Using viscosity approximation method, we study strong convergence to a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a finite family of multivalued mappings satisfying the condition \( (E) \) in the setting of Hilbert space. Our results improve and extend some recent results in the literature.

1. Introduction

Let \( H \) be a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). Let \( C \) be a nonempty closed convex subset \( H \). A subset \( C \subset H \) is called proximal if, for each \( x \in H \), there exists an element \( y \in C \) such that

\[
\| x - y \| = \text{dist}(x, C) = \inf \{ \| x - z \| : z \in C \}.
\]  

(1)

A single-valued mapping \( T : C \rightarrow C \) is said to be nonexpansive, if

\[
\| Tx - Ty \| \leq \| x - y \| , \quad \forall x, y \in C.
\]  

(2)

Let \( P_C \) be a nearest point projection of \( H \) into \( C \); that is, for \( x \in H \), \( P_Cx \) is a unique nearest point in \( C \) with the property

\[
\| x - P_Cx \| = \inf \{ \| x - y \| : y \in C \}.
\]  

(3)

We denote by \( CB(C) \), \( K(C) \), and \( P(C) \) the collection of all nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximal bounded subsets of \( C \) respectively. The Hausdorff metric \( H \) on \( CB(H) \) is defined by

\[
H(A, B) := \max \left\{ \sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(y, A) \right\},
\]  

(4)

for all \( A, B \in CB(H) \).

Let \( T : H \rightarrow 2^H \) be a multivalued mapping. An element \( x \in H \) is said to be a fixed point of \( T \), if \( x \in Tx \) and the set of fixed points of \( T \) is denoted by \( F(T) \).

A multivalued mapping \( T : H \rightarrow CB(H) \) is called

(i) nonexpansive if

\[
H(Tx, Ty) \leq \| x - y \| , \quad x, y \in H;
\]  

(5)

(ii) quasi-nonexpansive if \( F(T) \neq \emptyset \) and \( H(Tx, Tp) \leq \| x - p \| \) for all \( x \in H \) and all \( p \in F(T) \).

Recently, García-Falset et al. [1] introduced a new condition on single-valued mappings, called condition \( (E) \), which is weaker than nonexpansiveness.

Definition 1. A mapping \( T : H \rightarrow H \) is said to satisfy condition \( (E_\mu) \) provided that

\[
\| x - Ty \| \leq \mu \| x - Tx \| + \| x - y \| , \quad x, y \in H.
\]  

(6)

We say that \( T \) satisfies condition \( (E) \) whenever \( T \) satisfies \( (E_\mu) \) for some \( \mu \geq 1 \).

Recently, Abkar and Eslamian [2, 3] generalized this condition for multivalued mappings as follows.
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Definition 2. A multivalued mapping \( T : H \rightarrow CB(H) \) is said to satisfy condition \((E)\) provided that
\[
H(Tx,Ty) \leq \mu \text{ dist}(x,Tx) + \|x - y\|, \quad x, y \in H, \tag{7}
\]
for some \( \mu \geq 1 \).

It is obvious that every nonexpansive multivalued mapping \( T : H \rightarrow CB(H) \) satisfies the condition \((E)\), and every mapping \( T : H \rightarrow CB(H) \) which satisfies the condition \((E)\) with nonempty fixed point set \( F(T) \) is quasi-nonexpansive.

Example 3. Let us define a mapping \( T \) on \([0, 3]\) by
\[
T(x) = \begin{cases} 
0, & x \neq 3 \\
\frac{x}{3}, & x = 3.
\end{cases}
\tag{8}
\]
It is easy to see that \( T \) satisfies the condition \((E)\) but is not nonexpansive. Indeed, for \( x, y \in [0, 3], H(Tx,Ty) = |(x - y)/3| \leq |x - y| \). Let \( x = 0 \) and \( y = 3 \). Then \( H(Tx,Ty) = 2 \leq 3 = |x - y| \). If \( x \in (0,3) \) and \( y = 3 \), then, we have \( \text{dist}(x,Tx) = 2x/3 \) and \( \text{dist}(y,Ty) = 1 \); hence
\[
H(Tx,Ty) = 2 - \frac{x}{3} \leq 3 - x + \frac{4x}{3} = |x - y| + 2 \text{ dist}(x,Tx).
\tag{9}
\]
Thus, \( T \) satisfies the condition \((E)\). However, \( T \) is not nonexpansive; indeed for \( x = 3 \) and \( y = 7/3, H(Tx,Ty) = 11/9 > 2/3 = |x - y| \).

Let \( \Psi : C \times C \rightarrow \mathbb{R} \) be a bifunction. The equilibrium problem associated with the bifunction \( \Psi \) and the set \( C \) is:

find \( x \in C \) such that \( \Psi(x,y) \geq 0, \quad \forall y \in C \). \tag{10}

Such a point \( x \in C \) is called the solution of the equilibrium problem. The set of solutions is denoted by \( EP(\Psi) \).

A broad class of problems in optimization theory, such as variational inequality, convex minimization, and fixed point problems, can be formulated as an equilibrium problem; see [4, 5]. In the literature, many techniques and algorithms have been proposed to analyze the existence and approximation of a solution to equilibrium problem; see [6]. Many researchers have studied various iteration processes for finding a common element of the set of solutions of the equilibrium problems and the set of fixed points of a class of nonlinear mappings. For example, see [7–22].

Fixed points and fixed point iteration process for nonexpansive mappings have been studied extensively by many authors to solve nonlinear operator equations, as well as variational inequalities; see, for example, [23–28]. In the recent years, fixed point theory for multivalued mappings has been studied by many authors; see [29–40] and the references therein.

In this paper, using viscosity approximation method, we study strong convergence to a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a finite family of multivalued mappings satisfying the condition \((E)\) in the setting of Hilbert space. Our results improve and extend some recent results in the literature.

2. Preliminaries

For solving the equilibrium problem, we assume that the bifunction \( \Psi \) satisfies the following conditions:

(A1) \( \Psi(x,x) = 0 \) for any \( x \in C \);

(A2) \( \Psi \) is monotone; that is, \( \Psi(x,y) + \Psi(y,x) \leq 0 \) for any \( x, y \in C \);

(A3) \( \Psi \) is upper-hemicontinuous; that is, for each \( x, y, z \in C \),
\[
\lim_{t \to 0^+} \sup \Psi(tz + (1-t)x, y) \leq \Psi(x, y); \tag{11}
\]

(A4) \( \Psi(x,.) \) is convex and lower semicontinuous for each \( x \in C \).

Lemma 4 (see [4]). Let \( C \) be a nonempty closed convex subset of \( H \) and let \( \Psi \) be a bifunction of \( C \times C \) into \( \mathbb{R} \) satisfying \((A1)\)–\((A4)\). Let \( r > 0 \) and \( x \in H \). Then, there exists \( z \in C \) such that
\[
\Psi(z, y) + \frac{1}{r} (y - z, z - x) \geq 0, \quad \forall y \in C. \tag{12}
\]

Lemma 5 (see [6]). Assume that \( \Psi : C \times C \rightarrow \mathbb{R} \) satisfies \((A1)\)–\((A4)\). For \( r > 0 \) and \( x \in H \), define a mapping \( S_r : H \rightarrow C \) as follows:
\[
S_r x = \left\{ z \in C : \Psi(z, y) + \frac{1}{r} (y - z, z - x) \geq 0, \quad \forall y \in C \right\}. \tag{13}
\]

Then, the following hold:

(i) \( S_r \) is single valued;

(ii) \( S_r \) is firmly nonexpansive; that is, for any \( x, y \in H \),
\[
\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y \rangle; \tag{14}
\]

(iii) \( F(S_r) = EP(\Psi) \);

(iv) \( EP(\Psi) \) is closed and convex.

Lemma 6 (see [41]). Let \( H \) be a real Hilbert space. Then, for all \( x, y, z \in H \) and \( \alpha, \beta, \gamma \in [0, 1] \) with \( \alpha + \beta + \gamma = 1 \) one has
\[
\|ax + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \frac{\alpha \beta}{2}\|x - y\|^2 - \frac{\alpha \gamma}{2}\|x - z\|^2 - \frac{\beta \gamma}{2}\|y - z\|^2 \tag{15}
\]

Lemma 7. For every \( x \) and \( y \) in a Hilbert space \( H \), the following inequality holds:
\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle. \tag{16}
\]

Lemma 8 (see [42]). Let \( \{a_n\} \) be a sequence of nonnegative real numbers, \( \{a_n\} \) a sequence in \( (0, 1) \) with \( \sum_{n=1}^{\infty} a_n = \infty \), \( \{\gamma_n\} \) a sequence of nonnegative real numbers with \( \sum_{n=1}^{\infty} \gamma_n < \infty \), and \( \{\beta_n\} \) a sequence of real numbers with \( \lim \sup_{n \to \infty} \beta_n \leq 0 \). Suppose that the following inequality holds:
\[
a_{n+1} \leq (1 - a_n) a_n + a_n \beta_n + \gamma_n, \quad n \geq 0. \tag{17}
\]
Then, \( \lim_{n \to \infty} a_n = 0 \).
Lemma 9 (see [43]). Let \( \{u_n\} \) be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence \( \{u_{n_i}\} \) of \( \{u_n\} \) such that \( u_{n_i} < u_{n_i+1} \) for all \( i \geq 0 \).

Then, \( u_{n_i} \) converges weakly to

\[
\tau(n) = \max \{ k \leq n : u_k < u_{k+1} \}.  \tag{18}
\]

Then, \( \tau(n) \to \infty \) as \( n \to \infty \) and for all \( n \geq n_0 \),

\[
\max \{ u_{\tau(n)} : u_n \} \leq u_{\tau(n)+1}.  \tag{19}
\]

Lemma 10 (see [20]). Let \( C \) be a convex closed subset of a real Hilbert space \( H \).

\( \text{Let } T : C \to CB(C) \) be a quasi-nonexpansive multivalued mapping. If \( F(T) \neq \emptyset \) and \( T(p) = \{ p \} \) for all \( p \in F(T) \).

Then, \( F(T) \) is closed and convex.

Lemma 11 (see [20]). Let \( C \) be a convex closed subset of a real Hilbert space \( H \).

\( \text{Let } T : C \to K(C) \) be a multivalued mapping satisfying the condition \( (E) \).

If \( x_n \) converges weakly to \( v \) and \( \lim_{n \to \infty} \text{dist}(x_n, Tx_n) = 0 \), then \( v \in Tv \).

3. A Strong Convergence Theorem

Theorem 13. Let \( C \) be a nonempty convex closed subset of a real Hilbert space \( H \) and \( \Psi \) a bifunction of \( C \times C \) into \( R \) satisfying (A1)–(A4). Let \( T_i : C \to CB(C) (i = 1, 2, \ldots, m) \) be a finite family of multivalued mappings, each satisfying condition \( (E) \). Assume further that \( \mathcal{F} = \bigcap_{i=1}^{m} F(T_i) \) and \( T_i(p) = \{ p \} (i = 1, 2, \ldots, m) \) for each \( p \in \mathcal{F} \).

Let \( f \) be a \( k \)-contraction of \( C \) into itself. Let \( \{x_n\} \) and \( \{u_n\} \) be sequences generated by the following algorithm:

\[
x_0 \in C, \quad u_n \in C  \text{ such that } \Psi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C
\]

\[
y_{n,1} = a_{n,1}u_n + b_{n,1}x_n + c_{n,1}z_{n,1},
\]

\[
y_{n,2} = a_{n,2}u_n + b_{n,2}z_{n,1} + c_{n,2}z_{n,2},
\]

\[
y_{n,3} = a_{n,3}u_n + b_{n,3}z_{n,2} + c_{n,3}z_{n,3},
\]

\[\vdots\]

\[
y_{n,m} = a_{n,m}u_n + b_{n,m}z_{n,m-1} + c_{n,m}z_{n,m},
\]

\[
x_{n+1} = \theta_n f(x_n) + (1 - \theta_n)y_{n,m}
\]

\( \forall n \geq 0 \),

\[
\text{where } z_{n,1} \in T_1(u_n), z_{n,k} \in T_k(y_{n,k-1}) \text{ for } k = 2, \ldots, m, \text{ and } \{a_{n,i}\}, \{b_{n,i}\}, \{c_{n,i}\}, \{\theta_n\}, \text{ and } \{r_n\} \text{ satisfy the following conditions:}
\]

(i) \( \{a_{n,i}\}, \{b_{n,i}\}, \{c_{n,i}\} \subseteq [a, b] \subseteq (0, 1), a_{n,i} + b_{n,i} + c_{n,i} = 1, (i = 1, 2, \ldots, m), \)

(ii) \( \{\theta_n\} \subseteq (0, 1), \lim_{n \to \infty} \theta_n = 0, \sum_{n=1}^{\infty} \theta_n = \infty, \)

(iii) \( \{r_n\} \subseteq (0, \infty), \text{ and } \lim \inf_{n \to \infty} r_n > 0. \)

Then, the sequences \( \{x_n\} \) and \( \{u_n\} \) converge strongly to \( q \in \mathcal{F} \), where \( q = P_{\mathcal{F}} f(q) \).

Proof. Let \( Q = P_{\mathcal{F}} f \).

By Banach contraction principle, there exists a \( q \in \mathcal{F} \) such that \( q = P_{\mathcal{F}} f(q) \). From Lemma 5 for all \( n \geq 0 \), we have

\[
\|u_n - q\| = \left\| S_n x_n - S_n q \right\| \leq \|x_n - q\|.  \tag{21}
\]

We show that \( \{x_n\} \) is bounded. Since, for each \( i = 1, 2, \ldots, m, \)

\( T_i \) satisfies the condition \( (E) \) and we have

\[
\|y_{n,1} - q\| = \|a_{n,1}u_n + b_{n,1}x_n + c_{n,1}z_{n,1} - q\|
\]

\[
\leq a_{n,1}\|u_n - q\| + b_{n,1}\|x_n - q\| + c_{n,1}\|z_{n,1} - q\|
\]

\[
= a_{n,1}\|u_n - q\| + b_{n,1}\|x_n - q\| + c_{n,1}\text{dist}(z_{n,1}, T_1q)  \tag{22}
\]

\[
\leq a_{n,1}\|u_n - q\| + b_{n,1}\|x_n - q\| + c_{n,1}\text{dist}(z_{n,1}, T_1q)
\]

\[
\leq a_{n,1}\|u_n - q\| + b_{n,1}\|x_n - q\| + c_{n,1}\|u_n - q\|
\]

\[
\leq \|x_n - q\|
\]

\[
\|y_{n,2} - q\| = \|a_{n,2}u_n + b_{n,2}z_{n,1} + c_{n,2}z_{n,2} - q\|
\]

\[
\leq a_{n,2}\|u_n - q\| + b_{n,2}\|z_{n,1} - q\| + c_{n,2}\|z_{n,2} - q\|
\]

\[
= a_{n,2}\|u_n - q\| + b_{n,2}\text{dist}(z_{n,1}, T_1q) + c_{n,2}\text{dist}(z_{n,2}, T_2q)
\]

\[
\leq a_{n,2}\|u_n - q\| + b_{n,2}\text{dist}(z_{n,1}, T_1q) + c_{n,2}\text{dist}(z_{n,2}, T_2q)
\]

\[
\leq a_{n,2}\|u_n - q\| + b_{n,2}\|u_n - q\| + c_{n,2}\|y_{n,1} - q\|
\]

\[
\leq \|x_n - q\|.  \tag{23}
\]

By continuing this process, we obtain

\[
\|y_{nm} - q\| \leq \|x_n - q\|.  \tag{24}
\]
This implies that
\[
\|x_{n+1} - q\| \\
= \|\theta_n f(x_n) + (1 - \theta_n) y_n - q\| \\
\leq \theta_n \|fx_n - q\| + (1 - \theta_n) \|y_n - q\| \\
\leq \theta_n (\|fx_n - f(q)\| + \|f(q) - q\|) + (1 - \theta_n) \|x_n - q\| \\
\leq \theta_n k \|x_n - q\| + \theta_n \|f(q) - q\| + (1 - \theta_n) \|x_n - q\| \\
= (1 - \theta_n (1 - k)) \|x_n - q\| + \theta_n \|f(q) - q\| \\
\leq \max \left\{ \|x_n - q\|, \frac{\|f(q) - q\|}{1 - k} \right\}.
\]
(25)

By induction, we get
\[
\|x_n - q\| \leq \max \left\{ \|x_0 - q\|, \frac{\|f(q) - q\|}{1 - k} \right\},
\]
(26)
for all \( n \in \mathbb{N} \). This implies that \( \{x_n\} \) is bounded and we also obtain that \( \{u_n\}, \{y_n\}, \{f(x_n)\}, \) and \( \{z_n\} \) are bounded. Next, we show that \( \lim_{n \to \infty} \operatorname{dist}(u_n, T_1 u_n) = 0 \) for each \( i \in \mathbb{N} \). By Lemma 6, we have
\[
\|y_{n1} - q\|^2 \\
= \left\|a_{n,1}u_n + b_{n,1}x_n + c_{n,1}z_{n1} - q\right\|^2 \\
\leq a_{n,1}\|u_n - q\|^2 + b_{n,1}\|x_n - q\|^2 \\
+ c_{n,1}\|z_{n1} - q\|^2 \\
- a_{n,1}b_{n,1}\|x_n - u_n\|^2 - a_{n,1}c_{n,1}\|u_n - z_{n1}\|^2 \\
= a_{n,1}\|u_n - q\|^2 + b_{n,1}\|x_n - q\|^2 \\
+ c_{n,1}\|z_{n1} - T_1 q\|^2 \\
- a_{n,1}b_{n,1}\|x_n - u_n\|^2 - a_{n,1}c_{n,1}\|u_n - z_{n1}\|^2 \\
\leq a_{n,1}\|u_n - q\|^2 + b_{n,1}\|x_n - q\|^2 \\
+ c_{n,1}\|z_{n1} - T_1 q\|^2 \\
- a_{n,1}b_{n,1}\|x_n - u_n\|^2 - a_{n,1}c_{n,1}\|u_n - z_{n1}\|^2 \\
\leq \|x_n - q\|^2 - a_{n,1}b_{n,1}\|x_n - u_n\|^2 \\
- a_{n,1}c_{n,1}\|u_n - z_{n1}\|^2.
\]
(27)

Applying Lemma 6 once more, we have
\[
\|y_{n2} - q\|^2 \\
= \left\|a_{n,2}u_n + b_{n,2}z_{n2} + c_{n,2}z_{n,2} - q\right\|^2 \\
\leq a_{n,2}\|u_n - q\|^2 + b_{n,2}\|z_{n2} - q\|^2 + c_{n,2}\|z_{n,2} - q\|^2 \\
- a_{n,2}b_{n,2}\|z_{n2} - u_n\|^2 - a_{n,2}c_{n,2}\|u_n - z_{n,2}\|^2 \\
= a_{n,2}\|u_n - q\|^2 + b_{n,2}\|z_{n2} - T_1 q\|^2 \\
+ c_{n,2}\|z_{n,2} - T_2 q\|^2 - a_{n,2}c_{n,2}\|u_n - z_{n,2}\|^2 \\
\leq a_{n,2}\|u_n - q\|^2 + b_{n,2}\|y_{n2} - q\|^2 + c_{n,2}\|y_{n,2} - q\|^2 \\
- a_{n,2}c_{n,2}\|u_n - z_{n,2}\|^2 \\
\leq \|x_n - q\|^2 - a_{n,1}c_{n,1}\|u_n - z_{n,1}\|^2 - a_{n,1}b_{n,1}c_{n,2}\|x_n - u_n\|^2.
\]
(28)

By continuing this process we have
\[
\|y_{n,m} - q\|^2 \\
= \left\|a_{n,m}u_n + b_{n,m}z_{n,m-1} + c_{n,m}z_{n,m} - q\right\|^2 \\
\leq a_{n,m}\|u_n - q\|^2 + b_{n,m}\|z_{n,m-1} - q\|^2 + c_{n,m}\|z_{n,m} - q\|^2 \\
- a_{n,m}b_{n,m}\|z_{n,m-1} - u_n\|^2 - a_{n,m}c_{n,m}\|u_n - z_{n,m}\|^2 \\
= a_{n,m}\|u_n - q\|^2 + b_{n,m}\|y_{n,m-1} - q\|^2 \\
+ c_{n,m}\|y_{n,m} - T_1 q\|^2 \\
- a_{n,m}b_{n,m}\|y_{n,m-1} - u_n\|^2 - a_{n,m}c_{n,m}\|u_n - z_{n,m}\|^2 \\
\leq a_{n,m}\|u_n - q\|^2 + b_{n,m}\|y_{n,m-2} - q\|^2 \\
+ c_{n,m}\|y_{n,m-1} - q\|^2 - a_{n,m}c_{n,m}\|u_n - z_{n,m}\|^2 \\
\leq \|u_n - q\|^2 - a_{n,m}c_{n,m}\|u_n - z_{n,m}\|^2 \\
- a_{n,m-1}c_{n,m-1}\|u_n - z_{n,m-1}\|^2 \\
- \cdots - a_{n,1}b_{n,1}c_{n,2}\|u_n - x_n\|^2 \\
- a_{n,1}b_{n,1}c_{n,2}\cdots c_{n,m}\|u_n - x_n\|^2.
\]
(29)
which implies that
\[
\|x_{n+1} - q\|^2 = \|\theta_n f x_n + (1 - \theta_n) y_{n,m} - q\|^2
\leq \theta_n \|f x_n - q\|^2 + (1 - \theta_n) \|y_{n,m} - q\|^2
\leq \theta_n \|f x_n - q\|^2 + (1 - \theta_n) \|u_n - q\|^2
\leq (1 - \theta_n) a_{n,m} \|u_n - z_{n,m}\|^2
\leq (1 - \theta_n) a_{n,m-1} \|u_n - z_{n,m-1}\|^2
\leq \cdots (1 - \theta_n) a_{n,1} c_{n,2} \cdots c_{n,m} \|u_n - x_n\|^2
\leq (1 - \theta_n) a_{n,1} b_{n,1} c_{n,2} \cdots c_{n,m} \|u_n - x_n\|^2.
\]
\[\tag{30}\]
Therefore, we have that
\[
(1 - \theta_n) a_{n,1} b_{n,1} c_{n,2} \cdots c_{n,m} \|u_n - x_n\|^2
\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \theta_n \|y f x_n - q\|.
\]
\[\tag{31}\]
In order to prove that \(x_n \to q\) as \(n \to \infty\), we consider the following two cases.

Case 1. Suppose that there exists \(n_0\) such that \(|\|x_n - q\||\) is nonincreasing, for all \(n \geq n_0\). Boundedness of \(|\|x_n - q\||\) implies that \(|\|x_n - q\||\) is convergent. From (31) and conditions (i), (ii) we have that
\[
\lim_{n \to \infty} \|u_n - x_n\| = 0.
\]
\[\tag{32}\]
By a similar argument, for \(k = 1, 2, \ldots, m\), we obtain that
\[
\lim_{n \to \infty} \|u_n - z_{n,k}\| = 0.
\]
\[\tag{33}\]
Hence,
\[
\lim_{n \to \infty} \text{dist} (u_n, T_i u_n) \leq \lim_{n \to \infty} \|u_n - z_{n,1}\| = 0,
\]
\[
\lim_{n \to \infty} \text{dist} (u_n, T_k y_{n,k-1}) \leq \lim_{n \to \infty} \|u_n - z_{n,k}\| = 0,
\]
\[\tag{34}\]
(k = 2, \ldots, m).

Therefore, we have
\[
\lim_{n \to \infty} \|u_n - y_{n,1}\| \leq \lim_{n \to \infty} b_{n,1} \|u_n - x_n\|
+ \lim_{n \to \infty} c_{n,1} \|u_n - z_{n,1}\| = 0.
\]
\[\tag{35}\]
For \(k = 2, \ldots, m\), we have
\[
\lim_{n \to \infty} \|u_n - y_{n,k}\| \leq \lim_{n \to \infty} b_{n,k} \|u_n - z_{n,k-1}\|
+ \lim_{n \to \infty} c_{n,k} \|u_n - z_{n,k}\| = 0.
\]
\[\tag{36}\]
Using the previous inequality for \(k = 2, \ldots, m\), we have
\[
\text{dist} (u_n, T_k u_n) \leq \text{dist} (u_n, T_k y_{n,k-1}) + \|y_{n,k-1} - u_n\|
+ \|y_{n,k} - u_n\|
\leq (\mu + 1) \text{dist} (u_n, T_k y_{n,k-1}) + (\mu + 1) \|y_{n,k-1} - u_n\| \to 0,
\]
\[n \to \infty. \]
\[\tag{37}\]
Next, we show that
\[
\limsup_{n \to \infty} (q - f q, q - x_n) \leq 0,
\]
\[\tag{38}\]
where \(q = P_{\mathcal{F}} f (q)\). To show this inequality, we choose a subsequence \(\{x_{n_i}\}\) of \(\{x_n\}\) such that
\[
\lim_{i \to \infty} (q - f q, q - x_{n_i}) = \limsup_{n \to \infty} (q - f q, q - x_n) = 0.
\]
\[\tag{39}\]
Since \(\{x_{n_i}\}\) is bounded, there exists a subsequence \(\{x_{n_i}\}\) of \(\{x_n\}\) which converges weakly to \(v\). Without loss of generality, we can assume that \(x_{n_i}\) converges weakly to \(v\). Since \(\lim_{n \to \infty} \|x_n - u_n\| = 0\), we have \(u_{n_i}\) converges weakly to \(v\). We show that \(v \in \mathcal{F}\). Let us show \(v \in E P(Y)\). Since \(u_{n_i} = S_n x_{n_i}\), we have
\[
\Psi (u_{n_i}, y) + \frac{1}{r_{n_i}} \langle y - u_{n_i}, u_{n_i} - x_{n_i} \rangle \geq 0 \quad \forall y \in C.
\]
\[\tag{40}\]
From (A2), we have
\[
\frac{1}{r_{n_i}} \langle y - u_{n_i}, u_{n_i} - x_{n_i} \rangle \geq \Psi (y, u_{n_i}).
\]
\[\tag{41}\]
Replacing \(n\) with \(n_i\), we have
\[
\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle \geq \Psi (y, u_{n_i}).
\]
\[\tag{42}\]
From (A4), we have
\[
0 \geq \Psi (y, v), \quad \forall y \in C.
\]
\[\tag{43}\]
For \(t \in (0, 1]\) and \(y \in C\), let \(y_t = t y + (1 - t) v\). Since \(y, v \in C\), and \(C\) is convex, we have \(y_t \in C\) and hence \(\Psi (y_t, v) \leq 0\). So, from (A1) and (A4) we have
\[
0 = \Psi (y, y_t) \leq t \Psi (y, y) + (1 - t) \Psi (y, v) \leq t \Psi (y, y) + \Psi (y_t, y),
\]
\[\tag{44}\]
which gives \(0 \leq \Psi (y_t, y)\). Letting \(t \to 0\), we have, for each \(y \in C\), \(0 \leq \Psi (v, y)\) Also, since \(u_{n_i} \to v\) and \(\lim_{n \to \infty} \text{dist}(u_{n_i}, T_k u_{n_i}) = 0\), by Lemma 12 we have \(v \in \bigcap_{i \in I} F(T_i)\). Hence, \(v \in \mathcal{F}\). Since \(q = P_{\mathcal{F}} f (q)\) and \(v \in \mathcal{F}\), it follows that
\[
\limsup_{n \to \infty} (q - f q, q - x_n) = \lim_{i \to \infty} (q - f q, q - x_{n_i})
= (q - f q, q - v) \leq 0.
\]
\[\tag{45}\]
By using Lemma 7 and inequality (31) we have
\[
\|x_{n+1} - q\|^2 \\
\leq \|(1 - \theta_n) (y_{n,m} - q)\|^2 + 2\theta_n \langle f x_n - q, x_{n+1} - q \rangle \\
\leq \|(1 - \theta_n) (y_{n,m} - q)\|^2 + 2\theta_n \langle f x_n - f q, x_{n+1} - q \rangle \\
+ 2\theta_n \langle f q - q, x_{n+1} - q \rangle \\
\leq (1 - \theta_n)^2 \|x_n - q\|^2 + 2\theta_n k \|x_n - q\| \|x_{n+1} - q\| \\
+ 2\theta_n \langle f q - q, x_{n+1} - q \rangle.
\]
(46)

This implies that
\[
\|x_{n+1} - q\|^2 \leq \left( 1 - \frac{2(1-k)\theta_n}{1-\theta_n k} \right) \|x_n - q\|^2 \\
+ \frac{\theta_n^2}{1 - \theta_n k} \|x_n - q\|^2 \\
+ \frac{2\theta_n}{1 - \theta_n k} \langle f q - q, x_{n+1} - q \rangle.
\]
(47)

From Lemma 8, we conclude that the sequence \( \{x_n\} \) converges strongly to \( q \).

**Case 2.** Assume that there exists a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) such that
\[
\|x_{n_j} - q\| < \|x_{n_{j+1}} - q\|,
\]
(48)

for all \( j \in \mathbb{N} \). In this case, from Lemma 9, there exists a nondecreasing sequence \( \{\tau(n)\} \) of \( \mathbb{N} \) for all \( n \geq n_0 \) (for some \( n_0 \) large enough) such that \( \tau(n) \to \infty \) as \( n \to \infty \) and the following inequalities hold for all \( n \geq n_0 \):
\[
\|x_{\tau(n)} - q\| \leq \|x_{\tau(n)+1} - q\|, \quad \|x_{n} - q\| \leq \|x_{\tau(n)+1} - q\|.
\]
(49)

From (31) we obtain \( \lim_{n \to \infty} \|y_{\tau(n)} - T \| = 0 \), and \( \lim_{n \to \infty} \|u_{\tau(n)} - x_{\tau(n)}\| = 0 \). Following an argument similar to that in Case 1, we have
\[
\lim_{n \to \infty} \|x_{\tau(n)} - q\| = 0, \quad \lim_{n \to \infty} \|x_{\tau(n)+1} - q\| = 0.
\]
(50)

Thus, by Lemma 9 we have
\[
0 \leq \|x_n - q\| \leq \max \{\|x_{\tau(n)} - q\|, \|x_{n} - q\|\} \leq \|x_{\tau(n)+1} - q\|.
\]
(51)

Therefore, \( \{x_n\} \) converges strongly to \( q = P_{\mathcal{F}} f(q) \). This completes the proof.

Now, we remove the condition that \( T(p) = \{p\} \) for all \( p \in \mathcal{F} \), and state the following theorem.

**Theorem 14.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) and \( \Psi \) a bifunction of \( C \times C \) into \( \mathbb{R} \) satisfying (A1)–(A4). Let, for each \( 1 \leq i \leq m \), \( T_i : C \to P(C) \) be multivalued mappings such that \( T_i \) satisfies the condition (E). Assume that \( \mathcal{F} = \bigcap_{i=1}^{m} F(T_i) \bigcap EP(\Psi) \neq \emptyset \). Let \( f \) be a \( k \)-contraction of \( C \) into itself. Let \( \{x_n\} \) and \( \{u_n\} \) be sequences generated the following algorithm:
\[
x_0 \in C,
\]
\[
u_n \in C \text{ such that } \Psi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C
\]
\[
y_{n,1} = a_{n,1} u_n + b_{n,1} x_n + c_{n,1} z_{n,1},
\]
\[
y_{n,2} = a_{n,2} u_n + b_{n,2} z_{n,1} + c_{n,2} z_{n,2},
\]
\[
y_{n,3} = a_{n,3} u_n + b_{n,3} z_{n,2} + c_{n,3} z_{n,3}
\]
\[
\vdots
\]
\[
y_{n,m} = a_{n,m} u_n + b_{n,m} z_{n,m-1} + c_{n,m} z_{n,m},
\]
(52)

where \( z_{n,j} \in P_{T_j}(u_n), \) \( z_{n,k} \in P_{T_k}(y_{n,k-1}) \) for \( k = 2, \ldots, m \), and \( \{a_{n,i}\}, \{b_{n,i}\}, \{c_{n,i}\}, \{\theta_n\} \) and \( \{r_n\} \) satisfy the following conditions:

(i) \( \{a_{n,i}\}, \{b_{n,i}\}, \{c_{n,i}\} \subset [a,b] \subset (0,1), a_{n,j} + b_{n,j} + c_{n,j} = 1, (i = 1, 2, \ldots, m) \),

(ii) \( \{\theta_n\} \subset (0,1), \lim_{n \to \infty} \theta_n = 0, \sum_{n=1}^{\infty} \theta_n = \infty \),

(iii) \( \{r_n\} \subset (0,\infty) \), and \( \lim \inf_{n \to \infty} r_n > 0 \).

Then, the sequences \( \{x_n\} \) and \( \{u_n\} \) converge strongly to \( q \in \mathcal{F} \), where \( q = P_{\mathcal{F}} f(q) \).

Proof. Let \( p \in \mathcal{F} \); then \( P_{\mathcal{F}}(p) = \{p\} \). Now by substituting \( T_i \) instead of \( T_p \) and using a similar argument as in the proof of Theorem 13, the desired result follows.

\( \square \)

As a corollary for single-valued mappings, we obtain the following result.

**Corollary 15.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) and \( \Psi \) a bifunction of \( C \times C \) into \( \mathbb{R} \) satisfying (A1)–(A4). Let, for each \( 1 \leq i \leq m \), \( T_i : C \to C \) be a finite family of mappings satisfying condition (E). Assume that \( \mathcal{F} = \bigcap_{i=1}^{m} F(T_i) \bigcap EP(\Psi) \neq \emptyset \). Let \( f \) be a \( k \)-contraction
of $C$ into itself. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated the following algorithm:

$$x_0 \in C,$$

$$u_n \in C \text{ such that } \Psi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C$$

$$y_n,1 = a_{n,1}u_n + b_{n,1}x_n + c_{n,1}T_1u_n,$$

$$y_n,2 = a_{n,2}u_n + b_{n,2}T_1u_n + c_{n,2}T_2y_n,1$$

$$\vdots$$

$$y_n,m = a_{n,m}u_n + b_{n,m}T_{m-1}y_{n,m-2} + T_m y_{n,m-1},$$

$$x_{n+1} = \theta_{n,f}x_n + (1 - \theta_{n})y_{n,m}, \quad \forall n \geq 0,$$

(53)

where $\{a_{ij}\}, \{b_{rij}\}, \{c_{rij}\}$, and $\{r_n\}$ satisfy the following conditions:

(i) $\{a_{ij}\}, \{b_{rij}\}, \{c_{rij}\} \subset [a, b], (0, 1), a_{ij} + b_{rij} + c_{rij} = 1, (i = 1, 2, \ldots, m),$

(ii) $\{b_{rij}\} \subset (0, 1), \lim_{n \to \infty} b_{rij} = 0, \sum_{n=1}^{\infty} b_{rij} = \infty$

(iii) $\{r_n\} \subset (0, \infty), \text{ and lim inf}_{n \to \infty} r_n > 0.$

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $q \in \mathcal{F},$ where $q = P_{\mathcal{F}} f(q)$.

Remark 16. Our results generalize the corresponding results of S. Takahashi and W. Takahashi [9] from a single valued nonexpansive mapping to a finite family of multivalued mappings satisfying the condition (E). Our results also improve the recent results of Eslamian [16].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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