Research Article

The Global Weak Solution for a Generalized Camassa-Holm Equation

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A nonlinear generalization of the famous Camassa-Holm model is investigated. Provided that initial value $u_0 \in H^s(R)$ ($1 \leq s \leq 3/2$) and $(1 - \partial_x^2)u_0$ satisfies an associated sign condition, it is shown that there exists a unique global weak solution to the equation in space $u(t, x) \in L^2([0, +\infty), H^s(R))$ in the sense of distribution, and $u_x \in L^\infty([0, +\infty) \times R)$.

1. Introduction

In recent years, a lot of works have been carried out to investigate the Camassa-Holm equation [1],

$$u_t - u_{txx} + ku_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad (1)$$

which is a completely integrable equation. In fact, the Camassa-Holm equation arises as a model describing the unidirectional propagation of shallow water waves over a flat bottom [1–3]. The equation was originally derived much earlier as a bi-Hamiltonian generalization of the Korteweg-de Vries equation (see [4]). Johnson [2], Constantin and Lannes [5] derived models which include the Camassa-Holm equation (1). It has been found that (1) conforms with many conservation laws (see [6, 7]) and possesses smooth solitary wave solutions if $k > 0$ [3, 8] or peakons if $k = 0$ [3, 9]. Equation (1) is also regarded as a model of the geodesic flow for the H^1 right invariant metric on the Bott-Virasoro group if $k > 0$ and on the diffeomorphism group if $k = 0$ (see [10–14]). The well-posedness of local strong solutions for generalized forms of (1) has been given in [15–17]. The sharpest results for the global existence and blow-up solutions are found in Bressan and Constantin [18, 19].

Recently, Li et al. [20] studied the following generalized Camassa-Holm equation:

$$\begin{aligned} u_t - u_{txx} + ku^m u_x + (m+3)u^{m+1} u_x \\ = (m+2)u^m u_x u_{xx} + u^{m+1} u_{xxx}, \end{aligned} \quad (2)$$

where $m \geq 0$ is a natural number. Obviously, (2) reduces to (1) if $m = 0$. The authors applied the pseudoparabolic regularization technique to build the local well-posedness for (2) in Sobolev space $H^s(R)$ with $s > 3/2$ via a limiting procedure. Provided that the initial value u_0 satisfies a sign condition and $u_0 \in H^s(R)$ ($s > 3/2$), it is shown that there exists a unique global strong solution for (2) in space $C([0, \infty); H^s(R)) \cap C^1([0, \infty); H^{s-1}(R))$. However, the existence and uniqueness of the global weak solution for (2) is not investigated in [20].

The objective of this paper is to establish the well-posedness of global weak solutions for (2). Using the estimates in $H^q(R)$ with $0 \leq q \leq 1/2$, which are derived from the equation itself, we prove that there exists a unique global weak solution to (2) in space $H^s(R)$ with $1 \leq s \leq 3/2$ if $u_0 \in H^s(R)$, and $(1 - \partial_x^2)u_0$ satisfies an associated sign condition.

The structure of this paper is as follows. The main result is given in Section 2. Several lemmas are given in Section 3. Section 4 establishes the proof of the main result.

2. Main Results

Firstly, we give some notations.

The space of all infinitely differentiable functions $\phi(t, x)$ with compact support in $[0, +\infty) \times R$ is denoted by C_0^∞ . $L^p = L^p(R)$ ($1 \leq p < +\infty$) is the space of all measurable functions h such that $\|h\|_{L^p}^p = \int_R |h(t, x)|^p dx < \infty$. We define $L^\infty = L^\infty(R)$ with the standard norm

$\|h\|_{L^\infty} = \inf_{m(e)=0} \sup_{x \in R \setminus e} |h(t, x)|$. For any real number s , we let $H^s = H^s(R)$ denote the Sobolev space with the norm defined by

$$\|h\|_{H^s} = \left(\int_R (1 + |\xi|^2)^s |\hat{h}(t, \xi)|^2 d\xi \right)^{1/2} < \infty, \quad (3)$$

where $\hat{h}(t, \xi) = \int_R e^{-ix\xi} h(t, x) dx$.

For $T > 0$ and nonnegative number s , let $C([0, T]; H^s(R))$ denote the Frechet space of all continuous H^s -valued functions on $[0, T]$. We set $\Lambda = (1 - \partial_x^2)^{1/2}$.

Defining

$$\phi(x) = \begin{cases} e^{1/(x^2-1)}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases} \quad (4)$$

and letting $\phi_\varepsilon(x) = \varepsilon^{-(1/4)} \phi(\varepsilon^{-(1/4)} x)$ with $0 < \varepsilon < 1/4$ and $u_{\varepsilon 0} = \phi_\varepsilon * u_0$ (convolution of ϕ_ε and u_0), we know that $u_{\varepsilon 0} \in C^\infty$ for any $u_0 \in H^s$ with $s > 0$. Notation $(1 - \partial_x^2)u + k/2(m+1) \in N^+(R)$ (or equivalently $(1 - \partial_x^2)u + k/2(m+1) \in N^-(R)$) means that $(1 - \partial_x^2)u * \phi_\varepsilon + k/2(m+1) \geq 0$ (or equivalently $(1 - \partial_x^2)u * \phi_\varepsilon + k/2(m+1) \leq 0$) for an arbitrary sufficiently small $\varepsilon > 0$.

For the equivalent form of (2), we consider its Cauchy problem

$$\begin{aligned} u_t - u_{txx} &= -\frac{k}{m+1} (u^{m+1})_x - \frac{m+3}{m+2} (u^{m+2})_x \\ &\quad + \frac{1}{m+2} \partial_x^3 (u^{m+2}) - (m+1) \partial_x (u^m u_x^2) \\ &\quad + u^m u_x u_{xx}, \\ u(0, x) &= u_0(x). \end{aligned} \quad (5)$$

Definition 1. A function $u(t, x) \in L^2([0, +\infty), H^s(R))$ is called a global weak solution to problem (5) if for every $T > 0$, $u(t, x) \in H^s(R)$, $u_t(t, x) \in H^{s-1}(R)$, and all $\psi(t, x) \in C_0^\infty$, it holds that

$$\begin{aligned} \int_0^T \int_R \left[u_t - u_{txx} + ku^m u_x + (m+3)u^{m+1} u_x \right. \\ \left. - (m+2)u^m u_x u_{xx} - u^{m+1} u_{xxx} \right] \psi(t, x) dx dt = 0 \end{aligned} \quad (6)$$

with $u(0, x) = u_0(x)$.

Now, we give the main result of this work.

Theorem 2. Let $u_0(x) \in H^s(R)$, $1 \leq s \leq 3/2$, $(1 - \partial_x^2)u_0 + k/2(m+1) \in N^+(R)$, and $k \geq 0$ (or equivalently $(1 - \partial_x^2)u_0 + k/2(m+1) \in N^-(R)$, $k \leq 0$). Then, problem (5) has a unique global weak solution $u(t, x) \in L^2([0, +\infty), H^s(R))$ in the sense of distribution, and $u_x \in L^\infty([0, +\infty) \times R)$.

3. Several Lemmas

Lemma 3 (see [20]). Let $u_0(x) \in H^s(R)$ with $s > 3/2$. Then, the Cauchy problem (5) has a unique solution

$$u(t, x) \in C([0, T]; H^s(R)) \cap C^1([0, T]; H^{s-1}(R)), \quad (7)$$

where $T > 0$ depends on $\|u_0\|_{H^s(R)}$.

Lemma 4 (see [20]). Let $u_0(x) \in H^s$, $s > 3/2$, and $k \geq 0$, $(1 - \partial_x^2)u_0 + k/2(m+1) \geq 0$ (or equivalently $k \leq 0$, $(1 - \partial_x^2)u_0 + k/2(m+1) \leq 0$). Then, problem (5) has a unique solution satisfying

$$u(t, x) \in C([0, \infty); H^s(R)) \cap C^1([0, \infty); H^{s-1}(R)). \quad (8)$$

Using the first equation of system (5) derives

$$\frac{d}{dt} \int_R (u^2 + u_x^2) dx = 0, \quad (9)$$

from which one has the conservation law

$$\int_R (u^2 + u_x^2) dx = \int_R (u_0^2 + u_{0x}^2) dx. \quad (10)$$

Lemma 5 (see [20]). Let $s > 3/2$, and the function $u(t, x)$ is a solution of problem (5) and the initial data $u_0(x) \in H^s$. Then, the following inequality holds:

$$\|u\|_{H^1}^2 \leq \int_R (u^2 + u_x^2) dx = \int_R (u_0^2 + u_{0x}^2) dx. \quad (11)$$

For $q \in (0, s-1]$, there is a constant c such that

$$\begin{aligned} \int_R (\Lambda^{q+1} u)^2 dx &\leq \int_R (\Lambda^{q+1} u_0)^2 dx \\ &\quad + c \int_0^t \|u\|_{H^{q+1}}^2 (\|u_x\|_{L^\infty} \|u\|_{L^\infty}^m \\ &\quad + \|u\|_{L^\infty}^{m-1} \|u_x\|_{L^\infty}^2) d\tau. \end{aligned} \quad (12)$$

For $q \in [0, s-1]$, there is a constant c such that

$$\begin{aligned} \|u_t\|_{H^q} &\leq c \|u\|_{H^{q+1}} (\|u\|_{L^\infty}^m \|u\|_{H^1} + \|u\|_{L^\infty}^m \|u_x\|_{L^\infty} \\ &\quad + \|u\|_{L^\infty}^{m-1} \|u_x\|_{L^\infty}^2). \end{aligned} \quad (13)$$

For (2), consider the problem

$$\begin{aligned} p_t &= u^{m+1}(t, p), \quad t \in [0, T], \\ p(0, x) &= x. \end{aligned} \quad (14)$$

Lemma 6 (see [20]). Let $u_0 \in H^s$, $s \geq 3$, and let $T > 0$ be the maximal existence time of the solution to problem (5). Then, problem (14) has a unique solution $p \in C^1([0, T] \times R)$. Moreover, the map $p(t, \cdot)$ is an increasing diffeomorphism of R with $p_x(t, x) > 0$ for $(t, x) \in [0, T] \times R$.

Differentiating (14) with respect to x yields

$$\begin{aligned} \frac{d}{dt} p_x &= (m+1) u^m u_x(t, p) p_x, \quad t \in [0, T], \\ p_x(0, x) &= 1, \end{aligned} \quad (15)$$

which leads to

$$p_x(t, x) = \exp\left(\int_0^t (m+1) u^m u_x(\tau, p(\tau, x)) d\tau\right). \quad (16)$$

The next lemma is reminiscent of a strong invariance property of the Camassa-Holm equation (the conservation of momentum [21]).

Lemma 7 (see [20]). *Let $u_0 \in H^s$ with $s \geq 3$, and let $T > 0$ be the maximal existence time of the problem (5). It holds that*

$$y(t, p(t, x)) p_x^2(t, x) = y_0(x) e^{\int_0^t m u^m u_x d\tau}, \quad (17)$$

where $(t, x) \in [0, T] \times R$ and $y := u - u_{xx} + k/2(m+1)$.

Lemma 8. *If $u_0 \in H^s$, $s \geq 3$, such that $(1 - \partial_x^2)u_0 + k/2(m+1) \geq 0$, $k \geq 0$ (or equivalently, $(1 - \partial_x^2)u_0 + k/2(m+1) \leq 0$, $k \leq 0$), then the solution of problem (5) satisfies*

$$\|u_x\|_{L^\infty} \leq \|u\|_{L^\infty} + \frac{|k|}{2(m+1)} \leq c. \quad (18)$$

Proof. Using $u_0 - u_{0xx} + k/2(m+1) \geq 0$, it follows from Lemma 7 that $u - u_{xx} + k/2(m+1) \geq 0$. Letting $Y_1 = u - u_{xx}$, we have

$$u = \frac{1}{2} e^{-x} \int_{-\infty}^x e^{\eta} Y_1(t, \eta) d\eta + \frac{1}{2} e^x \int_x^{\infty} e^{-\eta} Y_1(t, \eta) d\eta, \quad (19)$$

from which we obtain

$$\begin{aligned} \partial_x u(t, x) &= -\frac{1}{2} \left(e^{-x} \int_{-\infty}^x e^{\eta} Y_1(t, \eta) d\eta + e^x \int_x^{\infty} e^{-\eta} Y_1(t, \eta) d\eta \right) \\ &\quad + e^x \int_x^{\infty} e^{-\eta} Y_1(t, \eta) d\eta \\ &= -u(t, x) + e^x \int_x^{\infty} e^{-\eta} Y_1(t, \eta) d\eta \\ &= -u(t, x) + e^x \int_x^{\infty} e^{-\eta} \left(Y_1(t, \eta) + \frac{k}{2(m+1)} \right) d\eta \\ &\quad - \frac{k}{2(m+1)} e^x \int_x^{\infty} e^{-\eta} d\eta \\ &= -u(t, x) + e^x \int_x^{\infty} e^{-\eta} (y(t, \eta)) d\eta - \frac{k}{2(m+1)} \\ &\geq -u(t, x) - \frac{k}{2(m+1)}. \end{aligned} \quad (20)$$

On the other hand, we have

$$\begin{aligned} \partial_x u(t, x) &= \frac{1}{2} \left(e^{-x} \int_{-\infty}^x e^{\eta} Y_1(t, \eta) d\eta + e^x \int_x^{\infty} e^{-\eta} Y_1(t, \eta) d\eta \right) \\ &\quad - e^{-x} \int_{-\infty}^x e^{\eta} Y_1(t, \eta) d\eta \\ &= u(t, x) - e^{-x} \int_{-\infty}^x e^{\eta} Y_1(t, \eta) d\eta \\ &= u(t, x) - e^{-x} \int_{-\infty}^x e^{\eta} \left(Y_1(t, \eta) + \frac{k}{2(m+1)} \right) d\eta \\ &\quad + \frac{k}{2(m+1)} e^{-x} \int_{-\infty}^x e^{\eta} d\eta \\ &= u(t, x) - e^{-x} \int_{-\infty}^x e^{\eta} y(t, \eta) d\eta + \frac{k}{2(m+1)} \\ &\leq u(t, x) + \frac{k}{2(m+1)}. \end{aligned} \quad (21)$$

The inequalities (19), (20), and (21) derive that inequality (18) is valid. Similarly, if $(1 - \partial_x^2)u_0 + k/2(m+1) \leq 0$, $k \leq 0$, we still know that (18) is valid. $\square$

Lemma 9. *For $s > 0$, $u_0 \in H^s$, it holds that*

$$\begin{aligned} \|u_{\varepsilon 0x}\|_{L^\infty} &\leq c \|u_{0x}\|_{L^\infty}, \\ \|u_{\varepsilon 0}\|_{H^q} &\leq c, \quad \text{if } q \leq s, \\ \|u_{\varepsilon 0}\|_{H^q} &\leq c \varepsilon^{s-q/4}, \quad \text{if } q > s, \\ \|u_{\varepsilon 0} - u_0\|_{H^q} &\leq c \varepsilon^{s-q/4}, \quad \text{if } q \leq s, \\ \|u_{\varepsilon 0} - u_0\|_{H^s} &= o(1), \end{aligned} \quad (22)$$

where c is a constant independent of ε .

The proof of this lemma can be found in Lai and Wu [15]. From Lemma 3, it derives that the Cauchy problem

$$\begin{aligned} u_t - u_{txx} &= -\frac{m+3}{m+2} (u^{m+2})_x + \frac{1}{m+2} \partial_x^3 (u^{m+2}) \\ &\quad - (m+1) \partial_x (u^m u_x^2) + u^m u_x u_{xx}, \\ u(0, x) &= u_{\varepsilon 0}(x), \quad x \in R, \end{aligned} \quad (23)$$

has a unique solution u depending on the parameter ε . We write $u_\varepsilon(t, x)$ to represent the solution of problem (23). Using Lemma 3 derives that $u_\varepsilon(t, x) \in C^\infty([0, T], H^\infty(R))$ since $u_{\varepsilon 0}(x) \in C_0^\infty(R)$.

Lemma 10. *Provided that $u_0 \in H^s$, $1 \leq s \leq 3/2$, $k \geq 0$, and $(1 - \partial_x^2)u_0 + k/2(m+1) \in N^+(R)$ (or equivalently $(1 - \partial_x^2)u_0 + k/2(m+1) \in N^-(R)$, $k \leq 0$), then there exists a constant $c_0 > 0$ independent of ε such that the solution of problem (23) satisfies*

$$\|u_{\varepsilon x}\|_{L^\infty} \leq \|u_x\|_{L^\infty} + \frac{|k|}{2(m+1)} \leq c_0. \quad (24)$$

Proof. Using identity (10) and Lemma 9, if $u_0 \in H^s(R)$ with $1 \leq s \leq 3/2$, we have

$$\|u_\varepsilon\|_{L^\infty} \leq \|u_\varepsilon\|_{H^1} = \|u_{\varepsilon 0}\|_{H^1} \leq c, \quad (25)$$

where c is independent of ε .

From Lemma 8, we have

$$\|u_{\varepsilon x}\|_{L^\infty} \leq \|u_\varepsilon\|_{L^\infty} + \frac{|k|}{2(m+1)} \leq c + \frac{|k|}{2(m+1)}, \quad (26)$$

which completes the proof. $\square$

Lemma 11. *For any $f_1 \in L^\infty$, $f_2 \in H^z$ with $z \leq 0$, it holds that*

$$\|f_1 f_2\|_{H^z} \leq c \|f_1\|_{L^\infty} \|f_2\|_{H^z} \quad \text{for any } z \leq 0. \quad (27)$$

The proof of this lemma can be found in [15].

4. Existence and Uniqueness of Global Weak Solution

Provided that $1 \leq s \leq 3/2$, for problem (23), applying Lemmas 5, 9, and 10, and the Gronwall's inequality, we obtain the inequalities

$$\begin{aligned} \|u_\varepsilon\|_{H^1} &\leq \|u_{\varepsilon 0}\|_{H^1} \leq c, \\ \|u_\varepsilon\|_{H^q} &\leq c \|u_{\varepsilon 0}\|_{H^q} \exp \left[\int_0^t (\|u_{\varepsilon x}\| + \|u_{\varepsilon x}\|_{L^\infty}^2) d\tau \right] \leq c e^{ct}, \\ \|u_{\varepsilon t}\|_{H^r} &\leq \|u_\varepsilon\|_{H^{r+1}} (1 + e^{ct}) \leq c (1 + e^{ct}), \end{aligned} \quad (28)$$

where $q \in (0, s]$, $r \in [0, s-1]$, and c is a constant independent of ε . It follows from the Aubin's compactness theorem that there is a subsequence of $\{u_\varepsilon\}$, denoted by $\{u_{\varepsilon_n}\}$, such that $\{u_{\varepsilon_n}\}$ and their temporal derivatives $\{u_{\varepsilon_n t}\}$ are weakly convergent to a function $u(t, x)$ and its derivative u_t in $L^2([0, T], H^s)$ and $L^2([0, T], H^{s-1})$, respectively, where T is an arbitrary fixed positive number. Moreover, for any real number $R_1 > 0$, $\{u_{\varepsilon_n}\}$ is convergent to the function u strongly in the space

$L^2([0, T], H^q(-R_1, R_1))$ for $q \in (0, s]$ and $\{u_{\varepsilon_n t}\}$ converges to u_t strongly in the space $L^2([0, T], H^r(-R_1, R_1))$ for $r \in [0, s-1]$.

4.1. The Proof of Existence for Global Weak Solution. For an arbitrary fixed $T > 0$, from Lemma 10, we know that $\{u_{\varepsilon_n x}\}(\varepsilon_n \rightarrow 0)$ is bounded in the space L^∞ . Thus, the sequences $\{u_{\varepsilon_n}\}$, $\{u_{\varepsilon_n x}\}$, $\{u_{\varepsilon_n x}^2\}$, and $\{u_{\varepsilon_n x}^3\}$ are weakly convergent to u , u_x , u_x^2 , and u_x^3 in $L^2([0, T], H^r(-R_1, R_1))$ for any $r \in [0, s-1]$, separately. Using $u^m(u_x^2)_x = (u^m u_x^2)_x - (u^m)_x u_x^2$, we know that u satisfies the equation

$$\begin{aligned} & - \int_0^T \int_R u (g_t - g_{xxt}) dx dt \\ & = \int_0^T \int_R \left[\left(\frac{m+3}{m+2} u^{m+2} + (m+1) u^m u_x^2 \right) g_x \right. \\ & \quad \left. - \frac{1}{m+2} u^{m+2} g_{xxx} - \frac{1}{2} u^m u_x^2 g_x \right. \\ & \quad \left. - \frac{m}{2} u^{m-1} u_x^3 g \right] dx dt, \end{aligned} \quad (29)$$

with $u(0, x) = u_0(x)$ and $g \in C_0^\infty$. Since $X = L^1([0, T] \times R)$ is a separable Banach space and $\{u_{\varepsilon_n x}\}$ is a bounded sequence in the dual space $X^* = L^\infty([0, T] \times R)$ of X , there exists a subsequence of $\{u_{\varepsilon_n x}\}$, still denoted by $\{u_{\varepsilon_n x}\}$, weakly star convergent to a function v in $L^\infty([0, T] \times R)$. As $\{u_{\varepsilon_n x}\}$ weakly converges to u_x in $L^2([0, T] \times R)$, it results that $u_x = v$ almost everywhere. Thus, we obtain $u_x \in L^\infty([0, T] \times R)$. Since $T > 0$ is an arbitrary number, we complete the global existence of weak solutions to problem (5).

Proof of Uniqueness. Suppose that there exist two global weak solutions $u(t, x)$ and $v(t, x)$ to problem (5) with the same initial value $u(0, x) \in H^s(R)$, $1 \leq s \leq 3/2$, we consider its associated regularized problem (23). Letting $w_\varepsilon = u_\varepsilon(t, x) - v_\varepsilon(t, x)$, from Lemma 10, we get $\|\partial u_{\varepsilon(t,x)}/\partial x\|_{L^\infty} \leq c$ and $\|\partial v_{\varepsilon(t,x)}/\partial x\|_{L^\infty} \leq c$ which is independent of ε . Still denoting $u = u_\varepsilon$, $v = v_\varepsilon$, and $w = w_\varepsilon$, it holds that

$$\begin{aligned} w_t &= (1 - \partial_x^2)^{-1} \left[-\partial_x (u^{m+2} - v^{m+2}) \right. \\ & \quad \left. - \partial_x [\partial_x (u^{m+1}) \partial_x w \right. \\ & \quad \left. + \partial_x (u^{m+1} - v^{m+1}) \partial_x v \right] \\ & \quad \left. + [u^m u_x u_{xx} - v^m v_x v_{xx}] \right] \\ & \quad - \frac{1}{m+2} \partial_x (u^{m+2} - v^{m+2}), \\ w(0, x) &= 0. \end{aligned} \quad (30)$$

Multiplying both sides of (30) by w , we get

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \int_R w^2 dx &\leq c \left| \int_R w(u^{m+2} - v^{m+2})_x dx \right| \\
 &\quad + \left| \int_R w \Lambda^{-2} (u^{m+2} - v^{m+2})_x dx \right| \\
 &\quad + \left| \int_R w \Lambda^{-2} [\partial_x (u^{m+1}) \partial_x w]_x dx \right| \\
 &\quad + \left| \int_R w \Lambda^{-2} [\partial_x (u^{m+1} - v^{m+1}) \partial_x v]_x dx \right| \\
 &\quad + \left| \int_R w \Lambda^{-2} [u^m u_x u_{xx} - v^m v_x v_{xx}] dx \right| \\
 &= I_1 + I_2 + I_3 + I_4 + I_5.
 \end{aligned} \tag{31}$$

Using $\|u\|_{L^\infty} \leq c$, $\|v\|_{L^\infty} \leq c$, $\|u_x\|_{L^\infty} \leq c$, $\|v_x\|_{L^\infty} \leq c$, we have

$$\begin{aligned}
 I_1 &\leq c \left| \int_R w \left[w \sum_{j=0}^{m+1} u^j v^{m+1-j} \right]_x dx \right| \\
 &= c \left| \int_R w \left[w_x \sum_{j=0}^{m+1} u^j v^{m+1-j} + w \sum_{j=0}^{m+1} (u^j v^{m+1-j})_x \right] dx \right| \\
 &= c \left| \int_R \left(\frac{1}{2} w^2 \right)_x \sum_{j=0}^{m+1} u^j v^{m+1-j} + w^2 \sum_{j=0}^{m+1} (u^j v^{m+1-j})_x dx \right| \\
 &= c \left| \int_R \left(\frac{-1}{2} w^2 \right) \sum_{j=0}^{m+1} (u^j v^{m+1-j})_x + w^2 \sum_{j=0}^{m+1} (u^j v^{m+1-j})_x dx \right| \\
 &= c \left| \int_R \left(\frac{1}{2} w^2 \right) \sum_{j=0}^{m+1} (u^j v^{m+1-j})_x dx \right| \\
 &\leq c \|w\|_{L^2}^2 \sum_{j=0}^{m+1} \|(u^j v^{m+1-j})_x\|_{L^\infty} \\
 &\leq c \|w\|_{L^2}^2.
 \end{aligned} \tag{32}$$

Applying Lemma 11 repeatedly, we have

$$\begin{aligned}
 I_2 &\leq c \|w\|_{L^2} \|\Lambda^{-2} (u^{m+2} - v^{m+2})_x\|_{L^2} \\
 &\leq c \|w\|_{L^2} \left\| w \sum_{j=0}^{m+1} u^j v^{m+1-j} \right\|_{L^2} \\
 &\leq c \|w\|_{L^2}^2 \sum_{j=0}^{m+1} \|u\|_{L^\infty}^j \|v\|_{L^\infty}^{m+1-j} \\
 &\leq c \|w\|_{L^2}^2,
 \end{aligned}$$

$$\begin{aligned}
 I_3 &\leq c \|w\|_{L^2} \|\Lambda^{-2} [\partial_x (u^{m+1}) \partial_x w]_x\|_{L^2} \\
 &\leq c \|w\|_{L^2} \|\partial_x (u^{m+1}) \partial_x w\|_{H^{-1}} \\
 &\leq c \|w\|_{L^2} \|\partial_x w\|_{H^{-1}} \|\partial_x (u^{m+1})\|_{L^\infty} \\
 &\leq c \|w\|_{L^2}^2,
 \end{aligned}$$

$$\begin{aligned}
 I_4 &\leq c \|w\|_{L^2} \|\partial_x (u^{m+1} - v^{m+1}) \partial_x v\|_{H^{-1}} \\
 &\leq c \|w\|_{L^2} \|\partial_x v\|_{L^\infty} \|\partial_x (u^{m+1} - v^{m+1})\|_{H^{-1}} \\
 &\leq c \|w\|_{L^2} \|u^{m+1} - v^{m+1}\|_{H^0} \\
 &\leq c \|w\|_{L^2} \left\| w \sum_{j=0}^m u^j v^{m-j} \right\|_{L^2} \\
 &\leq c \|w\|_{L^2}^2 \sum_{j=0}^m \|u\|_{L^\infty}^j \|v\|_{L^\infty}^{m-j} \\
 &\leq c \|w\|_{L^2}^2.
 \end{aligned} \tag{33}$$

For I_5 , using Lemma 11 derives

$$\begin{aligned}
 I_5 &\leq c \|w\|_{L^2} \|(u^m - v^m)(u_x^2)_x + v^m[u_x^2 - v_x^2]_x\|_{H^{-2}} \\
 &\leq c \|w\|_{L^2} \|(u^m - v^m)(u_x^2)_x\|_{H^{-2}} + \|v^m[u_x^2 - v_x^2]_x\|_{H^{-2}} \\
 &\leq c \|w\|_{L^2} \left(\|(u^m - v^m)(u_x^2)_x\|_{H^{-2}} - (u^m - v^m)_x u_x^2 \right)_{H^{-2}} \\
 &\quad + \|v\|_{L^\infty}^m \|(u - v)_x (u_x + v_x)\|_{H^{-1}} \\
 &\leq c \|w\|_{L^2} \left(\|(u^m - v^m) u_x^2\|_{H^{-1}} + \|(u^m - v^m)_x u_x^2\|_{H^{-2}} + c \|w\|_{L^2} \right) \\
 &\leq c \|w\|_{L^2} \left(\|u_x\|_{L^\infty}^2 \|w\|_{L^2} \sum_{j=0}^{m-1} \|u\|_{L^\infty}^j \|v\|_{L^\infty}^{m-1-j} + c \|w\|_{L^2} \right) \\
 &\leq c \|w\|_{L^2}^2.
 \end{aligned} \tag{34}$$

Using (32)–(34), we get

$$\frac{1}{2} \frac{d}{dt} \int_R w^2 dx \leq c \|w\|_{L^2}^2. \tag{35}$$

Applying $w(0) = 0$ results in $\|w\|_{L^2}^2 = 0$. Consequently, we know that the global weak solution is unique. $\square$

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