Research Article

Remarks on the Blow-Up Solutions for the Critical Gross-Pitaevskii Equation

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This paper is concerned with the blow-up solutions of the critical Gross-Pitaevskii equation, which models the Bose-Einstein condensate. The existence and qualitative properties of the minimal blow-up solutions are obtained.

1. Introduction and Main Results

In this paper, we deal with the Cauchy problem of the nonlinear Schrödinger equation with a harmonic potential

$$i\phi_t + \Delta\phi - |x|^2\phi + |\phi|^{4/N}\phi = 0, \quad x \in \mathbb{R}^N, \quad t \geq 0, \quad (1)$$

$$\phi(0, x) = \phi_0(x), \quad (2)$$

where $\phi = \phi(t, x): [0, T) \times \mathbb{R}^N \rightarrow \mathbb{C}$ is the wave function, N is the space dimension, and Δ denotes the Laplace operator on $\mathbb{R}^N$. Equation (1) is also called Gross-Pitaevskii equation (see [1, 2]), which models the Bose-Einstein condensate (see [3, 4]). The harmonic potential $|x|^2$ describes a magnetic field. With the nonlinear term $|\phi|^{4/N}\phi$ being replaced by $|\phi|^{p-1}\phi$, it is well known that the exponent $p = 1 + 4/N$ is the minimal value for the existence of blow-up solutions (see e.g., [5, 6]). Hence (1) is called critical Gross-Pitaevskii equation.

Let us recall the classical nonlinear Schrödinger equation

$$i\psi_t + \Delta\psi + |\psi|^{4/N}\psi = 0, \quad x \in \mathbb{R}^N, \quad t \geq 0, \quad (3)$$

$$\psi(0, x) = \psi_0(x). \quad (4)$$

For Cauchy problem (3)-(4), Ginibre and Velo [7] established the local existence in $H^1(\mathbb{R}^N)$. Glassey [8], Weinstein [9], and Zhang [10] proved that, for some initial data, the solutions of the Cauchy problem (3)-(4) blow up in finite time.

For the Cauchy problem (3)-(4), it is well known that there exists a minimum of L^2 norm for the initial data of blow-up solutions (see [9]). More precisely, let $Q(x)$ be the ground state, which is the unique, positive, radially symmetric solution (see [11]) of the semilinear elliptic equation

$$-\Delta u + u - |u|^{4/N}u = 0, \quad u \in H^1(\mathbb{R}^N). \quad (5)$$

Weinstein [9] proved that the solutions of the Cauchy problem (3)-(4) are globally defined if $\|\psi_0\|_{L^2} < \|Q\|_{L^2}$. On the other hand, for any $l \geq \|Q\|_{L^2}$, there exist blow-up solutions with $\|\psi_0\|_{L^2} = l$. Since then, much progress has been made on the blow-up rate and profile of the blow-up solutions of the Cauchy problem (3)-(4) (see [12-15]). In particular, based on the pseudoconformal invariance of (3) and the variational characterization of the ground, elaborate and interesting conclusions were established on the existence and profile of the minimal blow-up solution, which is the blow-up solution $\psi(t, x)$ such that $\|\psi_0\|_{L^2} = \|Q\|_{L^2}$ (see [13, 15, 16]). By using the pseudoconformal invariance of (3), Weinstein [15] constructed the explicit blow-up solution with critical mass ($\|\psi_0\|_{L^2} = \|Q\|_{L^2}$) for (3) in the form

$$(a + bt)^{-(N/2)} Q\left(\frac{x}{a + bt}\right) e^{i(b|x|^2)/4(a+bt)} e^{i(c+dt)/(a+bt)}, \quad (6)$$

where $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$, and $ab < 0$. Moreover, Weinstein proved that, for any minimal blow-up solution $\psi(t)$, the following holds:

$$\lim_{t \rightarrow T} \lambda(t)^{N/2} \psi(t, \lambda(t)(x + y(t))) = Q(x), \quad (7)$$

where T is the blow-up time and $y(t) \in \mathbb{R}^N$ and $\lambda(t) \in \mathbb{R}$ are some suitable functions.

Merle [13, 16] proved that $\psi(t, x)$ is a minimal blow-up solution of (3) if and only if there exist $\theta \in \mathbb{R}$, $\omega > 0$, $x_0 \in \mathbb{R}^N$, and $x_1 \in \mathbb{R}^N$ such that

$$\begin{aligned} \psi(t, x) = & \left(\frac{\omega}{T-t} \right)^{N/2} e^{i\theta + (i|x-x_1|^2/4(-T+t)) - (i\omega^2/(-T+t))} \\ & \times Q\left(\frac{\omega}{T-t} ((x-x_1) - (T-t)x_0) \right). \end{aligned} \quad (8)$$

For the Cauchy problem (1)-(2), local well-posedness in energy space was established in Cazenave [17]. Moreover, from the result of Carles [18] and Zhang [6, 19], it is known that $\phi(t)$ is globally defined if $\|\phi_0\|_{L^2} < \|Q\|_{L^2}$. In other words, $\|\phi_0\|_{L^2} \geq \|Q\|_{L^2}$ if $\phi(t)$ blows up in finite time.

Let $\phi(t)$ and $\psi(t)$ be the solutions of the Cauchy problems (1)-(2) and (3)-(4), respectively. Under the condition of $\phi_0(x) = \psi_0(x)$, Carles [18] established a formula, which reflects the relation between $\phi(t)$ and $\psi(t)$. According to the formula, Carles [18] established the following statements.

- (1) If $\phi(t)$ blows up at a finite time T_ϕ , then $T_\phi \leq \pi/2$.
- (2) If $\phi(t)$ blows up at $T_\phi < \pi/2$, $\psi(t)$ blows up at time $T_\psi < \infty$.
- (3) Conversely, $\psi(t)$ blows up at time $T_\psi < \infty$; then $\phi(t)$ blows up at $T_\phi < \pi/2$.
- (4) If $\phi(t)$ blows up at $T_\phi = \pi/2$, $\psi(t)$ exists globally ($T_\psi = \infty$).

Moreover, Carles studied the qualitative properties of minimal blow-up solutions $\phi(t)$ with $T_\phi < \pi/2$ (see [18, 20]). As for the minimal blow-up solutions with $T_\phi = \pi/2$, though the existence was established by the formula in [5], there is no further information on the qualitative properties obtained by the formula. Up to our knowledge, there is no result about the qualitative properties of the minimal blow-up solutions $\phi(t)$ of (1) with $T_\phi = \pi/2$.

The purpose of the present paper is to investigate the qualitative properties of the minimal blow-up solutions without any limit to the blow-up time. The formula presented in [18] is not used to carry out the objective. We follow the ideas of Merle [13, 16], as well as Weinstein [15], in which the profile and uniqueness of the minimal blow-up solutions for (3) were investigated. However, in contrast to (3), (1) loses the invariance of pseudoconformal invariance, which is very important in the arguments of [13, 15, 16]. Therefore, some appropriate modifications will be made in the argument of this work to reach our goal. In particular, we note that some techniques developed by Pang et al. [21] are adopted in this paper.

We state our main results.

Theorem 1. *There exist initial data ϕ_0 with $\|\phi_0\|_{L^2} = \|Q\|_{L^2}$ for which the solution of the Cauchy problem (1)-(2) blows up in a finite time.*

Theorem 2. *Let $\phi(t)$ be a blow-up solution of (1) with $\|\phi_0\|_{L^2} = \|Q\|_{L^2}$. Then there is $y_0 \in \mathbb{R}^N$ such that*

$$\phi(t, x) \longrightarrow \|Q\|_{L^2}^2 \delta_{y_0} \quad (9)$$

in the sense of distribution as $t \rightarrow T$.

Theorem 3. *There exists $C > 0$ such that*

$$\|\nabla \phi(t)\|_{L^2} \geq \frac{C}{T-t}, \quad \forall t \in [0, T). \quad (10)$$

Remark 4. For any blow-up solutions of (1), we know that $T \leq \pi/2$ (T is a blow-up time). When $T < \pi/2$, the formula presented in [18] is valid. For the minimal blow-up solutions with $T < \pi/2$, the conclusion of the above theorems can be found in [18]. However, there exist minimal blow-up solutions with $T = \pi/2$. For example, if the initial $\phi_0(x) = \psi_0(x) = Q(x)$, with $Q(x)$ being the solution of problem (5), the solution $\phi(t)$ of (1) will blow up at $T = \pi/2$, while the corresponding solution of (3) is a solitary wave $e^{it}Q(x)$. The minimal blow-up solutions with $T = \pi/2$ were sensible as pointed in [18].

In this paper, $L^q(\mathbb{R}^N)$, $\|\cdot\|_{L^q(\mathbb{R}^N)}$, and $\int_{\mathbb{R}^N} \cdot dx$ are denoted by L^q , $\|\cdot\|_{L^q}$, and $\int \cdot dx$, respectively. The various positive constants are also denoted by C .

This paper proceeds as follows. In Section 2, we establish some preliminaries. In Section 3, we give the proof of the existence and profile of the minimal blow-up solutions of (1) (Theorems 1 and 2). In Section 4, we derive the argument of the lower bound of the blow-up rate of the minimal blow-up solutions of (1) (Theorem 3).

2. Preliminaries

2.1. Local Wellposedness. The energy space of (1) was defined as

$$\Sigma := \{u \in H^1, |x|u \in L^2\}. \quad (11)$$

The inner product of the space Σ is defined as

$$\langle u, v \rangle := \int \nabla u \nabla \bar{v} + u \bar{v} + |x|^2 u \bar{v} dx. \quad (12)$$

The norm of Σ is denoted by $\|\cdot\|_\Sigma$. Moreover, we define an energy functional $\mathcal{E}$ on Σ by

$$\mathcal{E}(u) := \int |\nabla u|^2 + |x|^2 |u|^2 - \frac{1}{1+2/N} |u|^{2+4/N} dx. \quad (13)$$

From Cazenave [17], we have the local well-posedness for the Cauchy problem of (1) follows.

Proposition 5. *For any $\phi_0 \in \Sigma$, there exist $T > 0$ and a unique solution $\phi(t, x)$ of the Cauchy problem (1)-(2) in $C([0, T]; \Sigma)$*

such that either $T = \infty$ (global existence) or $T < \infty$ and $\lim_{t \rightarrow T} \|\phi(t)\|_{\Sigma} = \infty$ (blowup). Moreover, for any $t \in [0, T)$, it holds the conservation laws of mass

$$\|\phi(t)\|_{L^2} = \|\phi_0\|_{L^2} \quad (14)$$

and the energy

$$\mathcal{E}(\phi(t)) = \mathcal{E}(\phi_0). \quad (15)$$

2.2. Variational Characterization of the Ground State. Consider the equation

$$-\Delta u + \omega u - |u|^{4/N} u = 0, \quad u \in H^1(\mathbb{R}^N). \quad (16)$$

For (16), we set some notations such as $\mathcal{X}_\omega$ (the solution set), $\mathcal{G}_\omega$ (the ground solution set), and $\mathcal{S}$ as follows:

$$\begin{aligned} \mathcal{X}_\omega &= \{u \in H^1; u \neq 0, -\Delta u + \omega u - |u|^{4/N} u = 0\}, \\ \mathcal{G}_\omega &= \{u \in \mathcal{X}_\omega; S(u) \leq S(v), \forall v \in \mathcal{X}_\omega\}, \\ \mathcal{S} &= \bigcup_{\omega \in \mathbb{R}^+} \mathcal{G}_\omega, \end{aligned} \quad (17)$$

where $S(u) = \int (1/2)|\nabla u|^2 + (\omega/2)|u|^2 - (1/4N + 2)|u|^{2+(4/N)} dx$.

For any $u \in \mathcal{X}_\omega$, the following two identities hold true:

$$\begin{aligned} \int |\nabla u|^2 + \omega |u|^2 dx &= \int |u|^{2+(4/N)} dx, \\ \int (N-2)|\nabla u|^2 + N\omega |u|^2 dx &= \int \frac{N}{1+2/N} |u|^{2+(4/N)} dx \quad (\text{Pohozaev's identity}). \end{aligned} \quad (18)$$

The above two equalities imply

$$\mathcal{H}(u) = 0, \quad \forall u \in \mathcal{X}, \quad (19)$$

where

$$\mathcal{H}(u) := \int |\nabla u|^2 - \frac{1}{2/N+1} |u|^{2+(4/N)} dx. \quad (20)$$

Naturally, we get

$$u \in \mathcal{G}_\omega \iff \begin{cases} u \in \mathcal{X}_\omega, \\ \|u\|_{L^2} \leq \|v\|_{L^2}, \quad \forall v \in \mathcal{X}_\omega. \end{cases} \quad (21)$$

According to Cazenave [17], the set $\mathcal{G}_\omega$ can be described as

$$\mathcal{G}_\omega = \bigcup \{e^{i\theta} \varphi_\omega(\cdot - y); \theta \in \mathbb{R}, y \in \mathbb{R}^N\}, \quad (22)$$

where φ_ω is a positive, spherically symmetric, decreasing, and real valued function.

It is of importance that Kwong [11] proved the uniqueness for the solution $Q(x)$ of the problem

$$\begin{aligned} -\Delta u + u - |u|^{4/N} u &= 0, \quad u \in H^1, \\ u(x) &= u(|x|), \\ u(x) &> 0. \end{aligned} \quad (23)$$

Noticing the fact that $Q(x) = \varphi_\omega|_{\omega=1}$, it is easy to check that

$$\varphi_\omega = \omega^{N/4} Q(\omega^{1/2} x) \in \mathcal{G}_\omega, \quad \|\varphi_\omega\|_{L^2} = \|Q\|_{L^2}. \quad (24)$$

It follows from (21), (22), and (24) that

$$\begin{aligned} u \in \mathcal{G}_\omega &\iff \begin{cases} u \in \mathcal{X}_\omega, \\ \|u\|_{L^2} = \|Q\|_{L^2}, \end{cases} \\ \mathcal{G}_\omega &= \bigcup \{e^{i\theta} \varphi_\omega(\cdot - y); \theta \in \mathbb{R}, y \in \mathbb{R}^N\}, \\ &= \bigcup \{e^{i\theta} \omega^{N/4} Q(\omega^{1/2}(\cdot - y)); \theta \in \mathbb{R}, y \in \mathbb{R}^N\}. \end{aligned} \quad (25)$$

With functional $\mathcal{H}$ defined by (20), we now introduce the following constrained minimization problem

$$\mathcal{J}(\|Q\|_{L^2}) \equiv \inf \{ \mathcal{H}(f) \mid f \in H^1, \|f\|_{L^2} = \|Q\|_{L^2} \}. \quad (27)$$

Now, we claim that

$$u \in \mathcal{G} \iff u$$

is a solution to the minimization problem (27). (28)

In fact, $(N/(N+2))\|Q\|_{L^2}^{4/N}$ is the minimum of the functional (see Kwong [11] or Weinstein [9])

$$I(\psi) = \frac{\|\nabla \psi\|_{L^2}^2 \|\psi\|_{L^2}^{4/N}}{\|\psi\|_{L^{2+4/N}}^{2+4/N}}, \quad \psi \in H^1, \quad (29)$$

which derives the Gagliardo-Nirenberg inequality

$$\|\psi\|_{L^{2+4/N}}^{2+4/N} \leq \frac{N+2}{N} \left(\frac{\|\psi\|_{L^2}}{\|Q\|_{L^2}} \right)^{4/N} \|\nabla \psi\|_{L^2}^2. \quad (30)$$

The inequality (30) implies the following lemma on the functional $\mathcal{H}$.

Lemma 6 (see Weinstein [9]). *For any $f \in H^1$, one has*

$$\left[1 - \left(\frac{\|f\|_{L^2}}{\|Q\|_{L^2}} \right)^{4/N} \right] \|\nabla f\|_{L^2}^2 \leq \mathcal{H}(f). \quad (31)$$

Lemma 6 implies that

$$\mathcal{H}(f) \geq 0, \quad \text{if } \|f\|_{L^2} \leq \|Q\|_{L^2}. \quad (32)$$

It follows from (19), (27), and (32) that

$$\mathcal{J}(\|Q\|_{L^2}) = 0. \quad (33)$$

Hence, from (19) and (25), it holds that

$$\begin{aligned} u \in \mathcal{G} &\implies u \\ &\text{is a solution to the minimization problem (27).} \end{aligned} \quad (34)$$

On the other hand, if u is a minimizer of the variational problem of (27), it solves the Euler-Lagrange equation (16). So $u \in \mathcal{X}_\omega$ for some $\omega > 0$, and by (27) and (25), we know $u \in \mathcal{G}_\omega \subset \mathcal{G}$. This implies that

$$\begin{aligned} u \in \mathcal{G} &\Leftarrow u \\ &\text{is a solution to the minimization problem (27).} \end{aligned} \quad (35)$$

Hence (28) holds true.

Putting together (22), (25), and (28), we summarize the variational characterization.

Proposition 7. *Each of the following three statements is equivalent:*

- (i) $u \in \bigcup_{\omega \in \mathbb{R}^+} \mathcal{G}_\omega$,
- (ii) u is a solution to the minimizing problem $\min \{\mathcal{H}(u), \|u\|_{L^2} = \|Q\|_{L^2}\}$,
- (iii) $u = e^{i\theta} \omega^{N/4} Q(\omega^{1/2}(x - x_0))$, for some $\theta \in \mathbb{R}$, $\omega \in \mathbb{R}^+$, and $x_0 \in \mathbb{R}^N$.

2.3. Lemmas

Lemma 8 (see Zhang [6]). *Let $\phi_0 \neq 0$, the initial datum of Cauchy problem (1)-(2), satisfy*

$$\mathcal{E}(\phi_0) \leq \int |x|^2 |\phi_0|^2 dx; \quad (36)$$

then $\phi(t)$ blows up in a finite time.

Consider the constrained minimization problem

$$I(\alpha) \equiv \min \{\mathcal{H}(f) \mid f \in H^1, \|f\|_{L^2} = \alpha\}. \quad (37)$$

For $I(\alpha)$, we cite a lemma in [15].

Lemma 9 (see Weinstein [15]). *(a) Consider $I(\alpha) = 0$ or $I(\alpha) = -\infty$.*

(b) Let $\alpha < \|Q\|_{L^2}$ and u_n be a minimizing sequence; then it holds that $I(\alpha) = 0$ and $u_n \rightharpoonup 0$ weakly in H^1 .

Now, we recall some lemmas on the compactness.

Lemma 10 (see Brezis and Lieb [22]). *Let $f \in L^1_{\text{loc}}$, $\|\nabla f\|_{L^2} \leq C$, and $\mu(|f| > \varepsilon) \geq \delta > 0$. Then there exists a shift $T_y f(x) = f(x + y)$ such that, for some constant $\alpha = \alpha(C, \delta, \varepsilon)$,*

$$\mu\left(B(0, 1) \cap \left[T_y g > \frac{\varepsilon}{2}\right]\right) > \delta. \quad (38)$$

Lemma 11 (see Lieb [23]). *Let f_j be a uniformly bounded sequence of functions in $W^{1,p}$ with $1 < p < \infty$. Assume further that there are positive constant C and η satisfying $\mu(|f_j| > \eta) \geq C$. Then there exists a sequence $y_j \in \mathbb{R}^N$ such that*

$$f_j(\cdot + y_j) \rightharpoonup f \neq 0 \quad \text{weakly in } W^{1,p}. \quad (39)$$

Lemma 12. *Let θ be a real-valued function on $\mathbb{R}^N$ and $v \in H^1(\mathbb{R}^N)$ with $\|v\|_{L^2} \leq \|Q\|_{L^2}$. Then*

$$\left| \int \bar{v}(x) \nabla \theta(x) dx \right| \leq \left(2\mathcal{H}(v) \int |v(x)|^2 |\nabla \theta(x)|^2 dx \right)^{1/2}. \quad (40)$$

Proof. It follows from (30) and $\|v\|_{L^2} \leq \|Q\|_{L^2}$ that

$$\mathcal{H}(e^{i\alpha\theta} v) \geq 0 \quad (41)$$

for all real numbers α . On the other hand, it has

$$\mathcal{H}(e^{i\alpha\theta} v) = \alpha^2 \int |v|^2 |\nabla \theta|^2 dx - \alpha \int \Im(v \nabla \bar{v}) \nabla \theta dx + \mathcal{H}(v). \quad (42)$$

Thus the discriminant of the equation in α must be negative or null and the desired inequality follows. $\square$

Lemma 13. *There is a constant c_0 such that*

$$\int |x|^2 |\phi(t, x)|^2 dx \leq c_0. \quad (43)$$

Proof. Setting $J(t) = \int |x|^2 |\phi(t, x)|^2 dx$, we have

$$J'(t) = 2\Im \int \bar{\phi} \nabla \phi dx, \quad (44)$$

$$J''(t) = 4\mathcal{E}(\phi) - 4J(t).$$

It follows that

$$J(t) = (J(0) - \mathcal{E}(0)) \cos t + J'(0) \sin t + \mathcal{E}(0), \quad (45)$$

which implies the conclusion. $\square$

Lemma 14 (see [16, page 433]). *Let $u_n \in H^1$, $c_0 > 0$, and $R_0 > 0$, for arbitrary n , satisfy*

$$\begin{aligned} \mathcal{H}(u_n) &\leq c_0, \\ \|u_n\|_{L^2} &\leq \|Q\|_{L^2}, \\ \|\nabla u_n\|_{L^2} &\longrightarrow \infty, \end{aligned} \quad (H)$$

$$\int_{|x| > r_0} |u_n|^2 dx \leq \varepsilon(n),$$

where $\varepsilon(n) > 0$ depends only on n . Then, it holds that

$$\int_{|x| > 4r_0} |\nabla u_n|^2 dx \leq A, \quad (46)$$

with $A = A(r_0, c_0 > 0)$.

3. Profile of the Minimal Blow-Up Solution

Now we prove the existence of the minimal blow-up solutions.

Proof of Theorem 1. Setting $\phi_0 = \phi(c, \lambda) = c\lambda^{N/2}Q(\lambda x)$ with λ being arbitrary positive real number and c being complex number satisfying $|c| = 1$, then

$$\|\phi_0\|_{L^2} = \|Q\|_{L^2}. \quad (47)$$

From (15) and (19), the corresponding energy is

$$\begin{aligned} \mathcal{E}(\phi_0) &= (1 - |c|^{4/N})|c|^2\lambda^2 \int |\nabla Q|^2 dx \\ &\quad + \int |x|^2 |\phi_0|^2 dx = \int |x|^2 |\phi_0|^2 dx. \end{aligned} \quad (48)$$

Thus Lemma 8 infers that $\phi(t, x)$ blows up in a finite time. $\square$

Employing the concentration compactness lemma, we can prove the following proposition which is crucial to the study of the blow-up profile (Theorem 2).

Proposition 15. *Let $\phi(t) \in C([0, T], \Sigma)$ be a blow-up solution of the Cauchy problem (1)-(2) and T is the blow-up time. Set $\lambda(t) = \|\nabla Q\|_{L^2} / \|\nabla \phi(t)\|_{L^2}$ and $(S_\lambda \phi)(x, t) = \lambda^{N/2} \phi(\lambda x, t)$. If*

$$\|\phi_0\|_{L^2} = \|Q\|_{L^2}, \quad (49)$$

it holds that

$$S_{\lambda(t)} \phi(\cdot + y(t), t) e^{iy(t)} \longrightarrow Q(\cdot) \quad \text{in } H^1, \text{ as } t \longrightarrow T \quad (50)$$

with $y(t) \in \mathbb{R}^N$ and $\gamma(t) \in \mathbb{R}$.

Proof. Let $t_k \rightarrow T$. We choose $\lambda_k = \lambda(t_k)$ to satisfy

$$\|\nabla S_{\lambda_k} \phi(\cdot + y_k, t_k)\|_{L^2} = \lambda_k \|\nabla \phi(\cdot + y_k, t_k)\|_{L^2} = \|\nabla Q\|_{L^2}. \quad (51)$$

Setting $\phi_k \equiv S_{\lambda_k} \phi(\cdot + y_k, t_k)$, noticing that $\|\phi(t_k)\|_{L^2}$ tends to ∞ as $t_k \rightarrow T$, $\lambda_k \rightarrow 0$, and

$$\|\phi_k\|_{L^2} = \|\phi(t_k)\|_{L^2} = \|\phi_0\|_{L^2}, \quad (52)$$

we know that ϕ_k is uniformly bounded in H^1 and there is a weakly convergent subsequence ϕ_{k_j} such that

$$\begin{aligned} \phi_{k_j} &\rightharpoonup \phi \quad \text{in } L^2, \\ \phi_{k_j} &\rightharpoonup \phi \quad \text{in } H^1. \end{aligned} \quad (53)$$

We note that

$$\begin{aligned} \mathcal{H}(\phi_{k_j}) &= \lambda_{k_j}^2 \mathcal{H}(\phi(t_{k_j})) \leq \lambda_{k_j}^2 \mathcal{E}(\phi_0) \longrightarrow 0, \\ j &\longrightarrow \infty. \end{aligned} \quad (54)$$

Since we have assumed $\|\phi_0\|_{L^2} = \|Q\|_{L^2}$, by (52), (54), and (31), we know that ϕ_k is a minimizing sequence for the variational problem (27).

Next, we will prove that the minimizing sequence ϕ_k has a subsequence ϕ_{k_j} and a family y_j such that $\phi_{k_j}(\cdot - y_k)$ has a strong limit in H^1 . To see this, we need to make use of the concentration-compactness lemma (Lions [24]) which means that ϕ_{k_j} has one of three properties: vanishing, dichotomy, and compactness.

Vanishing. For every $M < \infty$, one has

$$\lim_{j \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{y_j + B_r(M)} |\phi_{k_j}(x)|^2 dx = 0. \quad (55)$$

Dichotomy. There exist a constant $\alpha \in (0, \|Q\|_{L^2})$ and sequences ψ_j^1 and ψ_j^2 , bounded in H^1 , such that, for all $\varepsilon > 0$, there exists $j_0 > 0$ such that for $j > j_0$

$$\begin{aligned} \|\psi_j^1\|_{L^2} - \alpha &\leq \varepsilon, \quad \|\psi_j^2\|_{L^2} - (\|Q\|_{L^2} - \alpha) \leq \varepsilon, \\ \|\phi_{k_j} - \psi_j^1 - \psi_j^2\|_{H^1} &\leq \varepsilon, \\ \|\phi_{k_j} - \psi_j^1 - \psi_j^2\|_{L^p} &\leq \varepsilon \quad \text{for } 2 \leq p < \frac{2N}{N-2}, \\ \text{distance}(\text{supp } \psi_j^1, \text{supp } \psi_j^2) &\longrightarrow \infty. \end{aligned} \quad (56)$$

Compactness. There exists y_j in $\mathbb{R}^N$. For any $\varepsilon > 0$, we can find $M < \infty$ such that

$$\int_{y_j + B_r(M)} |\phi_{k_j}|^2 dx \geq \|Q\|_{L^2}^2 - \varepsilon. \quad (57)$$

Now, we exclude the cases of vanishing and dichotomy.

Exclusion of Vanishing. By (52), (51), and (54) there are $C_1 > 0$ and $C_2 > 0$ such that

$$\|\phi_k\|_{L^2}^2 \leq C_1, \quad \|\phi_k\|_{L^{2+4/N}}^{2+4/N} \geq C_2 > 0. \quad (58)$$

By the boundness of $\|\phi_k\|_{H^1}$ and the Sobolev inequality, there exist $\gamma > 2 + 4/N$ and $C_3 > 0$ such that

$$\|\phi_k\|_{L^\gamma}^\gamma \leq C_3. \quad (59)$$

Now, we show the existence of positive constants ε and δ such that

$$\mu(|\phi_k| > \varepsilon) \geq \delta > 0. \quad (60)$$

Indeed, from (58) and (59), for sufficiently small $\varepsilon > 0$, we get

$$\begin{aligned}
 c_2 &\leq \int |\phi_k|^{2+4/N} dx \\
 &= \int_{\{|\phi_k| < \varepsilon\}} |\phi_k|^{2+4/N} dx \\
 &\quad + \int_{\{\varepsilon < |\phi_k| < (1/\varepsilon)\}} |\phi_k|^{2+4/N} dx + \int_{\{|\phi_k| > (1/\varepsilon)\}} |\phi_k|^{2+4/N} dx \\
 &\leq \frac{C_2}{4C_1} \int_{\{|\phi_k| < \varepsilon\}} |\phi_k|^2 dx \\
 &\quad + \int_{\{\varepsilon < |\phi_k| < (1/\varepsilon)\}} |\phi_k|^{2+4/N} dx + \frac{C_2}{4C_3} \int_{\{|\phi_k| > (1/\varepsilon)\}} |\phi_k|^\gamma dx \\
 &\leq \frac{C_2}{4C_1} \|\phi_k\|_{L^2}^2 + \int_{\{\varepsilon < |\phi_k| < (1/\varepsilon)\}} |\phi_k|^{2+4/N} dx \\
 &\quad + \frac{C_2}{4C_3} \|\phi_k\|_{L^\gamma}^\gamma dx \\
 &\leq \frac{C_2}{2} + \mu (|\phi_k| > \varepsilon) \left(\frac{1}{\varepsilon}\right)^{2+4/N}.
 \end{aligned} \tag{61}$$

Thus we know that (60) with $\delta = (C_2/2)\varepsilon^{2+4/N}$ is valid. From (60) and Lemma 10, there exist α and y_k satisfying

$$\mu \left(\{|x| \leq 1\} \cap \{|\phi_k(\cdot + y_k)|\} > \frac{\varepsilon}{2} \right) > \delta. \tag{62}$$

Thus,

$$\int_{|x| \leq 1} |\phi_{k_j}(\cdot + y_j)|^2 dx \geq \left(\frac{\varepsilon}{2}\right)^2 \delta, \tag{63}$$

which excludes the occurrence of vanishing.

Exclusion of Dichotomy. Suppose by contradiction that dichotomy occurs. Then, by the same argument as that in the case of *vanishing* we can get

$$0 < v < \mu \left\{ \theta < |\psi_j^1| \right\}, \tag{64}$$

where θ and v are two constants and ψ_j^1 is bounded in H^1 . Hence, by Lemma 11, there are a subsequence $\psi_{j_r}^1$ and a sequence y_r such that

$$\psi_{j_r}^1(\cdot + y_r) \rightharpoonup \psi \neq 0 \quad \text{in } H^1. \tag{65}$$

Using (56) gives rise to

$$\begin{aligned}
 0 &= I(\|Q\|_{L^2}) \geq \liminf_{r \rightarrow \infty} \mathcal{H}(\psi_{j_r}^1) + \liminf_{r \rightarrow \infty} \mathcal{H}(\psi_{j_r}^2) \\
 &= \liminf_{r \rightarrow \infty} \mathcal{H}(\psi_{j_r}^1).
 \end{aligned} \tag{66}$$

On the other hand, the fact $\|\psi_{j_r}^1\|_{L^2} < \|Q\|_{L^2}$ implies with Lemma 6 that

$$\liminf_{r \rightarrow \infty} \mathcal{H}(\psi_{j_r}^1) \geq 0. \tag{67}$$

Thus, for any fixed n^* , it has

$$\begin{aligned}
 0 &= I(\|Q\|_{L^2}) \geq \liminf_{r \rightarrow \infty} \mathcal{H}(\psi_{j_r}^1) = \sup_{n^*} \inf_{r \geq n^*} \mathcal{H}(\psi_{j_r}^1) \\
 &\geq \inf_{r \geq n^*} \mathcal{H}(\psi_{j_r}^1).
 \end{aligned} \tag{68}$$

We can then extract a minimizing subsequence, which we rename it by $\psi_{j_r}^1$; that is, $\lim_{r \rightarrow \infty} \mathcal{H}(\psi_{j_r}^1) = 0$. Using Lemma 9 yields

$$\psi_{j_r}^1 \longrightarrow 0, \tag{69}$$

which is impossible from (65).

Occurrence of Compactness. It follows from the previous arguments that compactness occurs. By (57), we get

$$\|Q\|_{L^2}^2 - \varepsilon \leq \int_{y_j + B(M)} |\phi_{k_j}|^2 dx \leq \int |\phi_{k_j}|^2 dx \leq \|Q\|_{L^2}^2. \tag{70}$$

For $\phi_{k_j}(\cdot + y_j)$ being bounded in $H^1(\mathbb{R}^N)$, there exist $\phi \in H^1(\mathbb{R}^N)$ and a subsequence, which we again label it by ϕ_{k_j} , such that

$$\phi_{k_j}(\cdot + y_j) \rightharpoonup \phi \quad \text{in } H^1. \tag{71}$$

Given $M > 0$, the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^2(\{|x| \leq r\})$ is compact and

$$\int_{|x| \leq r} |\phi|^2 dx = \lim_{j \rightarrow \infty} \int_{x_m + B(r)} |\phi_{k_j}|^2 dx. \tag{72}$$

Making use of (70) derives

$$\int_{\mathbb{R}^N} |\phi|^2 dx \geq \|Q\|_{L^2}^2 - \varepsilon \tag{73}$$

for any $\varepsilon > 0$. Hence, it holds that

$$\int_{\mathbb{R}^N} |\phi|^2 dx = \|Q\|_{L^2}^2. \tag{74}$$

It follows that

$$\phi_{k_j}(\cdot + y_j) \longrightarrow \phi \quad \text{in } L^2, \tag{75}$$

which implies with the Gagliardo-Nirenberg inequality (30) that

$$\phi_{k_j}(\cdot + y_j) \longrightarrow \phi \quad \text{in } L^{2+4/N}. \tag{76}$$

To show $\phi_{k_j} \rightarrow \phi$ in H^1 , we only need to show that $\|\nabla \phi\|_{L^2} = \|\nabla Q\|_{L^2}$.

From (51) and (54), we know that

$$\begin{aligned}
 0 &= \lim_{t \rightarrow T} \mathcal{H}(\phi_{\phi_{k_j}}) \\
 &= \|\nabla Q\|_{L^2} - \frac{1}{2/N + 1} \lim_{t \rightarrow T} \int |\phi_{k_j}|^{4/N+2} dx \\
 &= \|\nabla Q\|_{L^2} - \frac{1}{2/N + 1} \lim_{t \rightarrow T} \int |\phi|^{4/N+2} dx.
 \end{aligned} \tag{77}$$

Hence, $\|\nabla\phi\|_{L^2} < \|\nabla Q\|_{L^2}$ derives $\mathcal{E}(\phi) < 0$. This contradicts Lemma 6 and the fact $\phi \neq 0$.

Since ϕ solves the minimizing problem (27), it satisfies the Euler-Lagrange equation (16). Noticing the fact $\|\nabla|\phi|\|_{L^2} \leq \|\nabla\phi\|_{L^2}$, we infer that $|\phi|$ is also a solution to problem (27). Thus it is a nonnegative solution of (16). It follows from $\|\phi\|_{L^2} = \|Q\|_{L^2}$, $\|\nabla\phi\|_{L^2} = \|\nabla Q\|_{L^2}$, and Proposition 7 that

$$\phi = Q(\cdot + y_j) e^{i\gamma} \quad (78)$$

for some $y \in \mathbb{R}^N$ and $\gamma \in \mathbb{R}$. By redefining the sequence y_j , we can set $\gamma = 0$. $\square$

Proof of Theorem 2. It follows from Proposition 15 that

$$\lambda^N(t) |\phi(t, \lambda(t)(x + x(t)))|^2 \longrightarrow |Q(x)|^2 \quad \text{in } L^1 \text{ as } t \longrightarrow T, \quad (79)$$

$$|\phi(t, x + x(t))|^2 \longrightarrow \|Q\|_{L^2}^2 \delta_{x=0} \quad \text{as } t \longrightarrow T. \quad (80)$$

Using Lemma 13 derives that

$$\limsup_{t \rightarrow T} |x(t)| \leq \frac{\sqrt{c_0}}{\|Q\|_{L^2}}. \quad (81)$$

Hence we have a positive constant r_0 such that

$$\forall t \in [0, T), \quad |x(t)| \leq r_0. \quad (82)$$

$$\begin{aligned} & \int_{B(0,r)} |\phi(t, x)|^2 x dx \\ &= \int_{B(0,r)} |\phi(t, x)|^2 (x - x(t)) dx \\ &+ \int_{B(0,r)} |\phi(t, x)|^2 x(t) dx \\ &= \int_{B(-x(t), r)} |\phi(t, y + x(t))|^2 y dy \\ &+ \int_{B(-x(t), r)} |\phi(t, y + x(t))|^2 x(t) dy. \end{aligned} \quad (83)$$

From (82), for arbitrary $r > r_0$, there is a $\delta > 0$ such that $B(0, \delta) \subset B(-x(t), r)$. The formula (80) implies that

$$\int_{B(0,r)} |\phi(t, x)|^2 x dx - \int |Q(x)|^2 x(t) dx = 0. \quad (84)$$

On the other hand, Lemma 13 implies that

$$\int_{|x|>r} |\phi(t, x)|^2 x dx \leq \frac{c_0}{r}. \quad (85)$$

Thus

$$\lim_{t \rightarrow T} \left\{ \int |\phi(t, x)|^2 x dx - \int |Q(x)|^2 x(t) dx \right\} = 0. \quad (86)$$

By Lemma 12, we obtain

$$\begin{aligned} & \frac{d}{dt} \left| \int |\phi(t, x)|^2 x dx \right| \\ &= \left| 2 \Im \int \bar{\phi}(t, x) \nabla \phi(t, x) dx \right| \\ &= 2 \Im \sum_{j=1}^N \left| \int \bar{\phi}(t, x) \nabla \phi(t, x) \cdot \nabla \theta_j(x) dx \right| \\ &\leq 2 \sum_{j=1}^N \left(2 \mathcal{H}(\phi(t)) \int |\phi(t, x)|^2 |\nabla \theta_j(x)|^2 dx \right)^{1/2} \leq C, \end{aligned} \quad (87)$$

where $\theta_j(x) = x_j$. Hence there exists $x_1 \in \mathbb{R}^N$ such that

$$\lim_{t \rightarrow T} \int |\phi(t, x)|^2 x dx = - \left(\int |Q(x)|^2 dx \right) x_1. \quad (88)$$

Combining (86) with (88), we know that $x(t) \rightarrow -x_1$ as $t \rightarrow T$ and we have

$$|u(t, x)| \longrightarrow \|Q\|_{L^2}^2 \delta_{x=-x_1}. \quad (89)$$

$\square$

4. Blow-Up Rate

To establish the lower bound of the blow-up rate, we use the following proposition.

Proposition 16. *Letting y_0 be the blow-up point determined in Theorem 2, it has*

$$\lim_{t \rightarrow T} \int |x - y_0|^2 |\phi(t, x)|^2 dx = 0. \quad (90)$$

Proof. Let us define a positive function $h(x) \in C^1(\mathbb{R}^N)$ such that

$$h(x) = h(|x|) = \begin{cases} = 0, & |x| < 1, \\ > 0, & 1 < |x| < 2, \\ = \frac{|x|^2}{4}, & |x| > 2, \end{cases} \quad (91)$$

and $h_A(x) = A^2 h(x/A)$ for $A > 0$ and it is valid that

$$|\nabla h_A(x)|^2 \leq C h_A(x), \quad \forall x \in \mathbb{R}^N. \quad (92)$$

Carrying out direct computation and using Hölder's inequality, we have

$$\begin{aligned}
& \left| \frac{d}{dt} \int |\phi(t, x)|^2 h_A(x - y_0) dx \right| \\
&= \left| 2\Im \sum_{j=1}^N \int \bar{\phi}(t, x) \nabla \phi(t, x) \cdot \nabla h_A(x - y_0) dx \right| \\
&\leq C \left(\int_{|x-y_0| \geq A} |\nabla \phi(t, x)|^2 dx \right)^{1/2} \\
&\quad \times \left(\int |\phi(t, x)|^2 \nabla h_A(x - y_0) dx \right)^{1/2} \\
&\leq C \left(\int_{|x-y_0| \geq A} |\nabla \phi(t, x)|^2 dx \right)^{1/2} \\
&\quad \times \left(\int |\phi(t, x)|^2 h_A(x - y_0) dx \right)^{1/2},
\end{aligned} \tag{93}$$

which implies

$$\begin{aligned}
& \left| \frac{d}{dt} \left(\int |\phi(t, x)|^2 h_A(x - y_0) dx \right)^{1/2} \right| \\
&\leq C \left(\int_{|x-y_0| \geq A} |\nabla \phi(t, x)|^2 dx \right)^{1/2}.
\end{aligned} \tag{94}$$

Integrating on both sides gives rise to

$$\begin{aligned}
& \sup_{t \in [0, T]} \left(\int |\phi(t, x)|^2 h_A(x - y_0) dx \right)^{1/2} \\
&\leq \left(\int |\phi_0(x)|^2 h_A(x - y_0) dx \right)^{1/2} \\
&\quad + C \int_0^T \left(\int_{|x-y_0| \geq A} |\nabla \phi(s, x)|^2 dx \right)^{1/2} ds.
\end{aligned} \tag{95}$$

From the fact $\phi_0 \in \Sigma$, we have

$$\begin{aligned}
& \sup_{t \in [0, T]} \left(\int |\phi(t, x)|^2 h_A(x - y_0) dx \right)^{1/2} \\
&\leq \varepsilon(A) + C \int_0^T \left(\int_{|x-y_0| \geq A} |\nabla \phi(s, x)|^2 dx \right)^{1/2} ds.
\end{aligned} \tag{96}$$

By the virtue of Lemma 14 and Proposition 16, there exist A_1 and $C_2 > 0$ such that

$$\int_{|x-y_0| \geq A_1} (|\nabla \phi(s, x)|^2 dx)^{1/2} ds \leq C_2, \quad \forall s \in [0, T]. \tag{97}$$

Using the dominated convergence theorem, we infer that

$$\lim_{A \rightarrow \infty} \int_0^T \left(\int_{|x-y_0| \geq A} |\nabla \phi(s, x)|^2 dx \right)^{1/2} ds = 0. \tag{98}$$

Thus, it holds that

$$\lim_{A \rightarrow \infty} \sup_{t \in [0, T]} \left(\int_{|x-y_0| \geq A} |\phi(t, x)|^2 |x - y_0|^2 dx \right) = 0, \tag{99}$$

which implies that there is $a_\varepsilon > 0$ such that, for $\forall t \in [0, T]$,

$$\int_{|x-y_0| \geq a_\varepsilon} |x - y_0|^2 |\phi(t, x)|^2 dx \leq \frac{\varepsilon}{2}. \tag{100}$$

The identity $\|\phi(t)\|_{L^2} = \|\phi_0\|_{L^2} = \|Q\|_{L^2}$ shows that

$$\begin{aligned}
& \int_{|x-y_0| \leq b_\varepsilon} |x - y_0|^2 |\phi(t, x)|^2 dx \leq b_\varepsilon^2 \|Q\|_{L^2}^2 \\
&\leq \frac{\varepsilon}{2}, \quad \text{for } b_\varepsilon^2 = \frac{\varepsilon}{2\|Q\|_{L^2}^2}.
\end{aligned} \tag{101}$$

In addition, we have

$$\begin{aligned}
& \int_{b_\varepsilon \leq |x-y_0| \leq a_\varepsilon} |x - y_0|^2 |\phi(t, x)|^2 dx \\
&\leq a_\varepsilon^2 \int_{b_\varepsilon \leq |x-y_0| \leq a_\varepsilon} |\phi(t, x)|^2 dx.
\end{aligned} \tag{102}$$

Using Theorem 2 yields

$$\lim_{t \rightarrow T} \int_{b_\varepsilon \leq |x-y_0| \leq a_\varepsilon} |x - y_0|^2 |\phi(t, x)|^2 dx = 0. \tag{103}$$

In conclusion, for all $\varepsilon > 0$, we have shown that

$$\lim_{t \rightarrow T} \int |x - y_0|^2 |\phi(t, x)|^2 dx \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon. \tag{104}$$

□

Now, we establish the lower bound of the blow-up rate.

Proof of Theorem 3. Simple calculation yields

$$\begin{aligned}
& \frac{d}{dt} \int |x - y_0|^2 |\phi(t, x)|^2 dx \\
&= 4\Im \int (x - y_0) \phi(t, x) \nabla \bar{\phi}(t, x).
\end{aligned} \tag{105}$$

Therefore, the inequality (40) in the case $\theta(x) = |x - y_0|^2$ implies that

$$\left| \frac{d}{dt} \left(\int |x - y_0|^2 |\phi(t, x)|^2 dx \right)^{1/2} \right| \leq C. \tag{106}$$

Integrating from t to T , by Proposition 16, we obtain

$$\left| \left(\int |x - y_0|^2 |\phi(t, x)|^2 dx \right)^{1/2} \right| \leq C(T - t). \tag{107}$$

Combining the above inequality and the following inequality

$$\begin{aligned}
& \left(\int |\phi(t, x)|^2 dx \right)^2 \\
&\leq \left(\int |x - y_0|^2 |\phi(t, x)|^2 dx \right) \left(\int |\nabla \phi(t, x)|^2 dx \right),
\end{aligned} \tag{108}$$

we get the result

$$\|\nabla\phi(t)\|_{L^2} \geq \frac{\|Q\|_{L^2}}{C(T-t)}. \quad (109)$$

□

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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