization parameter \( \Delta t = 0.1 \) and list the three invariants \( Q_1, Q_2, \) and \( Q_3 \) and the optimal error estimate in \( L^2 \) and \( L^{\infty} \) norms for \( u \) at different times \( t = 0, 4, 8, 12, 16, \) and 20. At the same time, we show some numerical results at time \( t = 20 \) obtained by other numerical methods in Table 1. From Table 1, we find that our method is more accurate than the numerical methods in \([17, 18, 28]\) but is less than the numerical methods in \([23, 24]\). From the shown results in Table 1, we can see that \( Q_1, Q_2, \) and \( Q_3 \) keep almost constants, so the conservation for our method is very well.

In Tables 2 and 3, we take spatial mesh parameter \( h = 0.125 \) and obtain the optimal error results in \( L^2 \) and \( L^{\infty} \) norms for \( u \) at different times \( t = 0, 4, 8, 12, 16, \) and 20 with different time discretization parameters \( \Delta t = 0.1, 0.2, \) and \( 0.4 \). From Tables 2 and 3, we see easily that the convergence rate for time is close to order 2. Similarly, the results for \( q \) are shown in Tables 4 and 5.

In Tables 6 and 7, the optimal error results in \( L^2 \) and \( L^{\infty} \) norms for \( u \) at different times \( t = 0, 4, 8, 12, 16, \) and 20 with different spatial mesh parameters \( h = 0.2, 0.4, \) and \( 0.8 \) and time discretization parameter \( \Delta t = 0.01 \) are shown. It is easy to see that the convergence rate for space is close to order 2. The similar results for \( q \) are listed in Tables 8 and 9.

Figure 1 shows the surface for the exact solution \( u \) in space-time domain \((x, t) \in [-20, 60] \times [0, 20]\), and the corresponding surface for the numerical solution \( u_h \) with \( h = 2 \) and \( \Delta t = 0.4 \) is described in Figure 2. From Figures 1 and 2, we see easily that the exact solution \( u \) is approximated very well by the numerical solution \( u_h \). In Figures 3 and 4, we show the surface for the exact solution \( q \) and the numerical solution \( q_h \), respectively, with \( h = 2 \) and \( \Delta t = 0.4 \) and obtain a good approximation solution \( q_h \) for the exact solution \( q \).

The comparison between the exact solution \( u \) and the numerical solution \( u_h \) is described at different times \( t = 0, 4, 8, 12, 16, \) and 20 with \( h = 0.125 \) and \( \Delta t = 0.2 \) in Figure 5. The similar comparison for the exact solution \( q \) and the numerical solution \( q_h \) is shown in Figure 6. Figures 5 and 6 show that the solitary wave for \( u \) and \( q \) moves to the right with unchanged form and velocity, respectively. Furthermore, the exact solutions \( u \) and \( q \) are approximated well by the numerical solutions \( u_h \) and \( q_h \), respectively.

In Figure 7, we show the comparison between \( u \) and \( u_h \) at different spaces \( x = 0, 10, 20, \) and 30 with \( h = 0.125 \) and \( \Delta t = 0.2 \) to verify the efficiency for the proposed scheme in this paper. Figure 8 describes a similar result for \( q \).
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0

Numerical

Exact

0 5 10 15 20

\( q_{h} \text{ and } q \)

\( x = 0 \)

\( x = 30 \)

\( x = 20 \)

\( x = 10 \)

0

0.01

0.02

0.03

0.04

Figure 8: Comparison between \( q \) and \( q_{h} \) at spaces \( x = 0, 10, 20, \) and 30 with \( h = 0.125 \) and \( \Delta t = 0.2 \).

From the previous analysis in Tables 1–9 and Figures 1–8, we can see that the numerical results confirm the theoretical results of Theorem 5 and our method is efficient for RLW equation.

5. Concluding Remarks

In this paper, we propose and analyze an explicit multistep mixed finite element method, which combines spatial mixed finite element method and time explicit multistep method, for RLW equation. We discuss the numerical process for our method, prove the theoretical results for the fully discrete explicit multistep mixed scheme, obtain the optimal convergence order, and compare our method’s accuracy with some other numerical schemes. Compared with the numerical method in [20, 25, 26], our method can obtain the optimal error estimates in \( L^{2} \) and \( H^{1} \) norms for the scalar unknown \( u \) and its flux \( q = u_{x} \) simultaneously. From our numerical results, we can see that our method is efficient for RLW equation.

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