Research Article

Positive Solutions for Nonlinear Integro-Differential Equations of Mixed Type in Banach Spaces

Yan Sun\textsuperscript{1,2}

\textsuperscript{1} School of Mathematical Sciences, Fudan University, Shanghai 200433, China
\textsuperscript{2} Department of Mathematics, Shanghai Normal University, Shanghai 200234, China

Correspondence should be addressed to Yan Sun; ysun@shnu.edu.cn

Received 29 August 2013; Accepted 16 September 2013

Academic Editor: Geraldo Botelho

Copyright © 2013 Yan Sun. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We establish some new existence theorems on the positive solutions for nonlinear integro-differential equations which do not possess any monotone properties in ordered Banach spaces by means of Banach contraction mapping principle and cone theory based on some new comparison results.

1. Introduction

In this paper, we consider the existence of the unique positive solution and at least one positive solution for the following initial value problem (IVP) of the nonlinear integro-differential equation of mixed type in ordered Banach spaces $E$:

$$
\begin{align*}
&u''(t) = g(t, u(t), (Tu)(t), (Su)(t)), \quad \forall t \in I, \\
&u(0) = u_0, \quad u'(0) = u_1,
\end{align*}
$$

(1)

where $I = [0, a]$ ($a > 0$), $g(t, u(t), (Tu)(t), (Su)(t)) = f(t, u(t), (Tu)(t), (Su)(t))$, $u_0, u_1 \in E$, $f \in C[I \times P \times P \times P, P]$, and $f$ is nonincreasing with the second variable $u(t)$, with $f$ is nondecreasing with the third variable $u(t)$. $P$ is a positive cone in ordered Banach spaces $E$, and

$$
\begin{align*}
(Tu)(t) &= \int_0^t k(t, s) u(s) \, ds, \\
(Su)(t) &= \int_0^a h(t, s) u(s) \, ds, \quad t \in I.
\end{align*}
$$

(2)

In (2), $k \in C[D, R^+]$ and $h \in C[D_0, R^+]$, where $D = \{(t, s) \in R \times R : 0 \leq s \leq t \leq a\}$, $D_0 = \{(t, s) \in R \times R : (t, s) \in I \times I\}$, $R^+$ denotes the set of nonnegative real numbers, and $R$ denotes the set of real numbers. In the special case when $f$ does not contain the third argument, Guo [1] proved the minimal and maximal solution of the following initial value problem:

$$
u' = f(t, u, Tu), \quad t \in I, \quad u(t_0) = x_0,
$$

(3)

under some stronger conditions. He used the topological degree theory and the monotone iterative technique. When $f$ does not possess any monotone assumption, the problem of proving the existence results is an interesting and important question. The aim of the paper is to study this kind of problem. By means of Banach contraction mapping principle and cone theory based on some new comparison results, we obtain some new existence theorems of the solutions for the initial value problems of the nonlinear integro-differential equations of mixed type which does not possess any monotone properties in ordered Banach spaces. The new results obtained are quite general and are used to generalize, improve, and unify many recent results of [2–8]. Their results assert the existence of one solution, the minimal and maximal solution using monotone iterative technique with the stronger conditions. Our method is different from the method of mixed monotony, and the results obtained in the paper are new even if the space $E$ is finite dimensional.

Inspired and motivated greatly by the above work, the aim of the paper is to consider the existence of positive solutions for the boundary value problem (1) under simpler conditions. The main results of problem (1) are obtained by making use
of new fixed point theorem. More specifically, in the proof of these theorems, we establish new comparison theorem and construct a special cone for strict set contraction operator. Our main results in essence improve and generalize the corresponding results of [1, 9–12]. Moreover, our method is different from those in [5, 10, 13].

The rest of the paper is organized as follows. In Section 2, we present some known results and introduce conditions to be used in the next section. The main theorems are formulated and proved in Section 3.

2. Preliminaries and Lemmas

In this section, we will state some necessary definitions and preliminary results.

Definition 1. Let $E$ be a real Banach space. A nonempty closed set $P \subset E$ is called a cone if it satisfies the following two conditions:

1. $z \in P, \lambda > 0$ implies $\lambda z \in P$;
2. $z \in P, -z \in P$ implies $z = \theta$, where $\theta$ denotes the zero element of $E$.

A cone is said to be solid if it contains interior points, $P \neq \theta$. The positive cone $P$ is said to be generating if $E = P - P$; that is, every element $y \in E$ can be represented in the form $y = x - z$, where $x, z \in P$. Every cone with nonempty interior is generating. A cone $P$ induces a partial ordering in $E$ given by $u \leq v$ if $v - u \in P$. If $u \leq v$ and $u \neq v$, we write $u < v$; if cone $P$ is solid and $v - u \in \hat{P}$, we write $u \ll v$.

Definition 2. A cone $P \subset E$ is said to be normal if there exists a positive constant $\lambda$ such that $\|x + y\| \geq \lambda, \forall x, y \in P, \|x\| = 1, \|y\| = 1$.

Proposition 3. A positive cone $P$ is called normal if and only if there exists $N > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq N\|y\|$.

Theorem 1. Let $P$ be a Banach space and let $A : E \times E \to E$. The mapping $A$ is said to be monotone if $u \leq v$ implies $Au \leq Av$ for all $u, v \in E$.

Lemma 8 (see [2, 14]). A positive cone $P$ is normal and generating if and only if there exists a positive constant $\tau > 0$ and $y, z \in P$ such that $x = y - z$ for all $x \in P$ and $\|z\| \leq \tau \|x\|$.

The following lemma plays a key role for improving the main results.

Lemma 9. Let $P$ be a normal generate cone, and let $A : P \times P \to P$ be a nonlinear operator. If there exists a positive bounded linear operator $A : P \to P$ such that $r(L) < 1$ and

$$L(x - \bar{x} + \bar{y} - y) \leq A(x, y) - A(\bar{x}, \bar{y}) \leq L(x - \bar{x} + \bar{y} - y),$$

then $A$ has a unique positive fixed point $x^*$, and for any $x_0 \in P$, let $x_n = A(x_{n-1}, x_{n-1}) \ (n = 1, 2, \ldots), x_n \to x^* \ (n \to \infty)$.

Proof. Since $r(L) < 1$, then $r(L) = \lim_{n \to \infty} \|L^n\|^{1/n} < 1$. Thus, there exists a natural number $n_0$ and $0 < \alpha < 1$ such that

$$\|L^{n_0}\| \leq \alpha^{n_0}. \quad (8)$$
Since $P$ is generate, it follows from Lemma 8 that there exist $\tau > 0$, $y, z \in P$ such that $x = y - z$ and
\begin{equation}
\|y\| \leq \tau \|x\|, \quad \|z\| \leq \tau \|x\|, \quad \forall x \in P.
\end{equation}

Then $y + z \leq 2\tau \|x\|$, $\forall x \in P$. Suppose that $x, y, z \in P$ are not in the same line for any $0 \leq x - y \leq z$. Let $u_1 = (1/2)(x + y - z)$, $u_2 = x - y + z$, $u_3 = -x + y + z$. Then $u_1, u_2, u_3 \in P$ and $x \geq y \geq u_1, x - u_1 = (1/2)u_2, y - u_1 = (1/2)u_3, u_2 + u_3 = 2z$. It follows from (7) that
\begin{equation}
-L u_2 \leq A(x, x) - A(u_1, x) \leq L u_2,
\end{equation}
\begin{equation}
-L (u_2 + u_3) \leq A(u_1, x) - A(y, y) \leq L (u_2 + u_3),
\end{equation}
\begin{equation}
-L u_3 \leq A(y, u_1) - A(y, y) \leq L u_3.
\end{equation}
By virtue of (10) + (12) − (11), we have
\begin{equation}
-L u_2 \leq A(x, x) - A(y, y) \leq L u_3
\end{equation}
\begin{equation}
- L \cdot 2z \leq A(x, x) - A(y, y) \leq L \cdot 2z.
\end{equation}

We first define the operator $\widetilde{A} : P \to P$ as the following $\widetilde{A}x = A(x, x), \forall x \in P$. Let $w = 2z$. Then $-Lw \leq \widetilde{A}x - \widetilde{A}y \leq Luw$. Since $L$ is a positive bounded linear operator, then $Luw \in P$. By induction, it is easy to see that
\begin{equation}
-L^nw \leq \widetilde{A}^nx - \widetilde{A}^ny \leq L^nw, \quad L^nw \in P.
\end{equation}

Since $w \in P$ is arbitrary and $n_0$ is taken as in (8), we get
\begin{equation}
\left\|\widetilde{A}^nx - \widetilde{A}^ny\right\| \leq L^nw \left\|x - y\right\| \leq \alpha^n \left\|x - y\right\|.
\end{equation}

From $0 < \alpha < 1$ and Banach contract principle, we know that $\widetilde{A}^{n_0}$ has the unique positive fixed point $x^*$ in $P$; that is, $x^*$ is the unique positive solution of operator equation $x = A(x, x)$. And for any $x_n \in E$, let $x_n = A(x_{n-1}, x_{n-1}) (n = 1, 2, \ldots)$, we have $\|x_n - x^*\| \to 0 (n \to \infty)$. 

**Remark 10.** The results of Lemma 9 obtained are quite general and are used to improve, generalize, and unify many results of Chen [13], Guo [9], Krasnosel'skii and Zabreiko [16], Zhang [11], Su et al. [10], and Liu [12]. Not only do we obtain the existence and uniqueness of fixed points for mixed nonmonotone binary operators in ordered Banach spaces, but we also solve the difficult open problem that nonmonotone binary operator has unique fixed points under some weaker conditions.

**Lemma 11** (see [17]). Let $E$ be a Banach space and $H \subset C[I, E]$ if $H$ is a countable set of strongly measurable functions $y : I \to E$ such that there exists $\varphi \in L[I, R^+]$ such that $\|y(t)\| \leq \varphi(t), t \in I, y \in H$. Then $Y(H(t))$ is Lebesgue integrable on $I$, and
\begin{equation}
Y\left(\left\{\int_I y(s) \, ds : y \in H\right\}\right) \leq 2 \int_I Y(H(s)) \, ds.
\end{equation}

**Lemma 12** (see [1, 18]). Let $H \subset C[I, E]$ be bounded and equicontinuous; then $Y(H(t))$ is continuous on $I$, and
\begin{equation}
Y\left(\int_I H(s) \, ds\right) \leq \int_I Y(H(s)) \, ds.
\end{equation}

**Lemma 13** (see [2, 19]). Let $X$ be a Banach space, $K \subset X$ closed and convex, and $F : K \to K$ continuous with the further property that for some $x \in K$ we have $B \subset K$ countable. $B = \overline{co}\{\alpha \in F(B)\}$ implies that $B$ is relatively compact. Then $F$ has a fixed point in $K$.

**Lemma 14.** Assume that $m \in C[I, R^+]$ satisfies
\begin{equation}
m(t) \leq M \int_{t_0}^t m(s) \, ds + M^* \int_{t_0}^{t+a} m(s) \, ds, \quad t \in I,
\end{equation}
where $M > 0$ and $M^* \geq 0$ are constants. Then $m(t) \equiv 0$ for $t \in I$ provided one of the following two conditions hold:

(i) $M > M^* (e^{M^*t} - 1)$;
(ii) $\alpha (M + M^*) < 1$.

**Proof.** Suppose that (i) holds. Let $p(t) = \int_{t_0}^t m(s) \, ds$; then $p(t_0) = 0, p'(t) = m(t) \geq 0, \forall t \in I$. From (18), we know that
\begin{equation}
p'(t) \leq M p(t) + M^* p(t_0 + a), \quad t \in I.
\end{equation}

Since $e^{-M^*(t-t_0)} > 0$, we have
\begin{equation}
\left(p(t) e^{-M^*(t-t_0)}\right)' \leq M^* p(t_0 + a) e^{-M^*(t-t_0)}, \quad t \in I.
\end{equation}
Integrate from $t_0$ to $t$; noticing $p(t_0) = 0$, we have
\begin{equation}
0 \leq p(t) e^{-M^*(t-t_0)} \leq \int_{t_0}^t M^* p(t_0 + a) e^{-M^*(t-t_0)} \, ds
\end{equation}
\begin{equation}
= \frac{M^*}{M} p(t_0 + a) \left(1 - e^{-M^*(t-t_0)}\right), \quad t \in I.
\end{equation}
Hence,
\begin{equation}
0 \leq p(t) \leq \frac{M^*}{M} p(t_0 + a) \left(e^{M^*(t-t_0)} - 1\right), \quad t \in I.
\end{equation}
We claim that
\begin{equation}
p(t_0 + a) = 0.
\end{equation}
Otherwise, $p(t_0 + a) > 0$; then we take $t = t_0 + a$ in (22), we obtain $M \leq M^* (e^{M^*t} - 1)$, which contradicts assumption (i).

Therefore, from (22) and (23), we know that $p(t) = \int_{t_0}^t m(s) \, ds = 0$. Consequently, $m(t) \equiv 0$ a.e. on $t \in I$.

Let us suppose now that (ii) holds. Then
\begin{equation}
m(t) \leq (M + M^*) \int_{t_0}^{t+a} m(s) \, ds, \quad t \in I.
\end{equation}
It follows by integrating the above inequality that
\[
\int_{t_0}^{t_0+a} m(s) \, ds \leq a \left( M + M' \right) \int_{t_0}^{t_0+a} m(s) \, ds,
\]
which, by assumption (ii), implies that \( \int_{t_0}^{t_0+a} m(s) \, ds = 0 \) and so \( m(t) \equiv 0 \) a.e. on \( I \). The proof of Lemma 14 is therefore complete. \( \square \)

Lemma 15 (see [2, 19]). \( H \subset C^1[I, E] \) is relatively compact if and only if each element \( y(t) \in H \) and \( y'(t) \in H \) are uniformly bounded and equicontinuous on \( I \).

3. Main Results

We are now in a position to prove our main results concerning the unique positive solution and at least one positive solution.

Theorem 16. Suppose that \( P \) is a normal solid cone whenever \( t \in I \); there exists \( \bar{M} > 0 \), \( y_1, \bar{y}_1, y_2, \bar{y}_2 \in P \), \( y_1 \geq \bar{y}_1, y_2 \leq \bar{y}_2 \) such that
\[
f(t, y_1, y_2, T y_1, S y_1) - f(t, \bar{y}_1, \bar{y}_2, T \bar{y}_1, S \bar{y}_1) \\
\geq - \bar{M} \left[ y_1 - \bar{y}_1 + \bar{y}_2 - y_2 + T y_1 - T \bar{y}_1 \right],
\]
\[
f(t, y_1, y_2, T y_1, S y_1) - f(t, \bar{y}_1, \bar{y}_2, T \bar{y}_1, S \bar{y}_1) \\
\leq \bar{M} \left[ y_1 - \bar{y}_1 + \bar{y}_2 - y_2 + T y_1 - T \bar{y}_1 \right].
\]
Then IVP (1) has the unique positive solution \( x^* \) in \( C^2[I, P] \).

Proof. Define operator \( A : C[I, P] \times C[I, P] \rightarrow C[I, P] \) as the following:
\[
A(u, v) = u_0 + tu_1 + \int_0^t (t-s) f(s, u(s), v(s), (Tu)(s), (Sv)(s)) \, ds, \quad t \in I.
\]
Then \( u \in C^2[I, P] \) is a solution of IVP (1) if and only if \( u \in C[I, P] \) is a fixed point of \( A \); that is, \( u = A(u, u) \) for any \( u, v, \bar{u}, \bar{v} \in C[I, P], u \geq \bar{u}, v \leq \bar{v} \). From (26) and (28), we know that\[
A(u, v) - A(\bar{u}, \bar{v}) \\
\geq - \bar{M} \int_0^t [(u - \bar{u}) + (\bar{v} - v) + T (u - \bar{u})] \, ds \\\n\geq - \bar{M} \left[ (1 + a) \int_0^t (u - \bar{u}) \, ds + \int_0^t (\bar{v} - v) \, ds \right],
\]
(29)
\[
\geq - \bar{M} (1 + a) \int_0^t (u - \bar{u}) + (\bar{v} - v) \, ds,
\]
(30)
where
\[
A(u, v) - A(\bar{u}, \bar{v}) \leq \bar{M} \int_0^t [(u - \bar{u}) + (\bar{v} - v) + T (u - \bar{u})] \, ds \leq \bar{M} \left[ (1 + a) \int_0^t (u - \bar{u}) \, ds + \int_0^t (\bar{v} - v) \, ds \right].
\]
(31)
From (29) and (30), for any \( u, v, \bar{u}, \bar{v} \in C[I, P], u \geq \bar{u}, v \leq \bar{v} \) we know that
\[
-L (u - \bar{u}) + (\bar{v} - v) \leq A(u, v) - A(\bar{u}, \bar{v}) \leq L (u - \bar{u}) + (\bar{v} - v),
\]
(32)
where
\[
(Lu)(t) = \bar{M} (1 + a) \int_0^t w(s) \, ds.
\]
(33)
Now we prove that \( r(L) < 1 \), for \( t \in I \). From (32), we know that
\[
\| (Lu)(t) \| = \bar{M} (1 + a) t \| u \|.
\]
(34)
Denote \( \eta = \bar{M} (1 + a) \). From (32) and (33), we get
\[
\| (L^n u)(t) \| \leq \eta \int_0^t \| (Lu)(s) \| \, ds \leq \eta t \| u \|, \quad \forall t \in I.
\]
(35)
By induction, for any natural number \( n \), we have
\[
\| (L^n u)(t) \| \leq \eta^n t^n \| u \|, \quad \forall t \in I,
\]
(36)
Therefore, \( r(L) = \lim_{n \to \infty} \| L^n u \|^{1/n} = 0 < 1 \).

Since \( P \) is normal solid cone in \( C[I, E] \) and \( P \) is generate, by virtue of Lemma 9, we know that \( A \) has a unique positive fixed point \( x^* \) in \( P \); that is, IVP (1) has a unique positive solution \( x^* \) in \( C^2[I, P] \subset C^2[I, E] \). \( \square \)

Remark 17. The conditions of Theorem 16 cannot be obtained by Chen and Zhuang [3], Lakshmikantham et al. [8], and Sun and Liu [20]. We obtain a unique positive solution of IVP (1). The conditions imposed on nonlinear term \( f \) are sharper, and the result is new.

Denote \( M'(R) = \sup \{ f(t, u, v, w, z) : (t, u, v, w, z) \in I \times B_R \times B_R \times B_R \times B_R, B_R = \{ x \in P : \| x \| \leq R \} \} \).

Theorem 18. Assume that \( f : I \times P \times P \times P \times P \rightarrow P \) satisfying the following conditions.

\( (H_1) \) For any \( R > 0 \), \( f \) is uniformly continuous on \( I \times B_R \times B_R \times B_R \) and
\[
\lim_{R \to \infty} \frac{M'(R)}{R} < (a_0)_{-1}, \quad \text{where} \quad a_0 = \max \{ 1, a_k, a_h \}.
\]
(37)
(H2) There exists $L > 0$ such that
\[
Y(f(t, B(t), B(t), (TB)(t), (SB)(t)))
\leq \lambda_1 Y(B(t)) + \lambda_2 Y((TB)(t)) + \lambda_3 Y((SB)(t))
\tag{37}
\]
for any bounded set $S \subseteq C[I, P] \subseteq C[I, E]$, $t \in I$, and $2(\lambda_1 + \lambda_3 a_k) > 0$, $2\lambda_3 h_0 a \geq 0$ with $2a(\lambda_1 + \lambda_2 a_k + \lambda_3 a h_0) < 1$.

Then IVP (1) has at least one positive solution in $C^2[I, P] \subseteq C^2[I, E]$.

**Proof.** We first define the operator $F : C[I, P] \to C[I, P]$ by the formula
\[
(Fu)(t) = u_0 + tu_1 + \int_0^t (t-s)f(s, u(s), u(s), (Tu)(s), (Su)(s)) \, ds,
\tag{38}
\]
It is easy to know that $u$ is a solution of IVP (1) if and only if $u$ is a fixed point of $F$.

It follows from (H1) that there exist $0 < r < (aa_0)^{-1}$ and $R_1 > 0$ such that
\[
M'(R) < r, \quad \text{for } R \geq a_0 R_1.
\tag{39}
\]
Let
\[
R^* = \max \left\{ R_1, (\|u_0\| + a \|u_1\|)(1 - aa)^{-1} \right\},
\]
\[
B_f(R^*) = \{ u \in C[I, P] : \|u\| \leq R^* \}.
\tag{40}
\]
Then for any $u \in B_f(R^*)$, we have
\[
\|u\| = R^* \leq a_0 R^*, \quad \|Tu\| \leq a_k \|u\| \leq a_k R^* \leq a_0 R^*,
\]
\[
\|Su\| \leq a h_0 \|u\| \leq a h_0 R^* \leq a_0 R^*.
\tag{41}
\]
It follows from (38) and (39) that
\[
\|Fu\| \leq \|u_0\| + a \|u_1\| + aM'(a_0 R^*) \leq \|u_0\| + a \|u_1\| + aa_0 R^* r \leq R^*.
\tag{42}
\]
Set $K = \overline{\operatorname{co}}(B_f(R^*))$. Then $F$ is a continuous operator from $K$ into $K$. It is easy to see from (38), (H1), and the normality of $P$ that $K \subseteq B_f(R^*)$ is uniformly bounded and equicontinuous on $I$.

Let $B \subseteq K \subseteq C[I, P]$ be any countable subset satisfying $B = \overline{\operatorname{co}}(\{u(t)\} \cap (FB)(t))$ for any $t \in I$. By applying Lemma 12, we get
\[
Y(B(t)) = \overline{\operatorname{co}}(Y(B(t))) \leq 2 \int_0^t \left[ \lambda_1 Y(B(t)) + \lambda_2 Y((TB)(t)) + \lambda_3 Y((SB)(t)) \right] \, ds
\leq 2 \int_0^t \left[ \lambda_1 Y(B(t)) + \lambda_2 Y((TB)(t)) + \lambda_3 Y((SB)(t)) \right] \, ds
\]
where $M = (2\lambda_1 + 2\lambda_3 a_k)$, $M^* = 2\lambda_3 a h_0$. Thus, by Ascoli-Arzelà theorem, $B$ is relative compact in $X = C[I, E]$. It follows from Lemma 14 that $F$ has a fixed point $x^* \in K = \overline{\operatorname{co}}(F(B_f(R^*)))$ which is a solution of IVP (1). This completes the proof of Theorem 18.

**Remark 19.** The conditions imposed on nonlinear term do not possess any monotone properties. The results of the paper cannot be obtained by making use of the fixed point theorems on decreasing and increasing operators or mixed monotone operators.

**Conflict of Interests**

The author declares no conflict of interests.

**Acknowledgments**

The author is very grateful to Professor Lishan Liu, Professor Tijun Xiao, and Professor R. P. Agarwal for their many valuable comments. The work is supported financially by the Foundation of Shanghai Natural Science (13ZR1430100) and the Foundation of Shanghai Municipal Education Commission (DYL201020).

**References**


