Research Article

General Split Feasibility Problems in Hilbert Spaces

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Introducing a general split feasibility problem in the setting of infinite-dimensional Hilbert spaces, we prove that the sequence generated by the purposed new algorithm converges strongly to a solution of the general split feasibility problem. Our results extend and improve some recent known results.

1. Introduction

Let $H$ and $K$ be infinite-dimensional real Hilbert spaces, and let $A : H \to K$ be a bounded linear operator. Let $\{C_i\}_{i=1}^p$ and $\{Q_i\}_{i=1}^r$ be the families of nonempty closed convex subsets of $H$ and $K$, respectively.

(a) The convex feasibility problem (CFP) is formulated as the problem of finding a point $x^*$ with the property:

$$x^* \in \bigcap_{i=1}^p C_i.$$  \hspace{1cm} (1)

(b) The split feasibility problem (SEP) is formulated as the problem of finding a point $x^*$ with the property:

$$x^* \in C, \quad Ax^* \in Q,$$  \hspace{1cm} (2)

where $C$ and $Q$ are nonempty closed convex subsets of $H$ and $K$, respectively.

(c) The multiple-set split feasibility problem (MSSFP) is formulated as the problem of finding a point $x^*$ with the property:

$$x^* \in \bigcap_{i=1}^p C_i, \quad Ax^* \in \bigcap_{i=1}^r Q_i.$$  \hspace{1cm} (3)

Note that (MSSFP) reduces to (SEP) if we take $p = r = 1$.

There is a considerable investigation on CFP in view of its applications in various disciplines such as image restoration, computer tomograph, and radiation therapy treatment planning [1]. The split feasibility problem SFP in the setting of finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [2] for modelling inverse problems which arise from phase retrievals and in medical image reconstruction [3]. Since then, a lot of work has been done on finding a solution of SFP and MSSFP; see, for example, [2–25]. Recently, it is found that the SFP can also be applied to study the intensity-modulated radiation therapy; see, for example, [6,16] and the references therein. Very recently, Xu [8] considered the SFP in the setting of infinite-dimensional Hilbert spaces.

The original algorithm given in [2] involves the computation of the inverse $A^{-1}$ provided it exists. In [8], Xu studied some algorithm and its convergence. In particular, he applied Mann’s algorithm to the SFP and purposed an algorithm which is proved to be weakly convergent to a solution of the SFP. He also established the strong convergence result, which shows that the minimum-norm solution can be obtained. In [7], Wang and Xu purposed the following cyclic algorithm to solve MSSFP:

$$x_{n+1} = P_{C[n]} \left( x_n + \gamma A^* (P_{Q[n]} - I) Ax_n \right),$$  \hspace{1cm} (4)

where $[n] := n (\mod p)$, (mod function take values in $\{1,2,\ldots,p\}$), and $\gamma \in (0,2/\|A\|^2)$. They show that the sequence $\{x_n\}$ convergence weakly to a solution of MSSFP provided the solution exists. To study strong convergence to
a solution of MSSFP, first we introduce a general form of the split feasibility problem for infinite families as follows.

(d) General split feasibility problem (GSFP) is to find a point \( x^* \) such that

\[
 x^* \in \bigcap_{i=1}^{\infty} C_i, \quad Ax^* \in \bigcap_{i=1}^{\infty} Q_i. \tag{5}
\]

We denote by \( \Omega \) the solution set of GSFP.

In this paper, using viscosity iterative method defined by Moudafi [21], we propose an algorithm for finding the solutions of the general split feasibility problem in a Hilbert space. We establish the strong convergence of the proposed algorithm to a solution of GSFP.

2. Preliminaries

Throughout the paper, we denote by \( H \) a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). Let \( \{x_n\} \) be a sequence in \( H \) and \( x \in H \). Weak convergence of \( \{x_n\} \) to \( x \) is denoted by \( x_n \rightharpoonup x \), and strong convergence by \( x_n \rightarrow x \). Let \( C \) be a closed and a convex subset of \( H \). For every point \( x \in H \), there exists a unique nearest point in \( C \), denoted by \( P_C x \). This point satisfies

\[
\| x - P_C x \| \leq \| x - y \|, \quad \forall y \in C. \tag{6}
\]

The operator \( P_C \) is called the metric projection or the nearest point mapping of \( H \) onto \( C \). The metric projection \( P_C \) is characterized by the fact that \( P_C(x) \in C \) and

\[
\left\langle y - P_C(x), x - P_C(x) \right\rangle \leq 0, \quad \forall x \in H, \quad y \in C. \tag{7}
\]

Recall that a mapping \( T : C \rightarrow C \) is called nonexpansive if

\[
\| Tx - Ty \| \leq \| x - y \|, \quad \forall x, y \in C. \tag{8}
\]

It is well known that \( P_C \) is a nonexpansive mapping. It is also known that \( H \) satisfies Opial’s condition, that is, for any sequence \( \{x_n\} \) with \( x_n \rightharpoonup x \), the inequality

\[
\liminf_{n \to \infty} \| x_n - x \| < \liminf_{n \to \infty} \| x_n - y \| \tag{9}
\]

holds for every \( y \in H \) with \( y \neq x \).

Lemma 1. Let \( H \) be a Hilbert space. Then, for all \( x, y \in H \)

\[
\| x + y \|^2 \leq \| x \|^2 + 2 \langle y, x + y \rangle. \tag{10}
\]

Lemma 2 (see [22]). Let \( H \) be a Hilbert space, and let \( \{x_n\} \) be a sequence in \( H \). Then, for any given sequence \( \{\lambda_n\}_{n=1}^{\infty} \subset (0,1) \) with \( \sum_{n=1}^{\infty} \lambda_n = 1 \) and for any positive integer \( i, j \) with \( i < j \),

\[
\sum_{n=1}^{\infty} \lambda_n \| x_n - x \|^2 \leq \sum_{n=1}^{\infty} \lambda_n \| x_n - x_i \|^2 - \lambda_i \lambda_j \| x_i - x_j \|^2. \tag{11}
\]

Lemma 3 (see [23]). Assume that \( \{a_n\} \) is a sequence of nonnegative real numbers such that

\[
a_{n+1} \leq (1 - y_n) a_n + y_n \delta_n + \beta_n, \quad n \geq 0, \tag{12}
\]

where \( \{y_n\}, \{\beta_n\}, \) and \( \{\delta_n\} \) satisfy the following conditions:

(i) \( y_n \in [0,1], \sum_{n=1}^{\infty} y_n = \infty, \)

(ii) \( \lim \sup_{n \to \infty} \delta_n \leq 0 \) or \( \sum_{n=1}^{\infty} |y_n \delta_n| < \infty, \)

(iii) \( \beta_n \geq 0 \) for all \( n \geq 0 \) with \( \sum_{n=0}^{\infty} \beta_n < \infty. \)

Then, \( \lim_{n \to \infty} a_n = 0. \)

Lemma 4 (see [24]). Let \( \{t_n\} \) be a sequence of real numbers such that there exists a subsequence \( \{n_i\} \) of \( \{n\} \) such that \( t_{n_i} < t_{n_{i+1}} \) for all \( i \in \mathbb{N}. \) Then, there exists a nondecreasing sequence \( \{\tau(n)\} \subset \mathbb{N} \) such that \( \tau(n) \to \infty, \) and the following properties are satisfied by all (sufficiently large) numbers \( n \in \mathbb{N}: \)

\[
t_{\tau(n)} \leq t_{\tau(n)+1}, \quad t_n \leq t_{\tau(n)+1}. \tag{13}
\]

In fact

\[
\tau(n) = \max \{ k \leq n : t_k < t_{k+1} \}. \tag{14}
\]

Lemma 5 (demiclosedness principle [25]). Let \( C \) be a nonempty closed and convex subset of a real Hilbert space \( H. \) Let \( T : C \rightarrow C \) be a nonexpansive mapping such that \( \text{Fix}(T) \neq \emptyset. \) Then, \( T \) is demiclosed on \( C, \) that is, if \( y_n \rightharpoonup y \in C, \) and \( (y_n - Ty_n) \to y, \) then \( (I - T)z = y. \)

3. Main Result

In the following result, we propose an algorithm and prove that the sequence generated by the proposed method converges strongly to a solution of the GSFP.

Theorem 6. Let \( H \) and \( K \) be real Hilbert spaces, and let \( A : H \rightarrow K \) be a bounded linear operator. Let \( \{C_i\}_{i=1}^{\infty}, \) and \( \{Q_i\}_{i=1}^{\infty} \) be the families of nonempty closed convex subsets of \( H \) and \( K, \) respectively. Assume that GSFP (5) has a nonempty solution set \( \Omega. \) Suppose that \( f \) is a self \( k \)-contraction mapping of \( H, \) and let \( \{x_n\} \) be a sequence generated by \( x_0 \in H \) as

\[
x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \sum_{i=1}^{\infty} y_{n_i} P_{C_i} \left( I - \lambda_{n_i} A^* (I - P_{Q_i}) A \right) x_n, \quad n \geq 0, \tag{15}
\]

where \( \alpha_n + \beta_n + \sum_{i=1}^{\infty} y_{n_i} = 1. \) If the sequences \( \{\alpha_n\}, \{\beta_n\}, \{y_{n_i}\}, \) and \( \{\lambda_{n_i}\} \) satisfy the following conditions:

(i) \( \lim_{n \to \infty} \lambda_n = 0 \) and \( \sum_{n=0}^{\infty} \beta_n = \infty, \)

(ii) for each \( i \in \mathbb{N}, \lim \inf \alpha_i y_{n_i} > 0, \)

(iii) for each \( i \in \mathbb{N}, \{\lambda_{n_i}\} \subset (0, 2/\|A\|^2) \) and \( 0 < \lim \inf_{n \to \infty} \lambda_{n_i} \leq \lim \sup_{n \to \infty} \lambda_{n_i} < 2/\|A\|^2, \)

then, the sequence \( \{x_n\} \) converges strongly to \( x^* \in \Omega, \) where \( x^* = P_{\Omega} f(x^*). \)
Proof. First, we show that \( \{x_n\} \) is bounded. In fact, let \( z \in \Omega \). Since \( \{\lambda_n,i\} \subset (0,2/\|A\|^2) \), the operators \( P_C(I - \lambda_n,i A^*(I - P_Q)A) \) are nonexpansive, and hence we have

\[
\begin{align*}
\|x_{n+1} - z\| &= \|\alpha_n x_n + \beta_n f(x_n) \\
&+ \sum_{j=1}^{\infty} \gamma_{n,j} P_{C_j}(I - \lambda_n,j A^*(I - P_Q)A)x_n - z\| \\
&\leq \alpha_n \|x_n - z\| + \beta_n \|f(x_n) - z\| \\
&+ \sum_{j=1}^{\infty} \gamma_{n,j} \|P_{C_j}(I - \lambda_n,j A^*(I - P_Q)A)x_n - z\| \\
&\leq \alpha_n \|x_n - z\| + \beta_n \|f(x_n) - z\| \\
&+ \sum_{j=1}^{\infty} \gamma_{n,j} \|x_n - z\| \\
&\leq (1 - \beta_n) \|x_n - z\| + \beta_n \|f(x_n) - z\| \\
&\leq (1 - \beta_n) \|x_n - z\| + \beta_n \|f(x_n) - f(z)\| \\
&+ \beta_n \|f(z) - z\| \\
&\leq (1 - (1 - k)) \beta_n \|x_n - z\| \\
&+ (1 - k) \frac{\beta_n}{1 - k} \|f(z) - z\| \\
&\leq \max \left\{ \|x_n - z\|, \frac{1}{1 - k} \|f(z) - z\| \right\} \\
&\vdots \\
&\leq \max \left\{ \|x_0 - z\|, \frac{1}{1 - k} \|f(z) - z\| \right\},
\end{align*}
\]

which implies that \( \{x_n\} \) is bounded, and we also obtain that \( \{f(x_n)\} \) is bounded. Next, we show that for each \( i \in \mathbb{N} \),

\[
\lim_{n \to \infty} \|x_n - P_{C_i}(I - \lambda_n,i A^*(I - P_Q)A)x_n\| = 0.
\]

By using Lemma 2, for every \( z \in \Omega \) and \( i \in \mathbb{N} \), we have that

\[
\begin{align*}
\|x_{n+1} - z\|^2 &= \|\alpha_n x_n + \beta_n f(x_n) \\
&+ \sum_{j=1}^{\infty} \gamma_{n,j} P_{C_j}(I - \lambda_n,j A^*(I - P_Q)A)x_n - z\|^2 \\
&\leq \alpha_n \|x_n - z\|^2 + \beta_n \|f(x_n) - z\|^2 \\
&+ \sum_{j=1}^{\infty} \gamma_{n,j} \|P_{C_j}(I - \lambda_n,j A^*(I - P_Q)A)x_n - z\|^2 \\
&\leq \alpha_n \|x_n - z\|^2 + \beta_n \|f(x_n) - z\|^2 \\
&+ \sum_{j=1}^{\infty} \gamma_{n,j} \|x_n - z\|^2 \\
&\leq (1 - \alpha_n) \|x_n - z\|^2 + \beta_n \|f(x_n) - z\|^2 \\
&\leq (1 - \beta_n) \|x_n - z\|^2 + \beta_n \|f(x_n) - z\|^2.
\end{align*}
\]

Hence, for each \( i \in \mathbb{N} \), we have

\[
\alpha_n Y_{n,i} \|P_{C_i}(I - \lambda_n,i A^*(I - P_Q)A)x_n - x_i\|^2 \\
\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \beta_n \|f(x_n) - z\|^2.
\]

Next, we show that there exists a unique \( x^* \in \Omega \) such that \( x^* = P_Q(f(x^*)) \). We observe that for each \( n \geq 0, x^* \in \Omega \) solves the GSFP (5) if and only if \( x^* \) solves the fixed point equation

\[
x^* = P_{C_i}(I - \lambda_n,i A^*(I - P_Q)A)x^*, \quad i \in \mathbb{N},
\]

that is, the solution sets of fixed point equation (20) and GSFP (5) are the same (see for details [8]). Note that if \( \{\lambda_n\} \subset (0,2/\|A\|^2) \), then the operators \( P_C(I - \lambda_n,i A^*(I - P_Q)A) \) are nonexpansive. Since the fixed point set of nonexpansive operators is closed and convex, the projection onto the solution set \( \Omega \) is well defined whenever \( \Omega \neq \emptyset \). We observe that \( P_Q(f) \) is a contraction of \( H \) into itself. Indeed, since \( P_Q \) is nonexpansive,

\[
\|P_Q(f)(x) - P_Q(f)(y)\| \leq \|f(x) - f(y)\| \leq k \|x - y\|.
\]

Hence, there exists a unique element \( x^* \in \Omega \) such that \( x^* = P_Q(f(x^*)) \).

In order to prove that \( x_n \to x^* \) as \( n \to \infty \), we consider two possible cases.

Case 1. Assume that \( \{\|x_n - x^*\|\} \) is a monotone sequence. In other words, for \( \delta > 0 \) large enough, \( \|x_n - x^*\| \geq \delta \) is either nondecreasing or nonincreasing. Since \( \|x_n - x^*\| \) is bounded we have \( \|x_n - x^*\| \) is convergent. Since \( \lim_{n \to \infty} \beta_n = 0 \) and \( \{f(x_n)\} \) is bounded, from (19) we get that

\[
\lim_{n \to \infty} \alpha_n Y_{n,i} \|P_{C_i}(I - \lambda_n,i A^*(I - P_Q)A)x_n - x_i\|^2 = 0.
\]
By assuming that \( \liminf_i \alpha_n y_{n,i} > 0 \), we obtain
\[
\lim_{n \to \infty} \left\| P_{C_i} \left( I - \lambda_{n,j} A^* \left( I - P_Q \right) A \right) x_n - x_n \right\| = 0, \quad \forall i \in \mathbb{N}.
\] (23)

Now, we show that
\[
\limsup_{n \to \infty} \langle f(x^*) - x^*, x_n - x^* \rangle \leq 0.
\] (24)

To show this inequality, we choose a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that
\[
\lim_{k \to \infty} \langle f(x^*) - x^*, x_{n_k} - x^* \rangle = \limsup_{n \to \infty} \langle f(x^*) - x^*, x_n - x^* \rangle.
\] (25)

Since \( \{x_{n_k}\} \) is bounded, there exists a subsequence \( \{x_{n_{k_l}}\} \) of \( \{x_{n_k}\} \) which converges weakly to \( w \). Without loss of generality, we can assume that \( x_{n_{k_l}} \to w \) and \( \lambda_{n_{k_l}} \to \lambda_j \in (0, 2/\|A\|^2) \) for each \( i \in \mathbb{N} \). From (23), we have
\[
\left\| P_{C_i} \left( I - \lambda_{i,j} A^* \left( I - P_Q \right) A \right) x_{n_{k_l}} - x_n \right\|
\leq \left\| P_{C_i} \left( I - \lambda_{i,j} A^* \left( I - P_Q \right) A \right) x_n \right\|
+ \left\| P_{C_i} \left( I - \lambda_{i,j} A^* \left( I - P_Q \right) A \right) x_{n_{k_l}} - x_{n_{k_l}} \right\|
+ \left\| x_{n_{k_l}} - x_n \right\|
\leq \left\| \lambda_{i,j} A^* \left( I - P_Q \right) A x_n \right\|
+ \left\| \lambda_{i,j} A^* \left( I - P_Q \right) A x_{n_{k_l}} - x_{n_{k_l}} \right\|
+ \left\| x_{n_{k_l}} - x_n \right\|
\to 0 \quad \text{as } n \to \infty.
\] (26)

Notice that for each \( i \in \mathbb{N} \), \( P_{C_i} (I - \lambda_{i,j} A^* (I - P_Q)) A \) is nonexpansive. Thus, from Lemma 5, we have \( w \in \Omega \). Therefore, it follows that
\[
\limsup_{n \to \infty} \langle f(x^*) - x^*, x_n - x^* \rangle
= \lim_{k \to \infty} \langle f(x^*) - x^*, x_{n_{k_l}} - x^* \rangle
= \langle f(x^*) - x^*, w - x^* \rangle \leq 0.
\] (27)

Finally, we show that \( x_n \to P_{\Omega} f(x^*) \). Applying Lemma 1, we have that
\[
\| x_{n+1} - x^* \|^2
= \alpha_n \| x_n - x^* \|^2
+ \frac{\sum_{i=1}^{\infty} \gamma_n y_{n,i} \left( P_{C_i} \left( I - \lambda_{n,j} A^* \left( I - P_Q \right) A \right) x_n - x^* \right)^2}{\| P_{C_i} \left( I - \lambda_{n,j} A^* \left( I - P_Q \right) A \right) x_n - x^* \|^2 + 2 \beta_n \langle f(x_n) - x^*, x_{n+1} - x^* \rangle}
\leq (1 - \eta_n) \| x_n - x^* \|^2
+ \eta_n \| x_{n+1} - x^* \|^2.
\] (28)

This implies that
\[
\| x_{n+1} - x^* \|^2
\leq \frac{(1 - \beta_n)^2 + \beta_n k}{1 - \beta_n k} \| x_n - x^* \|^2
+ \frac{2 \beta_n}{1 - \beta_n k} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle
= \frac{1 - \beta_n}{1 - \beta_n k} \| x_n - x^* \|^2
+ \frac{\beta_n^2}{1 - \beta_n k} \| x_{n+1} - x^* \|^2
+ \frac{2 \beta_n}{1 - \beta_n k} \langle f(z) - x^*, x_{n+1} - x^* \rangle
\leq \left( 1 - \frac{2 (1 - k) \beta_n}{1 - \beta_n k} \right) \| x_n - x^* \|^2
+ \frac{2 (1 - k) \beta_n}{1 - \beta_n k} \left\{ \beta_n M \frac{1}{2 (1 - k)} \right\}
+ \frac{1}{1 - k} \| f(x^*) - x^*, x_{n+1} - x^* \|
\leq (1 - \eta_n) \| x_n - x^* \|^2 + \eta_n \delta_n.
\] (29)
where
\[ \delta_n = \frac{\beta_n M}{2(1-k)} + \frac{1}{1-k} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle, \]
(30)

\[ M = \sup \{ \|x_n - x^*\|^2 : n \geq 0 \} \]
and \( \eta_n = 2(1-k)\beta_n/(1-k) \). It is easy to see that \( \eta_n \to 0 \), \( \sum_{n=1}^{\infty} \eta_n = \infty \) and \( \limsup_{n \to \infty} \delta_n \leq 0 \). Hence, by Lemma 3, the sequence \( \{x_n\} \) converges strongly to \( x^* = P_{\Omega} f(x^*) \).

Case 2. Assume that \( \{\|x_n - x^*\|\} \) is not a monotone sequence. Then, we can define an integer sequence \( \{\tau(n)\} \) for all \( n \geq n_0 \) (for some \( n_0 \) large enough) by
\[ \tau(n) = \max \{k \in \mathbb{N} : k \leq n : \|x_k - x^*\| < \|x_k + 1 - x^*\| \}. \]
(31)

Clearly, \( \tau(n) \) is a nondecreasing sequence such that \( \tau(n) \to \infty \) as \( n \to \infty \) and for all \( n \geq n_0 \),
\[ \|x_{\tau(n)} - x^*\| < \|x_{\tau(n) + 1} - x^*\|. \]
(32)

From (19), we obtain that
\[ \lim_{n \to \infty} \|P_{C_i} (I - \lambda_{\tau(n)} A^* (I - P_{Q_j}) A) x_{\tau(n)} - x_{\tau(n)} \| = 0. \]
(33)

Following an argument similar to that in Case 1, we have
\[ \limsup_{n \to \infty} \langle f(x^*) - x^*, x_{\tau(n) + 1} - x^* \rangle \leq 0. \]
(34)

And by similar argument, we have
\[ \|x_{\tau(n) + 1} - x^*\|^2 \]
\[ \leq (1 - \eta_{\tau(n)}) \|x_{\tau(n)} - x^*\|^2 + \eta_{\tau(n)} \delta_{\tau(n)}, \]
(35)

where \( \eta_{\tau(n)} \to 0 \), \( \sum_{n=1}^{\infty} \eta_{\tau(n)} = \infty \) and \( \limsup_{n \to \infty} \delta_{\tau(n)} \leq 0 \). Hence, by Lemma 3, we obtain \( \lim_{n \to \infty} \|x_{\tau(n)} - x^*\| = 0 \) and \( \lim_{n \to \infty} \|x_{\tau(n) + 1} - x^*\| = 0 \). Now, from Lemma 4, we have
\[ 0 \leq \|x_n - x^*\| \]
\[ \leq \max \{ \|x_{\tau(n)} - x^*\|, \|x_n - x^*\| \} \]
\[ \leq \|x_{\tau(n) + 1} - x^*\|. \]
(36)

Therefore, \( \{x_n\} \) converges strongly to \( x^* = P_{\Omega} f(x^*) \).

For finite collections we have the following consequence of Theorem 6.

**Theorem 7.** Let \( H \) and \( K \) be real Hilbert spaces, and let \( A : H \to K \) be a bounded linear operator. Let \( \{C_i\}_{i=1}^{P} \) be a family of nonempty closed convex subsets in \( H \), and let \( \{Q_j\}_{j=1}^{P} \) be a family of nonempty closed convex subsets in \( K \). Assume that \( MSSFP \) has a nonempty solution set \( \Omega \). Let \( u \) be an arbitrary element in \( H \), and let \( \{x_n\} \) be a sequence generated by \( x_0 \in H \) and
\[ x_{n+1} = \alpha_n x_n + \beta_n u \]
\[ + \sum_{i=1}^{P} \gamma_{n,i} P_{C_i} (I - \lambda_{n,i} A^* (I - P_{Q_j}) A) x_n, \]
\[ n \geq 0, \]
(37)
where \( \alpha_n + \beta_n + \sum_{i=1}^{P} \gamma_{n,i} = 1 \). If the sequences \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_{n,i}\} \) and \( \{\lambda_{n,i}\} \) satisfy the following conditions:

(i) \( \lim_{n \to \infty} \beta_n = 0 \) and \( \sum_{n=0}^{\infty} \beta_n = \infty \),
(ii) for all \( i \in \{1, 2, \ldots, p\} \), \( \lim inf_n \alpha_n \gamma_{n,i} > 0 \),
(iii) for all \( i \in \{1, 2, \ldots, p\} \), \( \lambda_{n,i} \leq (0, 2/\|A\|^2) \) and
\[ 0 < \lim inf_{n \to \infty} \lambda_{n,i} \leq \lim sup_{n \to \infty} \lambda_{n,i} < \frac{2}{\|A\|^2}, \]
(38)

then the sequence \( \{x_n\} \) converges strongly to \( x^* \in \Omega \), where \( x^* = P_{\Omega} u \).

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**References**


