Research Article

Common Fixed Points of Generalized Cyclic Meir-Keeler-Type Contractions in Partially Ordered Metric Spaces

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The purpose of this paper is to prove some common point theorems for the generalized cyclic Meir-Keeler-type \((\alpha, \varphi, A, B)\)-contraction in partially ordered metric spaces. Our results generalize many recent common point theorems in the literature.

1. Introduction and Preliminaries

Throughout this paper, by \(\mathbb{R}^+\), we denote the set of all nonnegative real numbers, while \(\mathbb{N}\) is the set of all natural numbers. Let \((X, d)\) be a metric space, let \(D\) be a subset of \(X\), and let \(f: D \to X\) be a map. We say that \(f\) is contractive if there exists \(\alpha \in [0, 1)\) such that for all \(x, y \in D\),

\[d(fx, fy) \leq \alpha \cdot d(x, y).\]

The well-known Banach fixed point theorem asserts that if \(D = X\), \(f\) is contractive, and \((X, d)\) is complete, then \(f\) has a unique fixed point in \(X\). It is well known that the Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis. Also, this principle has many generalizations. For instance, a mapping \(f: X \to X\) is called a quasicontraction if there exists \(k < 1\) such that

\[d(fx, fy) \leq k \cdot \max\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\} .\]

for any \(x, y \in X\). In 1974, Ćirić [2] introduced these maps and proved an existence and uniqueness fixed point theorem.

The following definitions and results will be needed in the sequel. Let \(A\) and \(B\) be two nonempty subsets of a metric space \((X, d)\). A mapping \(f: A \cup B \to A \cup B\) is called a cyclic map if \(f(A) \subseteq B\) and \(f(B) \subseteq A\). In 2003, Kirk et al. [3, 4] proved the following fixed point theorem.

**Theorem 1** (see [3, 4]). Let \(A\) and \(B\) be two nonempty closed subsets of a complete metric space \((X, d)\), and suppose that \(f: A \cup B \to A \cup B\) satisfies

\[(i) \ f(A) \subseteq B\ and\ f(B) \subseteq A,
(ii) \ d(fx, fy) \leq k \cdot d(x, y)\ for\ all\ x \in A, y \in B, and \ k \in (0, 1).\]

Then \(A \cap B\) is nonempty, and \(f\) has a unique fixed point in \(A \cap B\).

Recently, many authors proved some fixed point theorems for cyclic maps satisfying various contractive conditions (see, [5–20]).

Let \(X\) be a nonempty set, and let \((X, \sqsubseteq)\) be a partially ordered set endowed with a metric \(d\). Then, the triple \((X, \sqsubseteq, d)\) is called a partially ordered metric space. Two elements \(x, y \in X\) are said to be comparable if either \(x \sqsubseteq y\) or \(y \sqsubseteq x\) holds. Altun et al. [21] introduced the notion of weakly increasing mappings and proved some existing theorems.

**Definition 2** (see [21]). Let \((X, \sqsubseteq)\) be a partially ordered set and \(f, g: X \to X\). Then \(f, g\) are said to be weakly increasing if \(fx \sqsubseteq gfx\) and \(gx \sqsubseteq fgx\) for all \(x \in X\).

And the following definition was introduced in [22].
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Definition 3 (see [22]). Let \((X, \sqsubseteq)\) be a partially ordered set, let \(A, B\) be closed subsets of \(X\) with \(X = A \cup B\), and let \(f, g : X \to X\). Then the pair \((f, g)\) is said to be \((A, B)\)-weakly increasing if \(fx \sqsubseteq gx\) for all \(x \in A\) and \(gx \sqsubseteq fx\) for all \(x \in B\).

In this paper, we introduce the new notion of generalized cyclic Meir-Keeler-type \((\alpha, \psi, A, B)\)-contraction. The purpose of this paper is to prove some common point theorems for the generalized cyclic Meir-Keeler-type \((\alpha, \psi, A, B)\)-contraction in partially ordered metric spaces. Our results generalize many recent common point theorems in the literature.

2. Main Results

In the sequel, we denote by \(\Psi\) the class of functions \(\psi : \mathbb{R}^+ \to \mathbb{R}^+\) satisfying the following conditions:

\[
\psi_1(\psi) = \text{an increasing, continuous function in each coordinate},
\]

\[
\psi_2(\psi) \text{ for all } t \in \mathbb{R}^+, \psi(t, t, t, 0, 2t) \leq t, \psi(t, t, t, 2t, 0) \leq t, \psi(0, 0, t, 0, t) \leq t, \psi(t, 0, 0, t, t) \leq t;
\]

\[
\psi_3(\psi) = 0 \text{ if and only if } t_1 = t_2 = t_3 = t_4 = t_5 = 0.
\]

We start with the following definition.

Definition 4 (see [23]). Let \(f : X \to X\) be a self-mapping of a set \(X\) and \(\alpha : X \times X \to \mathbb{R}^+\). Then \(f\) is called \(\alpha\)-admissible if

\[
x, y \in X, \quad \alpha(x, y) \geq 1 \implies \alpha(fx, fy) \geq 1. \tag{3}\]

Definition 5. Let \(A, B\) be two nonempty subsets of a set \(X\) with \(X = A \cup B\), let \(f : A \to B, g : B \to A\) with \(f(A) \subset B\) and \(g(B) \subset A\), and let \(\alpha : X \times X \to \mathbb{R}^+\). Then the pair \((f, g)\) is called \(\alpha\)-admissible if the following conditions hold:

\[
\alpha(fx, fx) \geq 1, \forall x \in A \implies \alpha(gfx, gfx) \geq 1, \tag{1}
\]

\[
\alpha(gy, gy) \geq 1, \forall y \in B \implies \alpha(fgy, fgy) \geq 1. \tag{2}
\]

In 1969, Meir and Keeler [24] introduced the following notion of Meir-Keeler-type contraction in a metric space \((X, d)\).

Definition 6. Letting \((X, d)\) be a metric space, \(f : X \to X\). Then \(f\) is called a Meir-Keeler-type contraction whenever for each \(\eta > 0\), there exists \(\gamma > 0\) such that

\[
\eta \leq d(x, y) < \eta + \gamma \implies d(fx, fy) < \eta. \tag{4}
\]

We now state the new notions of generalized cyclic Meir-Keeler-type \((\psi, A, B)\)-contractions and generalized Meir-Keeler-type \((\alpha, \psi, A, B)\)-contractions in partially ordered metric spaces as follows.

Definition 7. Let \((X, \sqsubseteq, d)\) be a partially ordered metric space, let \(A, B\) be two nonempty subsets of \(X\) with \(X = A \cup B\), and let \(f : A \to B, g : B \to A\) with \(f(A) \subset B\) and \(g(B) \subset A\). Then the pair \((f, g)\) is called a generalized cyclic Meir-Keeler-type \((\psi, A, B)\)-contraction; if for any comparable elements \(x, y \in X\) with \(x \in A\) and \(y \in B\), we have that for each \(\eta > 0\) there exists \(\delta > 0\) such that

\[
\eta \leq \psi(d(x, y), d(x, fx), d(y, gy), d(x, gy), d(y, fx)) < \eta + \delta \implies d(fx, gy) < \eta, \tag{5}
\]

where \(\psi \in \Psi\).

Definition 8. Let \((X, \sqsubseteq, d)\) be a partially ordered metric space, let \(A, B\) be two nonempty subsets of \(X\) with \(X = A \cup B\), and let \(f : A \to B, g : B \to A\) with \(f(A) \subset B\) and \(g(B) \subset A\), and \(\alpha : X \times X \to \mathbb{R}^+\). Then \((f, g)\) is called a generalized cyclic Meir-Keeler-type \((\alpha, \psi, A, B)\)-contraction if the following conditions hold:

\[
\alpha(fx, fx) \alpha(gy, gy) d(fx, gy) \leq \psi(d(x, y), d(x, fx), d(y, gy), d(x, gy), d(y, fx)). \tag{6}
\]

Further, if

\[
\psi(d(x, y), d(x, fx), d(y, gy), d(x, gy), d(y, fx)) = 0, \tag{8}
\]

then \(d(fx, gy) = 0\).

On the other hand, if

\[
\psi(d(x, y), d(x, fx), d(y, gy), d(x, gy), d(y, fx)) > 0, \tag{9}
\]

then

\[
\alpha(fx, fx) \alpha(gy, gy) d(fx, gy) < \psi(d(x, y), d(x, fx), d(y, gy), d(x, gy), d(y, fx)). \tag{10}
\]

We now state our first main result for the generalized cyclic Meir-Keeler-type \((\alpha, \psi, A, B)\)-contraction as follows.
Theorem 10. Let \((X, \sqsubseteq, d)\) be a partially ordered complete metric space, let \(A, B\) be nonempty closed subsets of \(X\) with \(X = A \cup B\), let \(\alpha : X \times X \to \mathbb{R}^+\), and let \(f, g : X \to X\) be two mappings such that the pair \((f, g)\) is a generalized cyclic Meir-Keeler-type \((\alpha, \psi, A, B)\)-contraction and \((A, B)\)-weakly increasing. Suppose that the following conditions hold:

(i) \(f\) or \(g\) is continuous;

(ii) there exists \(x_0 \in A\) with \(\alpha(fx_0, ftx_0) \geq 1\);

(iii) if \(\alpha(x_n, x_n) \geq 1\) for all \(n \in \mathbb{N}\) and \(\lim_{n \to \infty} x_n = v\), then \(\alpha(fv, fv) \geq 1\) and \(\alpha(gv, gv) \geq 1\).

Then \(f\) and \(g\) have a common fixed point in \(X\).

Proof. By (ii), there exists \(x_0 \in X\) with \(\alpha(fx_0, ftx_0) \geq 1\). Since \(f(A) \subset B\) and the pair \((f, g)\) is \(\alpha\)-admissible, there exists \(x_1 \in B\) such that

\[
x_1 = fx_0, \quad \alpha(gx_1, gx_1) = \alpha(gfx_0,gfx_0) \geq 1.
\]

(11)

Since \(g(B) \subset A\) and the pair \((f, g)\) is \(\alpha\)-admissible, there exists \(x_2 \in A\) such that

\[
x_2 = gx_1, \quad \alpha(fx_2, fx_2) = \alpha(fgx_1, fgx_1) \geq 1.
\]

(12)

Continuing this process, we construct the sequence \(\{x_n\}\) in \(X\) such that

\[
x_{2n+1} = fx_{2n}, \quad x_{2n+2} = gx_{2n+1}, \quad x_{2n} \in A, \quad x_{2n+1} \in B,
\]

(13)

and for all \(n \in \mathbb{N} \cup \{0\}\),

\[
\alpha(x_{2n+1}, x_{2n+1}) = \alpha(fx_{2n}, fx_{2n}) \geq 1,
\]

\[
\alpha(x_{2n+2}, x_{2n+2}) = \alpha(gx_{2n+1}, gx_{2n+1}) \geq 1.
\]

(14)

Since the pair \((f, g)\) is \((A, B)\)-weakly increasing, we have that

\[
x_1 = fx_0 \sqsubseteq gfx_0 = gx_1 \sqsubseteq fgx_1 = fx_2 = x_3 \sqsubseteq \cdots,
\]

(15)

and so we conclude that for all \(n \in \mathbb{N} \cup \{0\}\),

\[
gfx_{2n} = gx_{2n+1} = x_{2n+2} \sqsubseteq gfx_{2n+1} = fx_{2n+2} = x_{2n+3}.
\]

(16)

Case 1. Suppose that \(x_{2n} = x_{2n+1}\) for some \(n \in \mathbb{N}\) in the inequality (16). Since \(x_{2n}\) and \(x_{2n+1}\) are comparable in \(X\) with \(x_{2n} \in A\) and \(x_{2n+1} \in B\), by the Remark 9, we have

\[
d(x_{2n+1}, x_{2n+2}) = d(x_{2n+1}, x_{2n+2}) = d(fx_{2n}, gx_{2n+1})
\]

\[
\leq \alpha(fx_{2n}, fx_{2n}) \alpha(gx_{2n+1}, gx_{2n+1}) d(fx_{2n}, gx_{2n+1})
\]

\[
\leq \psi(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}), 0)
\]

\[
d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})
\]

\[
\leq \psi(0, 0, d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}), 0).
\]

(17)

If \(d(x_{2n+1}, x_{2n+2}) > 0\), then \(\psi(0, 0, d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}), 0) > 0\). By Remark 9, we get a contradiction. So we conclude that \(d(x_{2n+1}, x_{2n+2}) = 0\); that is, \(x_{2n+1} = x_{2n+2}\). Similarly, we may show that \(x_{2n+2} = x_{2n+3}\). Hence \(\{x_n\}\) is a constant sequence, and so \(\{x_n\}\) is a Cauchy sequence in \((X, \sqsubseteq, d)\).

Case 2. Suppose that \(x_{2n} \neq x_{2n+1}\) for all \(n \in \mathbb{N}\) in the inequality (16).

Substep 1. We show that the sequence \(\{d(x_n, x_{n+1}) : n \in \mathbb{N} \cup \{0\}\}\) is decreasing.

Subcase 1. If \(n\) is even, then we let \(n = 2m\) for some \(m \in \mathbb{N}\). Since \(x_{2m} \in A\), \(x_{2m+1} \in B\), and \(x_{2m}, x_{2m+1}\) are comparable in \(X\), we have

\[
d(x_{2m+1}, x_{2m+2}) = d(x_{2m+1}, x_{2m+2}) = d(fx_{2m}, gx_{2m+1})
\]

\[
\leq \alpha(fx_{2m}, fx_{2m}) \alpha(gx_{2m+1}, gx_{2m+1}) d(fx_{2m}, gx_{2m+1})
\]

\[
< \psi(d(x_{2m}, x_{2m+1}), d(x_{2m}, x_{2m+1}), d(x_{2m+1}, x_{2m+2}), d(x_{2m}, x_{2m+2}), 0)
\]

\[
d(x_{2m}, x_{2m+1}), d(x_{2m+1}, x_{2m+2})
\]

\[
\leq \psi(d(x_{2m}, x_{2m+1}), d(x_{2m}, x_{2m+1}), d(x_{2m+1}, x_{2m+2}), d(x_{2m}, x_{2m+2}), 0)
\]

\[
= \psi(d(x_{n}, x_{n+1}), d(x_{n}, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_{n}, x_{n+2}), 0).
\]

(18)
If \( d(x_{2m}, x_{2m+1}) < d(x_{2m+1}, x_{2m+2}) \), then the above inequality becomes
\[
d(x_{2m+1}, x_{2m+2}) < \psi(d(x_{2m+1}, x_{2m+2}), d(x_{2m+1}, x_{2m+2}), d(x_{2m+1}, x_{2m+2}), 0) \tag{19}
\]
\[
\leq d(x_{2m+1}, x_{2m+2}),
\]
which is a contradiction. So we have that
\[
d(x_{2m+1}, x_{2m+2}) \leq d(x_{2m+2}, x_{2m+1}). \tag{20}
\]

Subcase 2. If \( n \) is odd, then we let \( n = 2m + 1 \) for some \( m \in \mathbb{N} \).
Since \( x_{2m+2} \in A, x_{2m+3} \in B \) and \( x_{2m+2}, x_{2m+3} \) are comparable in \( X \), we have
\[
d(x_{n+2}, x_{n+1})
\]
\[
= d(x_{2m+3}, x_{2m+2}) = d(fx_{2m+2}, gx_{2m+1})
\]
\[
\leq \alpha(fx_{2m+2}, fx_{2m+2}, gx_{2m+1}, gx_{2m+1})
\]
\[
= \psi(d(x_{2m+2}, x_{2m+1}), d(x_{2m+1}, x_{2m+2}), d(x_{2m+1}, x_{2m+2}), d(x_{2m+1}, x_{2m+2}))
\]
\[
\leq \psi(d(x_{2m+2}, x_{2m+1}), d(x_{2m+2}, x_{2m+3}), d(x_{2m+2}, x_{2m+3}), 0, d(x_{2m+1}, x_{2m+2} + d(x_{2m+2}, x_{2m+1}))
\]
\[
= \psi(d(x_{n+1}, x_{n}), d(x_{n+1}, x_{n}), d(x_{n+1}, x_{n}), 0, d(x_{n+1}, x_{n}) + d(x_{n+1}, x_{n})))
\]
\[
\leq d(x_{2m+2}, x_{2m+3}),
\]
which is a contradiction. So we have that
\[
d(x_{2m+2}, x_{2m+3}) < d(x_{2m+1}, x_{2m+2}). \tag{21}
\]

If \( d(x_{2m+1}, x_{2m+2}) < d(x_{2m+2}, x_{2m+3}) \), then the above inequality becomes
\[
d(x_{2m+2}, x_{2m+3}) < \psi(d(x_{2m+2}, x_{2m+3}), d(x_{2m+2}, x_{2m+3}), d(x_{2m+2}, x_{2m+3}), 0, 2d(x_{2m+2}, x_{2m+3})) \tag{22}
\]
\[
\leq d(x_{2m+2}, x_{2m+3}),
\]
which is a contradiction. So we have that
\[
d(x_{2m+2}, x_{2m+3}) < d(x_{2m+1}, x_{2m+2}). \tag{23}
\]
From (20) and (23), we conclude that
\[
d(x_{n+1}, x_{n+2}) < d(x_{n+1}, x_{n}). \tag{24}
\]

From the above argument, we have that the sequence \( \{d(x_n, x_{n+1}) : n \in \mathbb{N} \} \) is decreasing, and it must converge to some \( \eta \geq 0 \); that is,
\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = \eta. \tag{25}
\]

**Substep 2.** We next claim that
\[
\lim_{n \to \infty} d(x_n, fx_{n+1}) = 0. \tag{26}
\]
Notice that \( \eta = \inf \{d(x_n, fx_{n+1}) : n \in \mathbb{N} \} \). We claim that \( \eta > 0 \). Suppose, to the contrary, that \( \eta = 0 \).
If \( n \) is even, by the argument of Subcase 1 and the inequality (25), we have
\[
limit_{n \to \infty} \psi(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), 0) = \eta. \tag{27}
\]
\[
d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}, 0) = \eta.
\]
Since \((f, g)\) is a generalized cyclic Meir-Keeler-type \((\alpha, \psi, A, B)\)-contraction, corresponding to \( \eta \) use and taking into account the above (27), there exist \( \delta > 0 \) and a natural number \( k \) such that
\[
\eta \leq \psi(d(x_{k}, x_{k+1}), d(x_{k}, x_{k+1}), d(x_{k+1}, x_{k+2}), d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+2}), 0) < \eta + \delta
\]
\[
\Rightarrow \alpha(fx_{k}, fx_{k}) \alpha(gx_{k+1}, gx_{k+1}) d(fx_{k}, gx_{k+1}) < \eta,
\]
which implies
\[
d(fx_k, gx_{k+1}) \leq \alpha(fx_k, fx_k) \alpha(gx_{k+1}, gx_{k+1}) d(fx_k, gx_{k+1}) < \eta. \tag{29}
\]
So we get a contradiction, since \( \eta = \inf \{d(x_n, x_{n+1}) : n \in \mathbb{N} \} \). Thus we have that
\[
\lim_{n \to \infty} d(x_n, fx_{n+1}) = 0. \tag{30}
\]
If \( n \) is odd, by the argument of Subcase 2 and the inequality (25), we have
\[
\lim_{n \to \infty} \psi(d(x_{n+1}, x_n), d(x_{n+1}, x_n), d(x_n, x_{n+1}), 0, d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})) = \eta. \tag{31}
\]
Similarly, we can prove that
\[
\lim_{n \to \infty} d(x_n, fx_{n+1}) = 0. \tag{32}
\]

**Substep 3.** We show that \( \{x_n\} \) is a Cauchy sequence in \( (X, \subseteq, d) \). It is sufficient to show that \( \{x_{2n}\} \) is a Cauchy sequence in \( (X, \subseteq, d) \).
Suppose, to the contrary, that \( \{x_{2n}\} \) is not a Cauchy sequence in \( (X, \subseteq, d) \). Then there exist \( \varepsilon > 0 \) and two
subsequences \( \{x_{2n(k)}\} \) and \( \{x_{2n(k)}\} \) of \( \{x_{2n}\} \) such that \( n(k) \) is the smallest integer for which \( n(k) > m(k) > k \),

\[
d(x_{2n(k)}, x_{2n(k)}) \geq \epsilon, \quad d(x_{2n(k)}, x_{2n(k)-1}) < \epsilon,
\]

and we get

\[
\epsilon \leq d(x_{2m(k)}, x_{2n(k)}) \\
\leq d(x_{2m(k)}, x_{2n(k)-2}) + d(x_{2n(k)-2}, x_{2n(k)-1}) \\
+ d(x_{2n(k)-1}, x_{2n(k)}) \\
< \epsilon + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}).
\]

Letting \( k \to \infty \) in the above inequality, we get

\[
\lim_{n \to \infty} d(x_{2n+1}, x_{2n}) = \epsilon.
\] (35)

On the other hand, we also obtain that

\[
\epsilon \leq d(x_{2m(k)}, x_{2n(k)}) \\
\leq d(x_{2m(k)}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}) \\
\leq d(x_{2m(k)+1}, x_{2n(k)-1}) + d(x_{2n(k)}, x_{2m(k)+1}) \\
+ d(x_{2n(k)-1}, x_{2n(k)}) \\
\leq d(x_{2n(k)}, x_{2n(k)-1}) + 2d(x_{2m(k)}, x_{2m(k)+1}) \\
+ d(x_{2n(k)-1}, x_{2n(k)}) \\
\leq d(x_{2n(k)}, x_{2n(k)-1}) + 2d(x_{2m(k)+1}, x_{2m(k)+1}) \\
+ 2d(x_{2n(k)-1}, x_{2n(k)}).
\] (36)

Letting \( k \to \infty \) in the above inequality, we get

\[
\lim_{n \to \infty} d(x_{2n+1}, x_{2n}) = \lim_{n \to \infty} d(x_{2m(k)}, x_{2n(k)-1}) = \lim_{n \to \infty} d(x_{2n(k)+1}, x_{2n(k)-1}) \\
= \epsilon.
\] (37)

Since \( x_{2m(k)} \in A, x_{2n(k)-1} \in B, \) and \( x_{2m(k)}, x_{2n(k)-1} \) are comparable in \( X \), we have

\[
d(x_{2m(k)+1}, x_{2n(k)}) \\
= d(fx_{2m(k)+1}, gx_{2n(k)-1}) \\
\leq \alpha(fx_{2m(k)}, fx_{2m(k)}), \alpha(gx_{2n(k)-1}, gx_{2n(k)-1}) \\
\times d(fx_{2m(k)}, gx_{2n(k)-1}) \\
< \psi(d(x_{2m(k)}, x_{2n(k)-1}), d(x_{2m(k)}, fx_{2m(k)})) \\
\leq \psi(d(x_{2m(k)}, x_{2n(k)-1}), d(x_{2n(k)-1}, gx_{2n(k)-1})) \\
= \psi(0, 0, d(y, g), d(y, g), 0) \\
\leq d(y, g).
\] (44)

This implies that \( y = g \). So we complete the proof.

\[\square\]
Applying Theorem 10 and if we let \( \alpha(x, y) = 1 \), then we immediately get the following theorem.

**Theorem 11.** Let \((X, \preceq, d)\) be a partially ordered complete metric space, let \(A, B\) be nonempty closed subsets of \(X\) with \(X = A \cup B\), and let \(f, g : X \rightarrow X\) be two mappings such that the pair \((f, g)\) is a generalized cyclic Meir-Keeler-type \((\psi, A, B)\)-contraction and \((A, B)\)-weakly increasing. If \(f\) or \(g\) is continuous, then \(f\) and \(g\) have a common fixed point in \(X\).

We next state our second main result for the generalized cyclic Meir-Keeler-type \((\alpha, \psi, A, B)\)-contraction as follows.

**Theorem 12.** Let \((X, \preceq, d)\) be a partially ordered complete metric space, let \(A, B\) be nonempty closed subsets of \(X\) with \(X = A \cup B\), let \(\alpha : X \times X \rightarrow \mathbb{R}_+\), and let \(f, g : X \rightarrow X\) be two mappings such that the pair \((f, g)\) is a generalized cyclic Meir-Keeler-type \((\alpha, \psi, A, B)\)-contraction and \((A, B)\)-weakly increasing. Suppose that the following conditions hold:

(i) if \(\{x_n\}\) is a nondecreasing sequence in \(X\) and \(\lim_{n \to \infty} x_n = v\), then \(x_n \preceq v\);

(ii) there exists \(x_0 \in A\) with \(\alpha(fx_0, fx_0) \geq 1\);

(iii) if \(\alpha(x_n, x_n) \geq 1\) for all \(n \in \mathbb{N}\) and \(\lim_{n \to \infty} x_n = v\), then \(\alpha(fv, fv) \geq 1\) and \(\alpha(gv, gv) \geq 1\).

Then \(f\) and \(g\) have a common fixed point in \(X\).

**Proof.** From the same proof’s process of Theorem 10, we can construct a nondecreasing sequence \(\{x_n\}\) in \(X\) with \(x_{2n+1} \in A\), and \(x_n \rightarrow v\) for some \(v \in X\). Since \(x_n \rightarrow v\) and \(A, B\) are nonempty closed subsets of \(X\), we have \(x_{2n} \rightarrow v\), \(x_{2n+1} \rightarrow v\), and \(v \in A \cap B\). By the condition (i), we get \(x_n \preceq v\) for all \(n \in \mathbb{N}\).

Since \(x_{2n} \in A\) and \(v \in B\), we have

\[
d(x_{2n+1}, gv) = d(fx_{2n}, gv) \\
\leq \alpha (fx_{2n}, fx_{2n}) \alpha (gv, gv) d(fx_{2n}, gv) \\
< \psi (d(x_{2n}, v), d(x_{2n}, fx_{2n}), d(v, gv)) \tag{45}
\]

Letting \(n \to \infty\) in the above inequality, we get

\[
d(v, gv) < \psi (0, 0, d(v, gv), d(v, gv), 0) \leq d(v, gv). \tag{46}
\]

This implies that \(d(v, gv) = 0\); that is, \(v = gv\). Similarly, we may show that \(v = fv\). So \(v\) is a common fixed point of \(f\) and \(g\). \(\Box\)

Applying Theorem 12, it is easy to get the following theorem.

**Theorem 13.** Let \((X, \preceq, d)\) be a partially ordered complete metric space, let \(A, B\) be nonempty closed subsets of \(X\) with \(X = A \cup B\), and let \(f, g : X \rightarrow X\) be two mappings such that the pair \((f, g)\) is a generalized cyclic Meir-Keeler-type \((\psi, A, B)\)-contraction and \((A, B)\)-weakly increasing. Suppose that the following condition holds:

if \(\{x_n\}\) is a nondecreasing sequence in \(X\) and \(\lim_{n \to \infty} x_n = v\), then \(x_n \preceq v\).

Then \(f\) and \(g\) have a common fixed point in \(X\).

**References**


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