Research Article

Some Difference Inequalities for Iterated Sums with Applications

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1. Introduction

Among these references, Bainov and Simeonov [4, P.107] considered the following interesting Gronwall-type inequality:

\[ u(t) \leq a(t) + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_{i}} f_i(t, t_{i-1}) u(t_{i-1}) dt_{i-1} \]

Kim [8] considered analogous Gronwall-type integral inequalities involving iterated integrals,

\[ u(t) \leq a + b(t) \left( \int_{t_1}^{t} f_1(t_1) u(t_1) \log u(t_1) dt_1 \right) \]

It has become one of the very few classic and most influential results in the theory and applications of inequalities. Because of its fundamental importance, over the years, many generalizations and analogous results of (2) have been established, such as [3–14].

Among these references, Bainov and Simeonov [4, P.107] considered the following interesting Gronwall-type inequality:

\[ u(t) \leq a(t) + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_{i}} f_i(t, t_{i-1}) u(t_{i-1}) dt_{i-1} \]

Kim [8] considered analogous Gronwall-type integral inequalities involving iterated integrals,

\[ u(t) \leq a + b(t) \left( \int_{t_1}^{t} f_1(t_1) u(t_1) \log u(t_1) dt_1 \right) \]

\[ + \sum_{i=2}^{n} \int_{t_{i-2}}^{t_{i-1}} g_i(t_{i-1}) \left( \int_{t_{i-2}}^{t_{i-1}} g_{i-1}(t_{i-2}) \right) \]

for some constant \( c \geq 0 \), then

\[ u(t) \leq c \exp \left( \int_{a}^{t} f(s) ds \right), \quad t \in [a, b] \]
\begin{align*}
&\times \left( \int_{t_i}^{t_{i-1}} f_i(t_i) u(t_i) \right) \\
&\quad \times \log(u(t_i)) \, dt_i \right) \, dt_{i-1} \right) \\
&\quad \cdots \right) \, dt_2 \right) \, dt_1 \right) .
\end{align*}

In 2011, Abdeldaim and Yakout [12] studied some new integral inequalities of Gronwall-Bellman-Pachpatte type such as

\[ u(t) \leq u(0) + \int_0^t f(s) u(s) \, ds + \int_0^t g(\xi) u(\xi) \, d\xi \, ds, \]

\begin{equation}(4)\end{equation}

Along with the development of the theory of integral inequalities and the theory of difference equations, more and more attentions are paid to discrete versions of Gronwall-type inequalities; for detailed information, please refer to the literatures [15–35]. For instance, Pachpatte [19] considered the following discrete inequality:

\[ u(n) \leq u(0) + \sum_{s=0}^{n-1} f(s) u(s) + \sum_{s=0}^{n-1} g(s) \left( \sum_{t=0}^{s-1} h(t) \left( \sum_{\tau=0}^{t-1} k(\tau) u^p(\tau) \right) \right) . \]

\begin{equation}(5)\end{equation}

In 2006, Cheung and Ren [24] studied

\[ u^p(m,n) \leq c + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} a(s,t) u^q(s,t) \]
\[ + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} b(s,t) u^q(s,t) u(u(s,t)). \]

\begin{equation}(7)\end{equation}

Later, Zheng et al. [31] discussed the following discrete inequality:

\[ u(n) \leq u(0) + \sum_{s=0}^{n-1} f_i(n,s) w_i(u(s)). \]

\begin{equation}(8)\end{equation}

In 2012, Zhou et al. [33] studied the following inequalities:

\[ u(n) \leq a(n) + \sum_{s=0}^{n-1} f_1(n,s) w(u(s)) \]
\[ + \sum_{s=0}^{n-1} f_1(n,s) \omega(s) \sum_{\tau=0}^{s-1} f_2(s,\tau) w(u(\tau)) \]
\[ + \sum_{s=0}^{n-1} f_1(n,s) w(u(s)) \]
\[ \times \sum_{\tau=0}^{s-1} f_2(s,\tau) \sum_{\xi=0}^{\tau-1} f_3(\tau,\xi) w(u(\xi)). \]

\begin{equation}(9)\end{equation}

However, the above results are not applicable to some certain inequalities with multiple iterated sums. Hence, it is desirable to consider more general difference inequalities of these extended types. They can be used in the study of certain classes of difference equations or applied in many practical engineering problems.

Motivated by the results given in [7, 8, 12, 19, 24, 25, 29, 33], in this paper we discuss the following two types of inequalities:

\[ u(n) \leq a(n) + \sum_{t_1=0}^{n-1} f_1(n,t_1) \]
\[ \times \left( \sum_{t_2=0}^{t_1-1} f_2(t_1,t_2) \cdots \left( \sum_{t_k=0}^{t_{k-1}-1} f_k(t_{k-1},t_k) u^p(t_k) \right) \right) . \]

\begin{equation}(10)\end{equation}
Abstract and Applied Analysis

\[ u(n) \leq a(n) + c(n) \sum_{t_1=n_0}^{n-1} f(t_1) g(u(t_1)) + \sum_{j=2}^{k} \sum_{i=1}^{n-1} f_i(n_1, t_i) \]
\times \left( \prod_{t=n_0}^{t_1-1} f(t_1, t_2) \cdots \prod_{i=n_0}^{t_i-1} f_i(t_{i-1}, t_i) \right) g(u(t_i)) \right) \cdots \right]. 

(11)

All the assumptions on (10) and (11) are given in the next sections. The inequalities (10) and (11) consist of multiple iterated sums. Under several practical assumptions, the inequalities are solved through rigorous analysis, and explicit bounds for the unknown functions are given clearly. Further, the derived results are applied to study the estimation of solutions of difference equations.

2. Main Results

Throughout this paper, let \( \mathbb{N}_{n_0} := \{n_0, n_0 + 1, n_0 + 2, \ldots \} \) and \( \mathbb{N}_{n_0}^b := \{n_0, n_0 + 1, n_0 + 2, \ldots, n_0 + b \} \) \( (n_0 \in \mathbb{N}, n, b \in \mathbb{N}) \). For function \( u(n) \), its difference is defined by

\[ \Delta u(n) = f(n) u(n+1) - u(n), \quad \forall n \in \mathbb{N}. \]

Lemma 1. Let \( u(n), a(n) \) and \( c(n) \) be real-valued nonnegative functions defined on \( \mathbb{N} \) and satisfy the inequality

\[ \Delta u(n) \leq a(n) u(n) + c(n), \quad \forall n \in \mathbb{N}. \]

where \( u(n) \) is a nonnegative constant. Then,

\[ u(n) \leq \left( u(n_0) + \sum_{s=n_0}^{n-1} c(s) \prod_{t=s}^{s} (1 + a(t))^{-1} \right) \prod_{t=n_0}^{n-1} (1 + a(s)), \quad \forall n \in \mathbb{N}_{n_0}. \]

(13)

Proof. From (12), we have

\[ u(n+1) - (1 + a(n)) u(n) \leq c(n), \quad \forall n \in \mathbb{N}. \]

Multiplying by \( \prod_{t=n_0}^{n-1} (1 + a(s))^{-1} \) on both sides of the above inequality (14) and summing up both sides from \( n_0 \) to \( n-1 \), we obtain

\[ u(n) \sum_{s=n_0}^{n-1} (1 + a(s))^{-1} - u(n_0) \leq \sum_{s=n_0}^{n-1} c(s) \prod_{t=s}^{s} (1 + a(t))^{-1}, \quad \forall n \in \mathbb{N}_{n_0}. \]

(15)

From (15), we obtain the desired estimate (13).

Theorem 2. Let \( u(n) \) and \( a(n) \) be nonnegative functions defined on \( \mathbb{N}_{n_0} \) with \( a(n) \) nondecreasing on \( \mathbb{N}_{n_0} \). Moreover, let \( f_i(n, s), \quad i = 1, 2, \ldots, k \), be nonnegative functions for \( n_0 \leq s \leq n \) \( (n_0, n, s \in \mathbb{N}_{n_0}) \) and nondecreasing in \( n \) for fixed \( s \in \mathbb{N}_{n_0} \). If \( p \geq 0 \) and \( p \) is not equal to 1, then the discrete inequality (10) gives

\[ u(n) \leq V_1(n, n), \quad \forall n \in \mathbb{N}_{n_0}^b, \]

(16)

where \( V_1(n, n) \) can be successively determined from the formulas

\[ V_k(M, n) = \exp \left( W_k \left( \ln(a(M)) + \sum_{s=n_0}^{M-1} \sum_{i=1}^{k-1} f_i(M, s) \right) \right) \]
\[ + \sum_{s=n_0}^{n-1} f_k(M, s) \right) \right), \]

(17)

\[ V_j(M, n) \leq \left( a(M) + \sum_{s=n_0}^{n-1} f_j(M, s) V_{j+1}(M, s) \right) \]
\[ \times \prod_{t=n_0}^{j-1} \left( 1 + \left( \sum_{i=1}^{j-1} f_i(M, t) - f_j(M, t) \right) \right)^{-1} \]
\[ \times \prod_{s=n_0}^{n-1} \left( 1 + \left( \sum_{i=1}^{p} f_i(M, s) - f_j(M, s) \right) \right) = V_j(M, n), \]

(18)

for \( j = k-1, 1, 2, 1, n \in \mathbb{N}_{n_0}^M \),

\[ W_1(x) = \int_{x_0}^{x} ds \exp ((p-1)s), \quad x_0 > 0, \]

(19)

where \( W_1^{-1} \) is the inverse functions of \( W_1, M \in \mathbb{N}_{n_0}, M \leq b_1 \) is chosen arbitrarily, and \( b_1 \) is the largest natural number such that

\[ W_1 \left( \ln(a(b_1)) + \sum_{s=n_0}^{b_1-1} \sum_{i=1}^{k-1} f_i(b_1, s) \right) \]
\[ + \sum_{s=n_0}^{b_1-1} f_k(b_1, s) \in \text{Dom}(W_1^{-1}). \]

Remark 3. Firstly, from (17) and (18), we obtain \( V_j(M, n) \); then let \( M = n \), and we get \( V_1(n, n) \) since \( M \) is chosen arbitrarily.

Remark 4. We can obtain \( b_1 \) using MATLAB program: firstly let \( b_1 = n_0 \), when \( W_1(\ln(a(b_1))) + \sum_{s=n_0}^{b_1-1} (\sum_{i=1}^{k-1} f_i(b_1, s)) \)
\[ \sum_{s=n_0}^{b_1-1} f_k(b_1, s) < W_1(\infty); \text{let } b_1 = b_1 + 1, \text{ and so on until } W_1(\ln(a(b_1))) + \sum_{s=n_0}^{b_1-1} f_k(b_1, s) < W_1(\infty); \]

Proof. Fix \( M \in \mathbb{N}_n \), where \( M \) is chosen arbitrarily and \( b_1 \) is defined by (20). For \( n \in \mathbb{N}_n \), from (10) we have

\[ u(n) \leq a(M) + \sum_{t_1=n_0}^{n-1} f_1(M, t_1) \times \left( \sum_{t_2=n_0}^{t_1-1} f_2(M, t_2) \right) \times \left( \sum_{t_3=n_0}^{t_2-1} f_3(M, t_3) u^p(t_3) \right) \cdots . \]

(21)

Now we introduce the functions

\[ v_j(n) = a(M) + \sum_{t_j=n_0}^{n-1} f_j(M, t_j) \times \left( \sum_{t_{j+1}=n_0}^{t_j-1} f_{j+1}(M, t_{j+1}) \right) \times \left( \sum_{t_{j+2}=n_0}^{t_{j+1}-1} f_{j+2}(M, t_{j+2}) u^p(t_{j+2}) \right) \cdots . \]

(22)

For \( n \in \mathbb{N}_n \) and \( j = 2, 3, \ldots, k \), then \( v_j, j = 1, 2, \ldots, k \), are all positive and nondecreasing functions on \( \mathbb{N}_n \) with \( v_j(n_0) = a(M), j = 1, 2, \ldots, k \), and the inequalities (22) and (23) imply that

\[ u(n) \leq v_1(n) \leq v_2(n) \leq \cdots \leq v_k(n), \quad \forall n \in \mathbb{N}_n. \]

(24)

From (22), we observe that

\[ \Delta v_j(n) = v_j(n + 1) - v_j(n) \]

\[ = f_j (M, n) \times \left( \sum_{j+1}^{n-1} f_{j+1}(M, t_{j+1}) \right) \times \left( \sum_{j+2}^{n-1} f_{j+2}(M, t_{j+2}) u^p(t_{j+2}) \right) \cdots . \]

(25)

We claim that

\[ \Delta v_j(n) \leq \left( \sum_{j+1}^{n-1} f_{j+1}(M, n) - f_{j+1}(M, n) \right) v_j(n) \]

\[ + f_j(M, n) v_{j+1}(n), \quad \forall n \in \mathbb{N}_n. \]

(26)

(27)

for \( n \in \mathbb{N}_n, j = 2, 3, \ldots, k - 1 \).

Now we prove (26) and (27) by induction. Obviously, (26) is true for \( j = 1 \) by (26). We make the inductive assumption that (26) is true for \( j - 1 \). By the inductive assumption and (24), from (23) we obtain

\[ \Delta v_j(n) \leq \Delta v_{j-1}(n) + f_j(M, n) \]

\[ \times \left( \sum_{j+1}^{n-1} f_j(M, t_{j+1}) \right) \times \left( \sum_{j+2}^{n-1} f_{j+2}(M, t_{j+2}) u^p(t_{j+2}) \right) \cdots . \]

(28)

\[ \leq \sum_{j+1}^{n-1} f_{j+1}(M, n) v_{j+1}(n) + f_j(M, n) v_j(n) \]

\[ \leq \sum_{j+1}^{n-1} f_{j+1}(M, n) v_{j+1}(n) + f_j(M, n) v_j(n) \]

\[ + f_j(M, n) v_{j+1}(n), \quad \forall n \in \mathbb{N}_n. \]
It actually proves (26) by induction. From (23) and (26), we have
\[
\Delta v_k(n) = \Delta v_{k-1}(n) + f_k(M,n) u^p(n)
\]
\[
\leq \left( \sum_{i=1}^{k-2} f_i(M,n) - f_{k-1}(M,n) \right) v_{k-1}(n)
\]
\[
+ f_{k-1}(M,n) v_k(n) + f_k(M,n) v_k^p(n)
\]
\[
\leq \left( \sum_{i=1}^{k-1} f_i(M,n) \right) v_k(n) + f_k(M,n) v_k^p(n),
\]  
for all \( n \in N_M \).

By the definition of \( w_1 \), we obtain
\[
\Delta w_1(n) = f_k(M,n) v_k^{p-1}(n)
\]
\[
\leq f_k(M,n) \exp ((p-1) w_1(n)),
\]  
for all \( n \in n_M \).

From (34) and (35), we get
\[
\frac{\Delta w_1(n)}{\exp ((p-1) w_1(n))} \leq f_k(M,n), \forall n \in n_M,
\]  
where \( W_i \) is defined by (19). By combining (33), (34), and (37), we can obtain that
\[
v_k(n) \leq \exp w_1(n)
\]
\[
\leq \exp \left( W_i^{-1} \left( W_i(w_1(n)) + \sum_{i=n_0}^{n-1} f_k(M,s) \right) \right)
\]
\[
\leq \exp \left( W_i^{-1} \left( \ln(a(M)) + \sum_{i=n_0}^{n-1} \sum_{s=0}^{k-1} f_i(M,s) \right) \right)
\]
\[
= V_i(M,n), \forall n \in n_M.
\]  
where \( V_i(M,n) \) is defined by (17). Applying Lemma 1 to (26) for \( j = k-1, \ldots, 2, 1 \), we have
\[
v_j(n) \leq \left( a(M) + \sum_{i=n_0}^{n-1} f_j(M,s) V_{j+1}(M,s) \right)
\]  
\[
\times \prod_{t=n_0}^{n-1} \left( 1 + \left( \sum_{i=1}^{j-1} f_i(M,t) - f_j(M,t) \right) \right)^{-1}
\]
\[
\times \prod_{t=n_0}^{n-1} \left( 1 + \left( \sum_{i=1}^{j-1} f_i(M,s) - f_j(M,s) \right) \right)
\]
\[
= V_j(M,n), \forall n \in n_M.
\]  
where \( V_j(M,n) \) is defined by (18). From (24) and (39), we have
\[
u(n) \leq v_1(n) \leq V_1(M,n), \forall n \in n_M.
\]
Since $M \in \mathbb{N}_{n_0}$ is arbitrary, from (40), we get the required estimate
\[ u(n) \leq V_1(n, n), \quad \forall n \in \mathbb{N}_{n_0}, \]  
where $b_1$ is defined by (20). Theorem 2 is proved. \(\square\)

**Theorem 5.** Let $u(n), a(n),$ and $c(n)$ be nonnegative functions defined on $\mathbb{N}_{n_0}$ with $a(n)$ and $c(n)$ nondecreasing on $\mathbb{N}_{n_0},$ and let $f_i(n, s), i = 1, 2, \ldots, k,$ be nonnegative functions for $n, s \in \mathbb{N}_{n_0}, n_0 \leq s \leq n,$ which are nondecreasing in $n$ for fixed $s \in \mathbb{N}_{n_0}$. Suppose that $g(u)$ is a nondecreasing continuous function on $[0, \infty)$ with $g(u) > 0$ for $u > 0$. The inequality (II) implies that
\[ u(n) \leq G^{-1} \left( G(a(n)) + c(n) \sum_{s=n_0}^{n-1} f_i(n, s) E(n, s) \right), \]  
where $G^{-1}$ is the inverse function of $G,
\[ G(u) = \int_{u_0}^{u} \frac{ds}{g(s)}, \quad u_0 > 0, \]  
$E(n, s)$
\[ := [1 + f_2(n, s)] [1 + f_3(n, s)] \]  
\[ \times (\cdots (1 + f_k(n, s)) \cdots \times (1 + f_k(n, s))) \cdots \right], \]  
and $b_2$ is the largest natural number such that
\[ G(a(b_2)) + c(b_2) \sum_{s=n_0}^{b_2-1} f_i(b_2, s) E(b_2, s) \in \text{Dom}(G^{-1}). \]  
(45)

**Remark 6.** We can obtain $b_2$ using Matlab program similar to Remark 4.

**Proof.** Let the function $a(n)$ be positive. Fix $M \in \mathbb{N}_{n_0}$, where $M$ is chosen arbitrarily and $b_2$ is defined by (45). For $n \in \mathbb{N}_{n_0}$, from (II) we have
\[ u(n) \]
\[ \leq a(M) + c(M) \left[ \sum_{t_1=n_0}^{n-1} f_1(M, t_1) g(u(t_1)) \right. \]
\[ + \sum_{i=2}^{k} \left[ \sum_{t_2=n_0}^{t_1-1} f_i(M, t_2) \right. \]
\[ \times \left( \sum_{t_3=n_0}^{t_2-1} f_2(M, t_3) \right) \cdots \]
\[ \times \left( \sum_{t_3=n_0}^{t_1-1} f_1(M, t_3) g(u(t_3)) \right) \cdots \right], \]  
(46)

We denote the right-hand side of (46) by $y(n)$ for $n \in \mathbb{N}_{n_0}$. Then $y(n_0) = a(M)$, the function $y(n)$ is positive and nondecreasing in $n \in \mathbb{N}_{n_0}$, $u(n) \leq y(n)$, and
\[ \Delta y(n) \]
\[ = c(M) f_1(M, n) \left[ g(u(n)) + \sum_{t_1=n_0}^{n-1} f_2(M, t_1) g(u(t_1)) \right. \]
\[ + \sum_{i=2}^{k} \left( \sum_{t_2=n_0}^{t_1-1} f_i(M, t_2) \right) \cdots \]
\[ \times \left( \sum_{t_3=n_0}^{t_2-1} f_1(M, t_3) g(u(t_3)) \right) \cdots \], \]  
(47)

Define a function $y_i(n)$ by
\[ y_i(n) = \sum_{t_2=n_0}^{n-1} f_2(M, t_2) g(u(t_2)) \]
\[ + \sum_{i=2}^{k} \left( \sum_{t_3=n_0}^{t_2-1} f_i(M, t_3) g(u(t_3)) \right) \cdots , \]  
(48)

for all $n \in \mathbb{N}_{n_0}$. From (47) and (48), we have
\[ \Delta y(n) = c(M) f_1(M, n) \left[ g(u(n)) + y_1(n) \right], \quad \forall n \in \mathbb{N}_{n_0}, \]  
(49)
From (48), we have
\[ \Delta y_1(n) = f_2(M, n) \left[ g(u(n)) + \sum_{t_3 = n_0}^{n-1} f_3(M, t_3) g(u(t_3)) \right. \\
+ \sum_{i=4}^{k} \left( \sum_{t_i = n_0}^{n-1} f_i(M, t_i) \right) \left( \sum_{t_1 = n_0}^{t_i-1} f_1(M, t_1) g(u(t_1)) \right) \ldots \left( \sum_{t_1 = n_0}^{t_i-1} f_1(M, t_1) g(u(t_1)) \right) \ldots \right], \]
\[ \forall n \in \mathbb{N}^M_{n_0}. \] (50)

From (50), we get
\[ \Delta y_1(n) = f_2(M, n) \left[ g(u(n)) + y_2(n) \right], \forall n \in \mathbb{N}^M_{n_0}, \] (51)
where
\[ y_2(n) = \sum_{t_3 = n_0}^{n-1} f_3(M, t_3) g(u(t_3)) \]
\[ + \sum_{i=4}^{k} \left( \sum_{t_i = n_0}^{n-1} f_i(M, t_i) \right) \left( \sum_{t_1 = n_0}^{t_i-1} f_1(M, t_1) g(u(t_1)) \right) \ldots \left( \sum_{t_1 = n_0}^{t_i-1} f_1(M, t_1) g(u(t_1)) \right) \ldots, \] (52)
for all \( n \in \mathbb{N}^M_{n_0}. \)

Continuing in this way, we obtain
\[ \Delta y_{k-2}(n) = f_{k-1}(M, n) \left[ g(u(n)) + y_{k-1}(n) \right], \forall n \in \mathbb{N}^M_{n_0}, \] (53)
where
\[ y_{k-1}(n) = \sum_{t_k = n_0}^{n-1} f_k(M, t_k) g(u(t_k)), \forall n \in \mathbb{N}^M_{n_0}. \] (54)

From (54) and the inequality \( u(n) \leq y(n) \), we have
\[ \frac{\Delta y_{k-1}(n)}{g(y(n))} \leq f_k(M, n), \forall n \in \mathbb{N}^M_{n_0}. \] (55)

We define the functions \( \bar{y}(s), \bar{y}_i(s) \) (\( i = 1, 2, \ldots, k-1 \)), which are nondecreasing and continuously differentiable on \([n_0, \infty)\) with \( \bar{y}(n) = y(n), \bar{y}_i(n) = y_i(n) \) (\( i = 1, 2, \ldots, k-1 \)) on \( \mathbb{N}^M_{n_0}. \)

On the other hand, by the formula of partial integration, we have
\[
\int_n^{n+1} \frac{\bar{y}_{k-1}(s)}{g'(\bar{y}(s))} ds = \frac{\bar{y}_{k-1}(n)}{g(y(n))} + \int_n^{n+1} \frac{\bar{y}_{k-1}(s) g'(\bar{y}(s))}{g^2(\bar{y}(s))} ds, \forall n \in \mathbb{N}^M_{n_0},
\] (56)

By the monotonicity of \( g, y \), from (56) we have
\[
\int_n^{n+1} \frac{\bar{y}_{k-1}(s)}{g'(\bar{y}(s))} ds \geq \frac{\bar{y}_{k-1}(n)}{g(y(n))}, \forall n \in \mathbb{N}^M_{n_0}. \] (57)

By the mean-value theorem for integrals, for arbitrarily given integers \( n, n+1 \in \mathbb{N}^M_{n_0} \), there exists \( \xi \) in the open interval \((n, n+1)\) such that
\[
\int_n^{n+1} \frac{\bar{y}_{k-1}(s)}{g'(\bar{y}(s))} ds = \int_n^{n+1} \frac{d(\bar{y}_{k-1}(s))}{g'(\bar{y}(s))} ds = \frac{1}{g(y(\xi))} \int_n^{n+1} d(\bar{y}_{k-1}(s)) \leq \frac{\Delta y_{k-1}(n)}{g(y(n))}, \forall n \in \mathbb{N}^M_{n_0}. \] (58)

From (55) and (57), we have
\[
\frac{\Delta y_{k-1}(n)}{g(y(n))} \leq \frac{\Delta y_{k-1}(n)}{g(y(n))} \leq f_k(M, n), \forall n \in \mathbb{N}^M_{n_0}. \] (59)

Next, from the inequalities (53) and (59), we have
\[
\frac{\Delta y_{k-2}(n)}{g(y(n))} \leq \frac{\Delta y_{k-2}(n)}{g(y(n))} \leq f_{k-1}(M, n) [1 + f_k(M, n)], \forall n \in \mathbb{N}^M_{n_0}. \] (60)

Once again, applying the same procedure from (56) to (59) to the inequality (60), we get
\[
\frac{\Delta y_{k-3}(n)}{g(y(n))} \leq \frac{\Delta y_{k-3}(n)}{g(y(n))} \leq f_{k-2}(M, n) [1 + f_k(M, n)], \forall n \in \mathbb{N}^M_{n_0}. \] (61)

Proceeding in this way, we obtain
\[
\frac{y_1(n)}{g(y(n))} \leq \frac{\Delta y_{k-1}(n)}{g(y(n))} \leq f_2(M, n) \cdot \left[ 1 + f_3(M, n) \cdot \left( 1 + f_4(M, n) \cdot \left( 1 + f_5(M, n) \ldots \left( 1 + f_k(M, n) \right) \ldots \right) \right], \forall n \in \mathbb{N}^M_{n_0}. \] (62)
Using the inequalities (49) and (62), we have
\[
\Delta y(n) \frac{g(y(n))}{g(y(n))} \leq c(M) f_1(M, n) \times \{1 + f_2(M, n) [1 + f_3(M, n) \\
\times \cdots (1 + f_{k-1}(M, n) \times 1 + f_k(M, n) \cdots)] \}
\]
\[= c(M) f_1(M, n) E(M, n), \quad \forall n \in \mathbb{N}^M \]

where \(E(M, n)\) is defined by (44).

Once again, using the mean-value theorem for integrals, for arbitrarily given integers \(n, n + 1 \in \mathbb{N}^M\), there exists \(\xi\) in the open interval \((y(n), y(n + 1))\) such that
\[
G(y(n + 1)) - G(y(n)) = \int_{y(n)}^{y(n+1)} \frac{ds}{g(s)} = \frac{\Delta y(n)}{g(\xi)} \leq \frac{\Delta y(n)}{g(y(n))}, \quad \forall n \in \mathbb{N}^M \tag{64}
\]
where \(G\) is defined by (43). Using (63), (64), and \(y(n_0) = a(M)\), we obtain
\[
u(n) \leq y(n) \leq G^{-1}\left(G(a(M)) + c(M) \sum_{s=n_0}^{n-1} f_1(M, s) E(M, s)\right), \quad \forall n \in \mathbb{N}^M \tag{65}
\]

In (65), let \(n = M\); we have
\[
u(M) \leq G^{-1}\left(G(a(M)) + c(M) \sum_{s=n_0}^{M-1} f_1(M, s) E(M, s)\right), \quad \forall n \in \mathbb{N}^M \tag{66}
\]

Due to the randomness of \(T\), (42) is achieved immediately from (66).

3. Application

In this section, we apply Theorem 5 to the following difference equation:
\[
\Delta x(n) = F\left(n, x(n), \sum_{s=n_0}^{n-1} z(s, x(s))\right), \quad \forall n \in \mathbb{N}^M \tag{67}
\]

**Corollary 7.** Assume that \(F\) is defined on \([\mathbb{N}^M] \times [0, \infty) \times [0, \infty)\), and there exist nonnegative functions \(d_i(n), i = 1, 2\), such that
\[
|F(n, x, y)| \leq d_1(n) g(|x|) + d_2(n) y, \quad |z(s, x)| \leq d_1(n) g(|x|), \tag{68}
\]
where the function \(g\) is as in Theorem 5. If \(x(n)\) is any solution of the problem (67), then
\[
|x(n)| \leq G^{-1}\left(G\left(|x(n_0)|\right) + \sum_{s=n_0}^{n-1} d_1(s) E(s)\right), \quad \forall n \in \mathbb{N}^M \tag{69}
\]
where the functions \(G, G^{-1}\) are as in Theorem 5,
\[
E(n) = 1 + d_1(n) (1 + d_2(n)), \tag{70}
\]
and \(b_3\) is the largest natural number such that
\[
G(|x(n_0)|) + b_3 - 1 \sum_{s=n_0}^{s_0} d_1(s) E(s) \in \text{Dom}\left(G^{-1}\right). \tag{71}
\]

**Proof.** The difference equation (67) is equivalent to
\[
x(n) = x(n_0) + \sum_{s=n_0}^{n-1} F\left(s, x(s), \sum_{t=n_0}^{s-1} z(t, x(t))\right), \quad \forall n \in \mathbb{N}^M \tag{72}
\]

Using (68), from (72), we have
\[
|x(n)| \leq |x(n_0)| + \sum_{s=n_0}^{n-1} d_1(s) g(|x(s)|) + \sum_{s=n_0}^{n-1} d_2(t) g(|x(s)|), \quad \forall n \in \mathbb{N}^M \tag{73}
\]

Notice that, by the assumption, all functions in (73) satisfy the conditions of Theorem 5. Applying Theorem 5 to the inequality (73), (69) is immediately derived. This completes the proof of Corollary 7.

**Conflict of Interests**

The authors declare that they have no competing interests.

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References


