Research Article

The Problem of Image Recovery by the Metric Projections in Banach Spaces

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Received 24 September 2012; Accepted 28 December 2012

1. Introduction

Let $C_1, C_2, \ldots, C_r$ be nonempty closed convex subsets of a real Hilbert space $H$ such that $\bigcap_{i=1}^{r} C_i \neq \emptyset$. Then, the problem of image recovery may be stated as follows: the original unknown image $z$ is known a priori to belong to the intersection of $\{C_i\}_{i=1}^{r}$; given only the metric projections $P_{C_i}$ of $H$ onto $C_i$ for $i = 1, 2, \ldots, r$, recover $z$ by an iterative scheme. Such a problem is connected with the convex feasibility problem and has been investigated by a large number of researchers.

Bregman [1] considered a sequence $\{x_n\}$ generated by cyclic projections, that is, $x_0 = x \in H$, $x_1 = P_{C_1}x$, $x_2 = P_{C_2}x_1$, $x_3 = P_{C_2}x_2$, $\ldots$, $x_r = P_{C_{r-1}}x_{r-1}$, $x_{r+1} = P_{C_r}x_r$, $x_{r+2} = P_{C_2}x_{r+1}$, $\ldots$. It was proved that $\{x_n\}$ converges weakly to an element of $\bigcap_{i=1}^{r} C_i$ for an arbitrary initial point $x \in H$.

Crombez [2] proposed a sequence $\{y_n\}$ generated by $y_0 = y \in H$, $y_{n+1} = \alpha_0 y_n + \sum_{i=1}^{r} \alpha_i (y_n + \lambda_i (P_{C_i} y_n - y_n))$ for $n = 0, 1, 2, \ldots$, where $0 < \alpha_i < 1$ for all $i = 0, 1, 2, \ldots, r$ with $\sum_{i=0}^{r} \alpha_i = 1$ and $0 < \lambda_i < 2$ for every $i = 1, 2, \ldots, r$. It was proved that $\{y_n\}$ converges weakly to an element of $\bigcap_{i=1}^{r} C_i$ for an arbitrary initial point $y \in H$.

Later, Kitahara and Takahashi [3] and Takahashi and Tamura [4] dealt with the problem of image recovery by convex combinations of nonexpansive retractions in a uniformly convex Banach space $E$. Alber [5] took up the problem of image recovery by the products of generalized projections in a uniformly convex and uniformly smooth Banach space $E$ whose duality mapping is weakly sequentially continuous (see also [6, 7]). On the other hand, using the hybrid projection method proposed by Haugazeau [8] (see also [9–11] and references therein) and the shrinking projection method proposed by Takahashi et al. [12] (see also [13]), Nakajo et al. [14] and Kimura et al. [15] considered this problem by the metric projections and proved convergence of the iterative sequence to a common point of countable nonempty closed convex subsets in a uniformly convex and smooth Banach space $E$ and in a strictly convex, smooth, and reflexive Banach space $E$ having the Kadec-Klee property, respectively. Kohsaka and Takahashi [16] took up this problem by the generalized projections and obtained the strong convergence to a common point of a countable family of nonempty closed convex subsets in a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable (see also [17, 18]). Although these results guarantee the strong convergence, they need to use metric or generalized projections onto different subsets for each step, which are not given in advance.

In this paper, we consider this problem by the metric projections, which are one of the most familiar projections to deal with. The advantage of our results is that we use projections onto the given family of subsets only, to generate
the iterative scheme. Our convergence theorems extend the results of [1, 2] to a Banach space \( E \), and they deduce the weak convergence to a common point of a countable family of nonempty closed convex subsets of \( E \).

There are a number of results dealing with the image recovery problem from the aspects of engineering using nonlinear functional analysis (see, e.g., [19]). Comparing with these researches, we may say that our approach is more abstract and theoretical; we adopt a general Banach space with several conditions for an underlying space, and therefore, the technique of the proofs can be applied to various mathematical results related to nonlinear problems defined on Banach spaces.

2. Preliminaries

Throughout this paper, let \( \mathbb{N} \) be the set of all positive integers, \( \mathbb{R} \) the set of all real numbers, \( E \) a real Banach space with norm \( \| \cdot \| \), and \( E^* \) the dual of \( E \). For \( x \in E \) and \( x^* \in E^* \), we denote by \( \langle x, x^* \rangle \) the value of \( x^* \) at \( x \). We write \( x_n \rightharpoonup x \) to indicate that a sequence \( \{x_n\} \) converges strongly to \( x \). Similarly, \( x_n \rightharpoonup x \) and \( x_n \rightarrow x \) will symbolize weak and weak\(^*\) convergence, respectively. We define the modulus \( \delta_E \) of convexity of \( E \) as follows: \( \delta_E \) is a function of \([0, 2]\) into \([0, 1]\) such that

\[
\delta_E(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : x, y \in E, \|x\| = 1, \|y\| = 1, \|x - y\| \geq \varepsilon \right\},
\]

for every \( \varepsilon \in [0, 2] \). \( E \) is called uniformly convex if \( \delta_E(\varepsilon) > 0 \) for each \( \varepsilon > 0 \). Let \( p > 1 \). \( E \) is said to be \( p \)-uniformly convex if there exists a constant \( c > 0 \) such that \( \delta_E(\varepsilon) \geq c \varepsilon^p \) for every \( \varepsilon \in [0, 2] \). It is obvious that a \( p \)-uniformly convex Banach space is uniformly convex. \( E \) is said to be strictly convex if \( \|x + y\|/2 < 1 \) for all \( x, y \in E \) with \( \|x\| = \|y\| = 1 \) and \( x \neq y \). We know that a uniformly convex Banach space is strictly convex and reflexive. For every \( p > 1 \), the (generalized) duality mapping \( J_p : E \rightharpoonup 2^{E^*} \) of \( E \) is defined by

\[
J_p x = \left\{ y^* \in E^* : \langle x, y^* \rangle = \|x\|^p, \|y^*\| = \|x\|^{p-1} \right\}
\]

for all \( x \in E \). When \( p = 2 \), \( J_2 \) is called the normalized duality mapping. We have that \( \phi_p, q > 1, \|x\|^p J_p x = \|x\|^q J_q x \) for all \( x \in E \). \( E \) is said to be smooth if the limit

\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\]

exists for every \( x, y \in E \) with \( \|x\| = \|y\| = 1 \). We know that the duality mapping \( J_p \) of \( E \) is single valued for each \( p > 1 \) if \( E \) is smooth. It is also known that if \( E \) is strictly convex, then the duality mapping \( J_p \) of \( E \) is one to one in the sense that \( x \neq y \) implies that \( J_p x \cap J_p y = \emptyset \) for all \( p > 1 \). If \( E \) is reflexive, then \( J_p \) is surjective, and \( J_p^{-1} \) is identical to the duality mapping \( J_p^*: E^* \rightharpoonup 2^E \) defined by

\[
J_p^* y^* = \left\{ x \in E : \langle x, y^* \rangle = \|y^*\|^q, \|x\| = \|y^*\|^{-q-1} \right\}
\]

for every \( y^* \in E^* \), where \( q \in \mathbb{R} \) satisfies \( 1/p + 1/q = 1 \). For \( p > 1 \), the duality mapping \( J_p \) of a smooth Banach space \( E \) is said to be weakly sequentially continuous if \( x_n \rightharpoonup x \) implies that \( J_p x_n \rightharpoonup J_p x \) (see [20, 21] for details). The following is proved by Xu [22] (see also [23]).

**Theorem 1** (Xu [22]). Let \( E \) be a smooth Banach space and \( p > 1 \). Then, \( E \) is \( p \)-uniformly convex if and only if there exists a constant \( c > 0 \) such that \( \|x + y\|^p \geq \|x\|^p + p\langle y, J_p x \rangle + c\|y\|^p \) holds for every \( x, y \in E \).

**Remark 2.** For a \( p \)-uniformly convex and smooth Banach space \( E \), we have that the constant \( c \) in the theorem above satisfies \( c \leq 1 \). Let

\[
c_0 = \sup \left\{ c > 0 : \|x + y\|^p \geq \|x\|^p + p\langle y, J_p x \rangle + c\|y\|^p \right\}.
\]

Then, there exists a positive real sequence \( \{c_n\} \) such that \( \lim_{n \to \infty} c_n = c_0 \) and \( \|x + y\|^p \geq \|x\|^p + p\langle y, J_p x \rangle + c_n\|y\|^p \) for all \( x, y \in E \) and \( n \in \mathbb{N} \). So, we get \( \|x + y\|^p \geq \|x\|^p + p\langle y, J_p x \rangle + c\|y\|^p \) for every \( x, y \in E \). Therefore, \( c_0 \) is the maximum of constants. In the case of Hilbert spaces, the normalized duality mapping \( J_2 \) is the identity mapping and \( c_0 = 1 \).

Let \( E \) be a smooth Banach space and \( p > 1 \). The function \( \Phi_p : E \times E \to \mathbb{R} \) is defined by

\[
\Phi_p(x, y) = \|y\|^p - p\langle y, J_p x \rangle + (p - 1)\|x\|^p.
\]

for every \( x, y \in E \). We have \( \Phi_p(x, y) \geq 0 \) for all \( x, y \in E \) and \( \Phi_p(x, z) + \Phi_p(x, y) = \Phi_p(x, y) + p(x - z, J_p x - J_p y) \) for all \( x, y, z \in E \). It is known that if \( E \) is strictly convex and smooth, then, for \( x, y \in E \), \( \Phi_p(x, y) = \Phi_p(x, y) = 0 \) if and only if \( x = y \). Indeed, suppose that \( \Phi_p(x, y) = \Phi_p(x, y) = 0 \). Then, since

\[
0 = \Phi_p(x, y) + \Phi_p(x, y) = \|y\|^p - \langle y, J_p x \rangle - \langle y, J_p x \rangle = \|y\|^p - 2\langle y, J_p x \rangle \geq p\langle y, J_p x \rangle - \|y\|^p \geq 0,
\]

we have \( \|x\| = \|y\| \). It follows that \( \langle y, J_p x \rangle = p^{-1}(\|y\|^p + (p - 1)\|x\|^p - \Phi_p(y, x)) = \|y\|^p \) and \( \|J_p x\| = \|x\|^{p-1} - \|y\|^{p-1} \), which implies that \( J_p y = J_p x \). Since \( J_p \) is one to one, we have \( x = y \) (see also [17]). We have the following result from Theorem 1.
**Lemma 3.** Let $p > 1$ and $E$ be a $p$-uniformly convex and smooth Banach space. Then, for each $x, y \in E$,
\[ \phi_p(x, y) \geq c_0 \|x - y\|^p \tag{8} \]
holds, where $c_0$ is maximum in Remark 2.

**Proof.** Let $x, y \in E$. By Theorem 1, we have
\[ \|x\|^p \geq \|y\|^p + p \langle x - y, J_p y \rangle + c_0 \|x - y\|^p, \tag{9} \]
where $c_0$ is maximum in Remark 2. Hence, we get
\[ \phi_p(x, y) = \|x\|^p - \|y\|^p - p \langle x - y, J_p y \rangle \geq c_0 \|x - y\|^p, \tag{10} \]
which is the desired result. \(\square\)

Let $C$ be a nonempty closed convex subset of a strictly convex and reflexive Banach space $E$, and let $x \in E$. Then, there exists a unique element $x_0 \in C$ such that $\|x_0 - x\| = \inf_{x \in C} \|y - x\|$. Putting $x_0 = P_Cx$, we call $P_C$ the metric projection onto $C$ (see [24]). We have the following result [25, p. 196] for the metric projection.

**Lemma 4.** Let $C$ be a nonempty closed convex subset of a strictly convex, reflexive, and smooth Banach space $E$, and let $x \in E$. Then, $y = P_Cx$ if and only if $\langle y - z, J_p(x - y) \rangle \geq 0$ for all $z \in C$, where $P_C$ is the metric projection onto $C$. \(\square\)

**Remark 5.** For $p > 1$, it holds that $\|x\|J_p x = \|x\|^{p-1} J_p x$ for every $x \in E$. Therefore, under the same assumption as Lemma 4, we have that $y = P_Cx$ if and only if $\langle y - z, J_p(x - y) \rangle \geq 0$ for all $z \in C$.

**3. Main Results**

Firstly, we consider the iteration of Crombez’s type and get the following result.

**Theorem 6.** Let $p, q > 1$ be such that $1/p + 1/q = 1$. Let $\{C_n\}_{n \in \mathbb{N}}$ be a family of nonempty closed convex subsets of a $p$-uniformly convex and smooth Banach space $E$ whose duality mapping $J_p$ is weakly sequentially continuous. Suppose that $\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$. Let $\lambda_{n,k} \in [0, (1 + 1/(p-1))^{p-1} c_0]$, and $\alpha_{n,k} \in [0, 1]$ for all $n \in \mathbb{N}$ and $k = 1, 2, \ldots, n$ with $\sum_{k=1}^{n} \alpha_{n,k} = 1$ for every $n \in \mathbb{N}$, where $c_0$ is maximum in Remark 2. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$ and
\[ x_{n+1} = I_p^n \left( \sum_{k=1}^{n} \alpha_{n,k} (J_p x_n - \lambda_{n,k} J_p (x_n - P_C x_n)) \right) \tag{11} \]
for every $n \in \mathbb{N}$. If $0 < \lim_{n \to \infty} \alpha_{n,k} \leq \lim_{n \to \infty} \alpha_{n,k} < (1 + 1/(p-1))^{p-1} c_0$ and $\lim_{n \to \infty} \alpha_{n,k} > 0$ for each $k \in \mathbb{N}$, then $\{x_n\}$ converges weakly to a point in $\bigcap_{n \in \mathbb{N}} C_n$.

**Proof.** Let $y_{n,k} = I_q^n (J_p x_n - \lambda_{n,k} J_p (x_n - P_C x_n))$ for $n \in \mathbb{N}$ and $k = 1, 2, \ldots, n$. Then, for $z \in \bigcap_{n \in \mathbb{N}} C_n$, we obtain
\[ \phi_p(z, y_{n,k}) - \phi_p(z, x_n) = -\phi_p(y_{n,k}, x_n) + p \langle y_{n,k} - z, J_p y_{n,k} - J_p x_n \rangle \]
\[ = -\phi_p(y_{n,k}, x_n) - p \lambda_{n,k} \langle y_{n,k} - z, J_p (x_n - P_C x_n) \rangle \]
\[ = -\phi_p(y_{n,k}, x_n) - p \lambda_{n,k} \langle y_{n,k} - x_n, J_p (x_n - P_C x_n) \rangle \]
\[ - p \lambda_{n,k} \langle x_n - z, J_p (x_n - P_C x_n) \rangle \]
\[ \tag{12} \]
for all $n \in \mathbb{N}$ and $k = 1, 2, \ldots, n$. Using Remark 5 with that $z \in C_k$, we get
\[ \langle x_n - z, J_p (x_n - P_C x_n) \rangle \]
\[ = \langle x_n - P_C x_n, J_p (x_n - P_C x_n) \rangle \]
\[ + \langle P_C x_n - z, J_p (x_n - P_C x_n) \rangle \]
\[ \geq \|x_n - P_C x_n\|^p \tag{13} \]
for every $n \in \mathbb{N}$ and $k = 1, 2, \ldots, n$. Thus, by Lemma 3 we have
\[ \phi_p(z, y_{n,k}) - \phi_p(z, x_n) \]
\[ \leq c_0 \|y_{n,k} - x_n\|^p \]
\[ - p \lambda_{n,k} \langle y_{n,k} - x_n, J_p (x_n - P_C x_n) \rangle \]
\[ - p \lambda_{n,k} \|x_n - P_C x_n\|^p \]
\[ \leq c_0 \|y_{n,k} - x_n\|^p + p \lambda_{n,k} \|y_{n,k} - x_n\| \|x_n - P_C x_n\|^{p-1} \]
\[ - p \lambda_{n,k} \|x_n - P_C x_n\|^p \]
\[ \tag{14} \]
for each $n \in \mathbb{N}$ and $k = 1, 2, \ldots, n$. Since it holds that
\[ st \leq \frac{s^p}{\beta} + \beta^{1/(p-1)} \frac{t^q}{q} \tag{15} \]
for $s, t \geq 0, p, q > 1$ with $1/p + 1/q = 1$, and $\beta > 0$, we have
\[ \|y_{n,k} - x_n\|^p \leq \frac{1}{\beta_k p} \|y_{n,k} - x_n\|^p \]
\[ + \beta_k^{1/(p-1)} \frac{p-1}{p} \|x_n - P_C x_n\|^p \tag{16} \]
for every $k \in \mathbb{N}$, $\beta_k > 0$ and $n \geq k$. Therefore, it follows that
\[ \phi_p(z, y_{n,k}) - \phi_p(z, x_n) \]
\[ \leq \left( \frac{\lambda_{n,k}}{\beta_k} - c_0 \right) \|y_{n,k} - x_n\|^p \]
\[ + \lambda_{n,k} \left( (p-1) \beta_k^{1/(p-1)} - p \right) \|x_n - P_C x_n\|^p \]
\[ \tag{17} \]
for every \( n \in \mathbb{N}, k = 1, 2, \ldots, n \), and \( \beta_k > 0 \). Since
\[
\phi_p(z, x_{n+1}) = \|z\|^p - p \left\langle z, \sum_{k=1}^{n} \alpha_{n,k} j P y_{n,k} \right\rangle + (p - 1) \left\langle \sum_{k=1}^{n} \alpha_{n,k} j P y_{n,k} \right\rangle^{p/(p-1)} \\
\leq \|z\|^p - p \left\langle z, j P y_{n,k} \right\rangle + (p - 1) \left\langle \sum_{k=1}^{n} \alpha_{n,k} y_{n,k} \right\rangle
\]
(18)

for every \( n \in \mathbb{N} \), we have
\[
\phi_p(z, x_{n+1}) - \phi_p(z, x_n) \leq \sum_{k=1}^{n} \alpha_{n,k} \left( \frac{\lambda_{n,k}}{\beta_k} - c_0 \right) \|y_{n,k} - x_n\|^p + \sum_{k=1}^{n} \alpha_{n,k} \lambda_{n,k} \left( (p - 1) \beta_k^{1/(p-1)} - p \right) \|x_n - P_{C_\infty} x_n\|^p
\]
(19)

for all \( n \in \mathbb{N} \) and \( \beta_1, \beta_2, \ldots, \beta_n > 0 \). Since \( \lambda_{n,k} \in [0,1/(p-1)] \), \( \alpha_{n,k} \in [0,1] \) for all \( n \in \mathbb{N} \) and \( k = 1, 2, \ldots, n \),
\[
0 < \lim_{n \to \infty} \lambda_{n,k} \leq \limsup_{n \to \infty} \lambda_{n,k} < \left( 1 + \frac{1}{(p-1)} \right)^{1/(p-1)} c_0, \quad \liminf_{n \to \infty} \alpha_{n,k} > 0
\]
(20)

for each \( k \in \mathbb{N} \), we can choose \( \beta_k > 0 \) for every \( k \in \mathbb{N} \) such that \( \alpha_{n,k} (\lambda_{n,k}/\beta_k - c_0) \leq 0 \), \( \alpha_{n,k} \lambda_{n,k} ((p - 1) \beta_k^{1/(p-1)} - p) \leq 0 \) for all \( n \geq k \) and
\[
\limsup_{n \to \infty} \alpha_{n,k} \left( \frac{\lambda_{n,k}}{\beta_k} - c_0 \right) < 0, \quad \limsup_{n \to \infty} \alpha_{n,k} \lambda_{n,k} \left( (p - 1) \beta_k^{1/(p-1)} - p \right) < 0
\]
(21)

for each \( k \in \mathbb{N} \). Hence, there exists \( \lim_{n \to \infty} \phi_p(z, x_n) \) for every \( z \in \bigcap_{n \in \mathbb{N}} C_n \) and
\[
\lim_{n \to \infty} \|y_{n,k} - x_n\| = \lim_{n \to \infty} \|x_n - P_{C_\infty} x_n\| = 0
\]
(22)

for all \( n \in \mathbb{N} \). It follows from Lemma 3 that \( \{x_n\} \) is bounded. Let \( \{x_{n_1}\} \) and \( \{x_{n_2}\} \) be subsequences of \( \{x_n\} \) such that \( x_{n_1} \to u_1 \) and \( x_{n_2} \to u_2 \). Then, we get \( \|x_n - P_{C_k} x_n\| \to 0 \) which implies that \( u_1 \in C_k \) for every \( k \in \mathbb{N} \). In the same way, we also have \( u_2 \in C_k \) for every \( k \in \mathbb{N} \). Let \( \lim_{n \to \infty} \phi_p(u_1, x_n) = \mu_1 \) and \( \lim_{n \to \infty} \phi_p(u_2, x_n) = \mu_2 \). Since
\[
\mu_1 - \mu_2 = \lim_{i \to \infty} \left( \phi_p(u_1, x_n) - \phi_p(u_2, x_n) \right)
\]
= \( \|u_1\|^p - \|u_2\|^p + \lim_{i \to \infty} \left\langle u_2 - u_1, j P x_n \right\rangle \)
(23)

and \( j P \) is weakly sequentially continuous, we have
\[
\mu_1 - \mu_2 = \|u_1\|^p - \|u_2\|^p + \left\langle u_2 - u_1, j P u_1 \right\rangle
\]
(24)

Similarly, we obtain \( \mu_2 - \mu_1 = -\phi_p(u_1, u_2) \). So, we get \( \phi_p(u_1, u_2) + \phi_p(u_2, u_1) = 0 \), that is, \( u_1 = u_2 \). Therefore, \( \{x_n\} \) converges weakly to a point in \( \bigcap_{n \in \mathbb{N}} C_n \).

Using the idea of [9, p. 256], we also have the following result by the iteration of Bregman’s type.

**Theorem 7.** Let \( p, q > 1 \) be such that \( 1/p + 1/q = 1 \). Let \( I \) be a countable set and \( \{C_j\}_{j \in I} \) a family of nonempty closed convex subsets of a \( p \)-uniformly convex and smooth Banach space \( E \) whose duality mapping \( j P \) is weakly sequentially continuous. Suppose that \( \bigcap_{j \in I} C_j \neq \emptyset \). Let \( \lambda_n \in [0, (1 + 1/(p-1))^{p-1}) \) for all \( n \in \mathbb{N} \), where \( c_0 \) is maximum in Remark 2, and let \( \{x_n\} \) be a sequence generated by \( x_1 = x \in E \) and
\[
x_{n+1} = J^q_{\phi_p} (J P x_n - \lambda_{n,q} j P x_n - P_{C_\infty} x_n)
\]
(25)

for every \( n \in \mathbb{N} \), where the index mapping \( i : J \to I \) satisfies that, for every \( j \in I \), there exists \( M_j \in \mathbb{N} \) such that \( j \in \{i(n), \ldots, i(n+M_j - 1)\} \) for each \( n \in \mathbb{N} \). If \( 0 < \liminf_{n \to \infty} \lambda_n \leq \limsup_{n \to \infty} \lambda_n < (1 + 1/(p-1))^{p-1} c_0 \), then, \( \{x_n\} \) converges weakly to a point in \( \bigcap_{j \in I} C_j \).

**Proof.** Let \( z \in \bigcap_{j \in I} C_j \). As in the proof of Theorem 6, we have
\[
\phi_p(z, x_{m+1}) - \phi_p(z, x_n) \leq \left( \frac{\lambda_n}{\beta} - c_0 \right) \|x_{m+1} - x_n\|^p
\]
+ \( \lambda_n \left( (p - 1) \beta^{1/(p-1)} - p \right) \|x_n - P_{C_\infty} x_n\|^p
\]
(26)

for all \( n \in \mathbb{N} \) and \( \beta > 0 \). Since \( \lambda_n \in [0, (1 + 1/(p-1))^{p-1}) \) for all \( n \in \mathbb{N} \) and \( 0 < \liminf_{n \to \infty} \lambda_n \leq \limsup_{n \to \infty} \lambda_n < (1 + 1/(p-1))^{p-1} c_0 \), we can find that \( \beta > 0 \) such that
\[
\limsup_{n \to \infty} \left( \frac{\lambda_n}{\beta} - c_0 \right) < 0, \quad \limsup_{n \to \infty} \left( (p - 1) \beta^{1/(p-1)} - p \right) < 0
\]
(27)

Then, there exists \( \lim_{n \to \infty} \phi_p(z, x_n) \) for every \( z \in \bigcap_{j \in I} C_j \) and
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} \|x_n - P_{C_\infty} x_n\| = 0.
\]
(28)
So, we have that \( \{x_n\} \) is bounded from Lemma 3. Let \( \{x_n\} \) be a subsequence of \( \{x_n\} \) such that \( x_{n_k} \to u \). For fixed \( j \in I \), there exists a strictly increasing sequence \( \{m_k\} \subset \mathbb{N} \) such that \( n_k \leq m_k \leq n_k + M_j - 1 \) and \( i(m_k) = j \) for every \( k \in \mathbb{N} \). It follows that

\[
\left\| x_{m_k} - x_n \right\| \leq \sum_{l=m_k}^{n+M_j-1} \left\| x_{l+1} - x_l \right\| \quad (29)
\]

for all \( k \in \mathbb{N} \) which implies that \( x_{m_k} \to u \). Since \( \lim_{n \to \infty} \|x_{m_k} - P_C x_{m_k}\| = 0, u \in C_j \) for every \( j \in I \). So, we get \( u \in \bigcap_{j \in I} C_j \). As in the proof of Theorem 6, using that \( J_p \) is weakly sequentially continuous, we get that \( \{x_n\} \) converges weakly to a point in \( \bigcap_{j \in I} C_j \).

Suppose that the index set \( I \) is a finite set \( \{0, 1, 2, \ldots, N - 1\} \). For the cyclic iteration, the index mapping \( i \) is defined by \( i(j) = j \mod N \) for each \( j \in I \). Clearly it satisfies the assumption in Theorem 7. In the case where the index set \( I \) is countably infinite, that is, \( I = \mathbb{N} \), one of the simplest examples of \( i : \mathbb{N} \to \mathbb{N} \) can be defined as follows:

\[
i(n) = \begin{cases} 
1 & (n = 2m - 1 \text{ for some } m \in \mathbb{N}), \\
2 & (n = 2m - 2 \text{ for some } m \in \mathbb{N}), \\
3 & (n = 4(2m - 1) - 1 \text{ for some } m \in \mathbb{N}), \\
\ldots, \\
k & (n = 2^{k-1}(2m - 1) \text{ for some } m \in \mathbb{N}), \\
\ldots
\end{cases} \quad (30)
\]

Then, the assumption in Theorem 7 is satisfied by letting \( M_j = 2^j \) for each \( j \in I = \mathbb{N} \).

4. Deduced Results

Since a real Hilbert space \( H \) is 2-uniformly convex and the maximum \( c_0 \) in Remark 2 is equal to 1, we get the following results. At first, we have the following theorem which generalizes the results of [2] by Theorem 6.

**Theorem 8.** Let \( \{C_n\}_{n \in \mathbb{N}} \) be a family of nonempty closed convex subsets of \( H \) such that \( \bigcap_{n \in \mathbb{N}} C_n \neq \emptyset \). Let \( \lambda_{n_k} \in [0, 2] \) and \( \alpha_{n_k} \in [0, 1] \) for all \( n \in \mathbb{N} \) and \( k = 1, 2, \ldots, n \) with \( \sum_{k=1}^{n} \alpha_{n_k} = 1 \) for every \( n \in \mathbb{N} \). Let \( \{x_n\} \) be a sequence generated by \( x_1 \in H \) and

\[
x_{n+1} = \sum_{k=1}^{n} \alpha_{n_k} (x_n - \lambda_{n_k} (x_n - P_{C_k} x_n)) \quad (31)
\]

for every \( n \in \mathbb{N} \). If it holds that \( 0 < \liminf_{n \to \infty} \lambda_{n_k} \leq \limsup_{n \to \infty} \alpha_{n_k} < 2 \) and \( \liminf_{n \to \infty} \alpha_{n_k} > 0 \) for each \( k \in \mathbb{N} \), then \( \{x_n\} \) converges weakly to a point in \( \bigcap_{n \in \mathbb{N}} C_n \).

Next, we have the following theorem which extends the result of [1] by Theorem 7.

**Theorem 9.** Let \( I \) be a countable set and \( \{C_j\}_{j \in I} \) a family of nonempty closed convex subsets of \( H \) such that \( \bigcap_{j \in I} C_j \neq \emptyset \). Let \( \lambda_n \in [0, 2] \) for all \( n \in \mathbb{N} \) and let \( \{x_n\} \) be a sequence generated by \( x_1 = x \in H \) and

\[
x_{n+1} = x_n - \lambda_n (x_n - P_{C_{i(n)}} x_n) \quad (32)
\]

for every \( n \in \mathbb{N} \), where the index mapping \( i : \mathbb{N} \to I \) satisfies that, for every \( j \in I \), there exists \( M_j \in \mathbb{N} \) such that \( i(j) \in \{i(n), \ldots, i(n+M_j-1)\} \) for each \( n \in \mathbb{N} \). If \( 0 < \liminf_{n \to \infty} \lambda_n \leq \limsup_{n \to \infty} \lambda_n < 2 \), then \( \{x_n\} \) converges weakly to a point in \( \bigcap_{j \in I} C_j \).

**Acknowledgment**

The first author was supported by the Grant-in-Aid for Scientific Research no. 22540175 from the Japan Society for the Promotion of Science.

**References**


