Research Article

New Convergence Definitions for Sequences of Sets

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Several notions of *convergence* for subsets of metric space appear in the literature. In this paper, we define *Wijsman I-convergence* and *Wijsman I*-convergence* for sequences of sets and establish some basic theorems. Furthermore, we introduce the concepts of *Wijsman I-Cauchy* sequence and *Wijsman I*-Cauchy* sequence and then study their certain properties.

1. Introduction and Background

The concept of convergence of sequences of points has been extended by several authors (see [1–9]) to the concept of convergence of sequences of sets. The one of these such extensions that we will consider in this paper is Wijsman convergence. We will define *I-convergence* for sequences of sets and establish some basic results regarding these notions.

Let us start with fundamental definitions from the literature. The natural density of a set K of positive integers is defined by

$$\delta(K) := \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : k \in K\}|, \quad (1)$$

where $|\{k \leq n : k \in K\}|$ denotes the number of elements of K not exceeding n ([10]).

Statistical convergence of sequences of points was introduced by Fast [11]. In [12], Schoenberg established some basic properties of statistical convergence and also studied the concept as a summability method.

A number sequence $x = (x_k)$ is said to be statistically convergent to the number ξ if, for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k - \xi| \geq \varepsilon\}| = 0. \quad (2)$$

In this case, we write $st\text{-}\lim x_k = \xi$. Statistical convergence is a natural generalization of ordinary convergence. If $\lim x_k = \xi$, then $st\text{-}\lim x_k = \xi$. The converse does not hold in general.

Definition 1 (see [13]). A family of sets $I \subseteq 2^{\mathbb{N}}$ is called an ideal on $\mathbb{N}$ if and only if

- (i) $\emptyset \in I$;
- (ii) for each $A, B \in I$ one has $A \cup B \in I$;
- (iii) for each $A \in I$ and each $B \subseteq A$ one has $B \in I$.

An ideal is called nontrivial if $\mathbb{N} \notin I$, and nontrivial ideal is called admissible if $\{n\} \in I$ for each $n \in \mathbb{N}$.

Definition 2 (see [14]). A family of sets $F \subseteq 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$ if and only if

- (i) $\emptyset \notin F$;
- (ii) for each $A, B \in F$ one has $A \cap B \in F$;
- (iii) for each $A \in F$ and each $B \supseteq A$ one has $B \in F$.

Proposition 3 (see [14]). I is a nontrivial ideal in $\mathbb{N}$ if and only if

$$F = F(I) = \{M \subseteq \mathbb{N} : A \in I\} \quad (3)$$

is a filter in $\mathbb{N}$.

Definition 4 (see [14]). Let I be a nontrivial ideal of subsets of $\mathbb{N}$. A number sequence $(x_n)_{n \in \mathbb{N}}$ is said to be *I-convergent* to ξ ($\xi = I\text{-}\lim_{n \rightarrow \infty} x_n$) if and only if for each $\varepsilon > 0$ the set

$$\{k \in \mathbb{N} : |x_k - \xi| \geq \varepsilon\} \quad (4)$$

belongs to I . The element ξ is called the *I* limit of the number sequence $x = (x_n)_{n \in \mathbb{N}}$.

The concept of I -convergence of real sequences is a generalization of statistical convergence which is based on the structure of the ideal I of subsets of the set of natural numbers. Kostyrko et al. [14] introduced the concept of I -convergence of sequences in a metric space and studied some properties of this convergence. I -convergence of real sequences coincides with the ordinary convergence if I is the ideal of all finite subsets of $\mathbb{N}$ and with the statistical convergence if I is the ideal of subsets of $\mathbb{N}$ of natural density zero.

Definition 5 (see [14]). An admissible ideal $I \subseteq 2^{\mathbb{N}}$ is said to have the property (AP) if for any sequence $\{A_1, A_2, \dots\}$ of mutually disjoint sets of I , there is sequence $\{B_1, B_2, \dots\}$ of sets such that each symmetric difference $A_i \Delta B_i$ ($i = 1, 2, \dots$) is finite and $\bigcup_{i=1}^{\infty} B_i \in I$.

Definition 5 is similar to the condition (APO) used in [15].

In [14], the concept of I^* -convergence which is closely related to I -convergence has been introduced.

Definition 6 (see [14]). A sequence $x = (x_n)$ of elements of X is said to be I^* -convergence to ξ if and only if there exists a set $M \in F(I)$,

$$M = \{m = (m_i) : m_i < m_{i+1}, i \in \mathbb{N}\} \subset \mathbb{N} \quad (5)$$

such that $\lim_{k \rightarrow \infty} x_{m_k} = \xi$.

In [14], it is proved that I -convergence and I^* -convergence are equivalent for admissible ideals with property (AP).

Also, in order to prove that I -convergent sequence coincides with I^* -convergent sequence for admissible ideals with property (AP), we need the following lemma.

Lemma 7 (see [13]). Let $\{P_i\}_{i=1}^{\infty}$ be a countable collection of subsets of $\mathbb{N}$ such that $P_i \in F(I)$ is a filter which associates with an admissible ideal I with property (AP). Then there exists a set $P \subset \mathbb{N}$ such that $P \in F(I)$ and the set $P \setminus P_i$ is finite for all i .

Theorem 8 (see [13]). Let $I \subseteq 2^{\mathbb{N}}$ be an admissible ideals with property (AP) and $x = (x_n)$ be a number sequence. Then $I - \lim_{n \rightarrow \infty} x_n = \xi$ if and only if there exists a set $P \in F(I)$, $P = \{p = (p_i) : p_i < p_{i+1}, i \in \mathbb{N}\}$ such that $\lim_{k \rightarrow \infty} x_{p_k} = \xi$.

Definition 9 (see [9]). Let (X, d) be a metric space. For any nonempty closed subsets $A, A_k \subseteq X$, one says that the sequence $\{A_k\}$ is Wijsman convergent to A :

$$\lim_{k \rightarrow \infty} d(x, A_k) = d(x, A) \quad (6)$$

for each $x \in X$. In this case one writes $W - \lim_{k \rightarrow \infty} A_k = A$.

As an example, consider the following sequence of circles in the (x, y) -plane: $A_k = \{(x, y) : x^2 + y^2 + 2kx = 0\}$. As $k \rightarrow \infty$ the sequence is Wijsman convergent to the y -axis $A = \{(x, y) : x = 0\}$.

Definition 10 (see [16]). Let (X, d) be a metric space. For any nonempty closed subsets $A, A_k \subseteq X$, one says that the

sequence $\{A_k\}$ is Wijsman statistical convergent to A if for $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0. \quad (7)$$

In this case one writes $st - \lim_W A_k = A$ or $A_k \rightarrow A(WS)$. Consider

$$WS := \left\{ \{A_k\} : st - \lim_W A_k = A \right\}, \quad (8)$$

where WS denotes the set of Wijsman statistical convergence sequences.

Also the concept of bounded sequence for sequences of sets was given by Nuray and Rhoades [16] as follows.

Let (X, ρ) be a metric space. For any nonempty closed subsets A_k of X , one says that the sequence $\{A_k\}$ is bounded if $\sup_k d(x, A_k) < \infty$ for each $x \in X$.

2. Wijsman I -Convergence

In this section, we will define Wijsman I -convergence and Wijsman I^* -convergence of sequences of sets, give the relationship between them, and establish some basic theorems.

Definition 11. Let (X, d) be a metric space and $I \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$. For any nonempty closed subsets $A, A_k \subset X$, one says that the sequence $\{A_k\}$ is Wijsman I -convergent to A , if, for each $\varepsilon > 0$ and for each $x \in X$, the set

$$A(x, \varepsilon) = \{k \in \mathbb{N} : |d(x, A_k) - d(x, A)| \geq \varepsilon\} \quad (9)$$

belongs to I . In this case, one writes $I_W - \lim A_k = A$ or $A_k \rightarrow A(I_W)$, and the set of Wijsman I -convergent sequences of sets will be denoted by

$$I_W = \left\{ \{A_k\} : \{k \in \mathbb{N} : |d(x, A_k) - d(x, A)| \geq \varepsilon\} \in I \right\}. \quad (10)$$

Example 12. $I \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$, (X, d) a metric space, and $A, A_k \subset X$ nonempty closed subsets. Let $X = \mathbb{R}^2$, $\{A_k\}$ be following sequence:

$$A_k = \begin{cases} \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 2ky = 0\} & \text{if } k \neq n^2 \\ \{(x, y) \in \mathbb{R}^2 : y = -1\} & \text{if } k = n^2, \end{cases} \quad (11)$$

$$A = \{(x, y) \in \mathbb{R}^2 : y = 0\}.$$

For $k = n^2$, $d((x, y), A_{n^2}) = |y+1| \neq d((x, y), A) = |y|$. Let us take a point (x^*, y^*) outside $x^2 + y^2 - 2ky = 0$. For $k \neq n^2$, we write $d((x^*, y^*), A_k) \rightarrow d((x^*, y^*), A) = |y^*|$. Since the line equation is

$$\frac{x-0}{x^*} = \frac{y-k}{y^*-k}, \quad (12)$$

where the line is passing from $(0, k)$ the center point of the circle and (x^*, y^*) the outside of the circle, we write $y = k +$

$((y^* - k)/x^*) \cdot x$. If we write this $y = k + ((y^* - k)/x^*) \cdot x$ value on the circle equation $x^2 + y^2 - 2ky = 0$, we can get

$$x = \frac{|k| \cdot x^*}{\sqrt{(x^*)^2 + (y^* - k)^2}}. \quad (13)$$

For $k \rightarrow \infty$, if we take limit, it will be $x \rightarrow x^*$. If we write $x = (|k| \cdot x^*) / \sqrt{(x^*)^2 + (y^* - k)^2}$ on the $y = k + ((y^* - k)/x^*) \cdot x$, we get $y \rightarrow 0$ ($k \rightarrow \infty$). Thus, for $k \neq n^2$

$$d((x^*, y^*), A_k) = \sqrt{(x - x^*)^2 + (y - y^*)^2} \rightarrow |y^*|. \quad (14)$$

So we get $d((x^*, y^*), A_k) \rightarrow d((x^*, y^*), A) = |y^*|$, for $k \neq n^2$.

For $k = n^2$ and $k \neq n^2$, the set sequence $\{A_k\}$ has two different limits. Thus $\{A_k\}$ is not Wijsman convergent to set A , but

$$\begin{aligned} \{k \in \mathbb{N} : |d((x, y), A_k) - d((x, y), A)| \geq \varepsilon\} \\ = \{k \in \mathbb{N} : k = n^2\} \subset I_d. \end{aligned} \quad (15)$$

Thus, suppose that

$$A(x, y, \varepsilon) = \{k \in \mathbb{N} : |d((x, y), A_k) - d((x, y), A)| \geq \varepsilon\} \quad (16)$$

for $\varepsilon > 0$ and for each $(x, y) \in \mathbb{R}^2$.

Since $\lim_{k \rightarrow \infty} [|d((x, y), A_k) - d((x, y), A)|] = 0$, for $k \neq n^2$, for each $\varepsilon > 0$,

$$\exists k_\varepsilon \in \mathbb{N} : \forall k > k_\varepsilon : |d((x, y), A_k) - d((x, y), A)| < \varepsilon. \quad (17)$$

Define the set $A_{k_\varepsilon}(x, y)$ as

$$A_{k_\varepsilon}(x, y) := \{k \in \mathbb{N} : |d((x, y), A_k) - d((x, y), A)| > \varepsilon\}. \quad (18)$$

Thus, since $A(x, y, \varepsilon) = A_{k_\varepsilon}(x, y) \cup \{k \in \mathbb{N} : k = n^2\}$ and $A_{k_\varepsilon}(x, y) \in I_d$ and $\{k \in \mathbb{N} : k = n^2\} \in I_d$, we can write

$$\begin{aligned} A(x, y, \varepsilon) \\ := \{k \in \mathbb{N} : |d((x, y), A_k) - d((x, y), A)| > \varepsilon\} \in I_d, \end{aligned} \quad (19)$$

where $I_d = \{A : \delta(A) = 0\}$. So the set sequence $\{A_n\}$ is Wijsman I -convergent to set A .

Example 13. Let $I \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$, (X, d) a metric space, and $A, A_n \subset X$ nonempty closed subsets. Let $X = \mathbb{R}^2$, $\{A_n\}$ be following sequence:

$$\begin{aligned} A_n \\ = \begin{cases} \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq n, 0 \leq y \leq \frac{1}{n} \cdot x\}, & \text{if } n \neq k^2 \\ \{(x, y) \in \mathbb{R}^2 : x \geq 0, y = 1\}, & \text{if } n = k^2, \end{cases} \\ A = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y = 0\}. \end{aligned} \quad (20)$$

Since

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |d((x, y), A_n) - d((x, y), A)| \geq \varepsilon\}| = 0, \quad (21)$$

the set sequence $\{A_n\}$ is Wijsman statistical convergent to set A . Thus we can write $st - \lim_W A_n = A$, but this sequence is not Wijsman convergent to set A . Because for $n \neq k^2$, $\lim_{n \rightarrow \infty} d((x, y), A_n) = d((x, y), A)$, but for $n = k^2$, $\lim_{n \rightarrow \infty} d((x, y), A_n) \neq d((x, y), A)$. Let $I_d \subset 2^{\mathbb{N}}$ be proper ideal. Define set K as

$$K = K(\varepsilon) = \{n \in \mathbb{N} : |d((x, y), A_n) - d((x, y), A)| \geq \varepsilon\}. \quad (22)$$

If we take I_d for I , Wijsman ideal convergent coincides with Wijsman statistical convergent. Really, one has

$$\begin{aligned} \{n \in \mathbb{N} : |d((x, y), A_n) - d((x, y), A)| \geq \varepsilon\} \\ = \{n \in \mathbb{N} : n = k^2\} \subset I_d. \end{aligned} \quad (23)$$

Since the Wijsman topology is not first countable in general, if $\{A_k\}$ is convergent to the set A Wijsman sense, every subsequence of $\{A_k\}$ may not be convergent to A . But if X is separable, then every subsequence of a convergent set sequence is convergent to the same limit.

Definition 14. Let $I \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$ and (X, d) be a separable metric space. For any nonempty closed subsets $A, A_k \subset X$, one says that the sequence $\{A_k\}$ is Wijsman I^* -convergent to A , if and only if there exists a set $M \in F(I)$, $M = \{m = (m_i) : m_i < m_{i+1}, i \in \mathbb{N}\} \subset \mathbb{N}$ such that for each $x \in X$

$$\lim_{k \rightarrow \infty} d(x, A_{m_k}) = d(x, A). \quad (24)$$

In this case, one writes $I_W^* - \lim A_k = A$.

Definition 15. Let $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal in $\mathbb{N}$ and (X, d) be a separable metric space. For any nonempty closed subset A_n in X , one says that the sequence $\{A_n\}$ is Wijsman I -Cauchy sequence if for each $\varepsilon > 0$ and for each $x \in X$, there exists a number $N = N(\varepsilon)$ such that

$$\{n \in \mathbb{N} : |d(x, A_n) - d(x, A_N)| \geq \varepsilon\} \quad (25)$$

belongs to I .

Definition 16. Let $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal in $\mathbb{N}$ and (X, d) be a separable metric space. For any nonempty closed subsets $A_k \subset X$, one says that the sequence $\{A_k\}$ is Wijsman I^* -Cauchy sequences if there exists a set $M = \{m = (m_i) : m_i < m_{i+1}, i \in \mathbb{N}\} \subset \mathbb{N}$, $M \in F(I)$ such that the subsequence $A_M = \{A_{m_k}\}$ is Wijsman Cauchy in X ; that is,

$$\lim_{k, p \rightarrow \infty} |d(x, A_{m_k}) - d(x, A_{m_p})| = 0. \quad (26)$$

Now we will prove that Wijsman I -convergence implies the Wijsman I -Cauchy condition.

Theorem 17. Let I be an arbitrary admissible ideal and let X be a separable metric space. Then $I_W\text{-}\lim A_n = A$ implies that $\{A_n\}$ is Wijsman I -Cauchy sequence.

Proof. Let I be an arbitrary admissible ideal and $I_W\text{-}\lim A_n = A$. Then for each $\varepsilon > 0$ and for each $x \in X$, we have

$$A(x, \varepsilon) = \{n \in \mathbb{N} : |d(x, A_n) - d(x, A)| \geq \varepsilon\} \quad (27)$$

that belongs to I . Since I is an admissible ideal, there exists an $n_0 \in \mathbb{N}$ such that $n_0 \notin A(x, \varepsilon)$.

Let $B(x, \varepsilon) = \{n \in \mathbb{N} : |d(x, A_n) - d(x, A_{n_0})| \geq 2\varepsilon\}$. Taking into account the inequality

$$\begin{aligned} & |d(x, A_n) - d(x, A_{n_0})| \\ & \leq |d(x, A_n) - d(x, A)| + |d(x, A_{n_0}) - d(x, A)|, \end{aligned} \quad (28)$$

we observe that if $n \in B(x, \varepsilon)$, then

$$|d(x, A_n) - d(x, A)| + |d(x, A_{n_0}) - d(x, A)| \geq 2\varepsilon. \quad (29)$$

On the other hand, since $n_0 \notin A(x, \varepsilon)$, we have $|d(x, A_{n_0}) - d(x, A)| < \varepsilon$. Here we conclude that $|d(x, A_n) - d(x, A)| \geq \varepsilon$; hence $n \in A(x, \varepsilon)$. Observe that $B(x, \varepsilon) \subset A(x, \varepsilon) \in I$ for each $\varepsilon > 0$ and for each $x \in X$. This gives that $B(x, \varepsilon) \in I$; that is $\{A_n\}$ is Wijsman I -Cauchy sequence. $\square$

Theorem 18. Let I be an admissible ideal and let X be a separable metric space. If $\{A_n\}$ is Wijsman I^* -Cauchy sequence, then it is Wijsman I -Cauchy sequence.

Proof. Let $\{A_n\}$ be Wijsman I^* -Cauchy sequence; then by the definition, there exists a set $M = \{m = (m_i) : m_i < m_{i+1}, i \in \mathbb{N}\} \subset \mathbb{N}$, $M \in F(I)$ such that

$$|d(x, A_{m_k}) - d(x, A_{m_p})| < \varepsilon \quad (30)$$

for each $\varepsilon > 0$, for each $x \in X$, and for all $k, p > k_0 = k_0(\varepsilon)$.

Let $N = N(\varepsilon) = m_{k_0+1}$. Then for every $\varepsilon > 0$, we have

$$|d(x, A_{m_k}) - d(x, A_N)| < \varepsilon, \quad k > k_0. \quad (31)$$

Now let $H = \mathbb{N} \setminus M$. It is clear that $H \in I$ and that

$$\begin{aligned} A(x, \varepsilon) &= \{n \in \mathbb{N} : |d(x, A_n) - d(x, A_N)| \geq \varepsilon\} \\ &\subset H \cup \{m_1, m_2, \dots, m_{k_0}\} \end{aligned} \quad (32)$$

belongs to I . Therefore, for every $\varepsilon > 0$, we can find a $N = N(\varepsilon)$ such that $A(x, \varepsilon) \in I$; that is, $\{A_n\}$ is Wijsman I -Cauchy sequence. Hence the proof is complete. $\square$

In order to prove that Wijsman I -convergent sequence coincides with Wijsman I^* -convergent sequence for admissible ideals with property (AP), we need the following lemma.

Lemma 19. Let $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal with property (AP) and (X, d) a separable metric space. If $I_W\text{-}\lim_{n \rightarrow \infty} d(x, A_n) = d(x, A)$, then there exists a set $P \in F(I)$ $P = \{p = (p_i) : p_i < p_{i+1}, i \in \mathbb{N}\}$ such that $I_W\text{-}\lim_{k \rightarrow \infty} d(x, A_{p_k}) = d(x, A)$.

Theorem 20. Let $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal with property (AP), let (X, d) be an arbitrary separable metric space and $x = (x_n) \in X$. Then, $I_W\text{-}\lim_{n \rightarrow \infty} d(x, A_n) = d(x, A)$, if and only if there exists a set $P \in F(I)$, $P = \{p = (p_i) : p_i < p_{i+1}, i \in \mathbb{N}\}$ such that $I_W\text{-}\lim_{k \rightarrow \infty} d(x, A_{p_k}) = d(x, A)$.

Now we prove that, a Wijsman I -Cauchy sequence coincides with a Wijsman I^* -Cauchy sequence for admissible ideals with property (AP).

Theorem 21. If $I \subseteq 2^{\mathbb{N}}$ is an admissible ideal with property (AP) and if (X, d) is a separable metric space, then the concepts Wijsman I -Cauchy sequence and Wijsman I^* -Cauchy sequence coincide.

Proof. If a sequence is Wijsman I^* -Cauchy, then it is Wijsman I -Cauchy by Theorem 18 where I does not need to have the (AP) property. Now it is sufficient to prove that $\{A_n\}$ is Wijsman I^* -Cauchy sequence in X under assumption that $\{A_n\}$ is a Wijsman I -Cauchy sequence. Let $\{A_n\}$ be a Wijsman I -Cauchy sequence. Then by definition, there exists a $N = N(\varepsilon)$ such that

$$A(x, \varepsilon) = \{n \in \mathbb{N} : |d(x, A_n) - d(x, A_N)| \geq \varepsilon\} \in I \quad (33)$$

for each $\varepsilon > 0$ and for each $x \in X$.

Let $P_i = \{n \in \mathbb{N} : |d(x, A_n) - d(x, A_{m_i})| < 1/i, i = 1, 2, \dots\}$ where $m_i = N(1/i)$. It is clear that $P_i \in F(I)$ for $i = 1, 2, \dots$. Since I has (AP) property, then by Lemma 7 there exists a set $P \subset \mathbb{N}$ such that $P \in F(I)$ and $P \setminus P_i$ is finite for all i . Now we show that

$$\lim_{n, m \rightarrow \infty} |d(x, A_n) - d(x, A_m)| = 0. \quad (34)$$

To prove this, let $\varepsilon > 0$, $x \in X$, and $j \in \mathbb{N}$ such that $j > 2/\varepsilon$. If $m, n \in P$ then $P \setminus P_j$ is finite set, therefore there exists $k = k(j)$ such that

$$\begin{aligned} & |d(x, A_n) - d(x, A_{m_j})| < \frac{1}{j}, \\ & |d(x, A_m) - d(x, A_{m_j})| < \frac{1}{j} \end{aligned} \quad (35)$$

for all $m, n > k(j)$. Hence it follows that

$$\begin{aligned} & |d(x, A_n) - d(x, A_m)| < |d(x, A_n) - d(x, A_{m_j})| \\ & \quad + |d(x, A_m) - d(x, A_{m_j})| < \varepsilon \end{aligned} \quad (36)$$

for $m, n > k(j)$.

Thus, for any $\varepsilon > 0$, there exists $k = k(\varepsilon)$ and $n, m \in P \in F(I)$:

$$|d(x, A_n) - d(x, A_m)| < \varepsilon. \quad (37)$$

This shows that the sequences $\{A_n\}$ is a Wijsman I^* -Cauchy sequence. $\square$

Theorem 22. Let I be an admissible ideal and (X, d) a separable metric space. Then $I_W^* - \lim A_k = A$ implies that $\{A_n\}$ is a Wijsman I -Cauchy sequence.

Proof. Let $I_W^* - \lim A_k = A$. Then by definition there exists a set $M \in F(I)$, $M = \{m = (m_i) : m_i < m_{i+1}, i \in \mathbb{N}\} \subset \mathbb{N}$ such that

$$\lim_{k \rightarrow \infty} d(x, A_{m_k}) = d(x, A) \quad (38)$$

for each $\varepsilon > 0$ and for each $x \in X$, and $k, p > k_0$,

$$\begin{aligned} & \left| d(x, A_{m_k}) - d(x, A_{m_p}) \right| \\ & < \left| d(x, A_{m_k}) - d(x, A) \right| + \left| d(x, A_{m_p}) - d(x, A) \right| \quad (39) \\ & < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore,

$$\lim_{k, p \rightarrow \infty} \left| d(x, A_{m_k}) - d(x, A_{m_p}) \right| = 0. \quad (40)$$

Hence, $\{A_n\}$ is a Wijsman I -Cauchy sequence. $\square$

Theorem 23. Let I be an admissible ideal and (X, d) a separable metric space. If the ideal I has property (AP) and if (X, d) is an arbitrary metric space, then for arbitrary sequence $\{A_n\}_{n \in \mathbb{N}}$ of elements of X $I_W - \lim A_n = A$ implies $I_W^* - \lim A_n = A$.

Proof. Suppose that I satisfies condition (AP). Let $I_W - \lim A_n = A$. Then

$$T(\varepsilon, x) = \{n \in \mathbb{N} : |d(x, A_n) - d(x, A)| \geq \varepsilon\} \in I \quad (41)$$

for each $\varepsilon > 0$ and for each $x \in X$. Put

$$\begin{aligned} T_1 &= \{n \in \mathbb{N} : |d(x, A_n) - d(x, A)| \geq 1\}, \\ T_n &= \left\{n \in \mathbb{N} : \frac{1}{n} \leq |d(x, A_n) - d(x, A)| < \frac{1}{n-1}\right\} \end{aligned} \quad (42)$$

for $n \geq 2$, and $n \in \mathbb{N}$. Obviously $T_i \cap T_j = \emptyset$ for $i \neq j$. By condition (AP) there exists a sequence of sets $\{V_n\}_{n \in \mathbb{N}}$ such that $T_j \Delta V_j$ are finite sets for $j \in \mathbb{N}$ and $V = \bigcup_{j=1}^{\infty} V_j \in I$. It is sufficient to prove that for $M = \mathbb{N} \setminus V$, $M = \{m = (m_i) : m_i < m_{i+1}, i \in \mathbb{N}\} \in F(I)$, we have $\lim_{k \rightarrow \infty} d(x, A_{m_k}) = d(x, A)$.

Let $\gamma > 0$. Choose $k \in \mathbb{N}$ such that $1/(k+1) < \gamma$. Then

$$\{n \in \mathbb{N} : |d(x, A_n) - d(x, A)| \geq \gamma\} \subset \bigcup_{j=1}^{k+1} T_j. \quad (43)$$

Since $T_j \Delta V_j$, $j = 1, 2, \dots$ are finite sets, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} & \left(\bigcup_{j=1}^{k+1} V_j \right) \cap \{n \in \mathbb{N} : n > n_0\} \\ &= \left(\bigcup_{j=1}^{k+1} T_j \right) \cap \{n \in \mathbb{N} : n > n_0\}. \end{aligned} \quad (44)$$

If $n > n_0$ and $n \notin V$, so $n \notin \bigcup_{j=1}^{k+1} V_j$ and by (44) $n \notin \bigcup_{j=1}^{k+1} T_j$. But then $|d(x, A_n) - d(x, A)| < 1/(n+1) < \gamma$ for each $x \in X$, so we have $\lim_{k \rightarrow \infty} d(x, A_{m_k}) = d(x, A)$. $\square$

3. Wijsman I -Limit Points and Wijsman I -Cluster Points Sequences of Sets

In this section, we introduce Wijsman I -limit points of sequences of sets and Wijsman I -cluster points of sequences of sets, prove some basic properties of these concepts, and establish some basic theorems.

Definition 24. Let $I \subseteq 2^{\mathbb{N}}$ a proper ideal in $\mathbb{N}$ and (X, d) a separable metric space. For any nonempty closed subsets $A_n, B_n \subset X$, one says that the sequences $\{A_n\}$ and $\{B_n\}$ are almost equal with respect to I if

$$\{n \in \mathbb{N} : A_n \neq B_n\} \in I, \quad (45)$$

and we write I -a.a.n $A_n = B_n$.

Definition 25. Let $I \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$ and let (X, d) be a separable metric space; A_n is nonempty closed subset of X . If $\{A_n\}_K$ is subsequence of $\{A_n\}$ and $K := \{n(j) : j \in \mathbb{N}\}$, then we abbreviate $\{A_{n_j}\}$ by $\{A_n\}_K$. If $K \in I$, then $\{A_n\}_K$ subsequence is called thin subsequence of $\{A_n\}$. If $K \notin I$, then $\{A_n\}_K$ subsequence is called nonthin subsequence of $\{A_n\}$.

Definition 26. Let $I \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$ and let (X, d) be a separable metric space, for any nonempty closed subsets $A_k \subset X$. One has the following.

- (i) $A \in X$ is said to be a Wijsman I -limit point of $\{A_n\}$ provided that there is a set $M = \{m = (m_i) : m_i < m_{i+1}, i \in \mathbb{N}\} \subset \mathbb{N}$ such that $M \notin I$ and for each $x \in X$ $\lim_{k \rightarrow \infty} d(x, A_{m_k}) = d(x, A)$.
- (ii) $A \in X$ is said to be a Wijsman I -cluster point of $\{A_n\}$ if and only if for each $\varepsilon > 0$, for each $x \in X$, we have

$$\{n \in \mathbb{N} : |d(x, A_n) - d(x, A)| < \varepsilon\} \notin I. \quad (46)$$

Denote by $I_W(\Lambda_{\{A_n\}})$, $I_W(\Gamma_{\{A_n\}})$, and $L_{\{A_n\}}$ the set of all Wijsman I -limit, Wijsman I -cluster, and Wijsman limit points of $\{A_n\}$, respectively.

For the sequences $\{A_n\}$, $I_W(\Gamma_{\{A_n\}}) \subseteq I_W(L_{\{A_n\}})$. Let $A \in I_W(\Gamma_{\{A_n\}})$. Then for each sequence $\{A_n\} \subset X$, we have $\lim_{k \rightarrow \infty} d(x, A_{m_k}) = d(x, A)$ which means that $A \in L_{\{A_n\}}$.

Theorem 27. Let $I \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$ and let (X, d) be a separable metric space. Then for each sequence $\{A_n\} \subset X$ one has $I_W(\Lambda_{\{A_n\}}) \subset I_W(\Gamma_{\{A_n\}})$.

Proof. Let $A \in I_W(\Lambda_{\{A_n\}})$. Then, there exists $M = \{m_1 < m_2 < \dots\} \subset \mathbb{N}$ such that $M = \{m = (m_i) : m_i < m_{i+1}, i \in \mathbb{N}\} \notin I$ and

$$\lim_{k \rightarrow \infty} d(x, A_{m_k}) = d(x, A). \quad (47)$$

According to (47), there exists $k_0 \in \mathbb{N}$ such that for each $\varepsilon > 0$, for each $x \in X$ and $k > k_0$, $|d(x, A_{m_k}) - d(x, A)| < \varepsilon$. Hence,

$$\begin{aligned} & \{k \in \mathbb{N} : |d(x, A_{m_k}) - d(x, A)| < \varepsilon\} \\ & \supseteq M \setminus \{m_1, m_2, \dots, m_{k_0}\}. \end{aligned} \quad (48)$$

Then, the set on the right hand side of (48) does not belong to I ; therefore

$$\{k \in \mathbb{N} : |d(x, A_{m_k}) - d(x, A)| < \varepsilon\} \notin I \quad (49)$$

which means that $A \in I_W(\Gamma_{\{A_n\}})$. $\square$

Theorem 28. Let $I \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$ and let (X, d) be a separable metric space. Then for each sequence $\{A_n\} \subset X$ one has $I_W(\Gamma_{\{A_n\}}) \subseteq L_{\{A_n\}}$.

Proof. Let $A \in I_W(\Gamma_{\{A_n\}})$. Then for each $\varepsilon > 0$ and for each $x \in X$, we have

$$\{n \in \mathbb{N} : |d(x, A_n) - d(x, A)| < \varepsilon\} \notin I. \quad (50)$$

Let

$$K_n := \left\{n \in \mathbb{N} : |d(x, A_n) - d(x, A)| < \frac{1}{n}\right\} \quad (51)$$

for $n \in \mathbb{N}$. $\{K_n\}_{n=1}^{\infty}$ is decreasing sequence of infinite subsets of $\mathbb{N}$. Hence $K = \{n = (n_i) : n_i < n_{i+1}, i \in \mathbb{N}\} \notin I$ such that $\lim_{n \rightarrow \infty} d(x, A_{n_i}) = d(x, A)$ which means that $A \in L_{\{A_n\}}$. $\square$

Theorem 29. Let $I \subseteq 2^{\mathbb{N}}$ a proper ideal in $\mathbb{N}$, (X, d) a separable metric space, and A_k, B_k nonempty subsets of X . If $\{A_k\} = \{B_k\}$ I -a.a.k for $k \in \mathbb{N}$, then $I_W(\Gamma_{\{A_k\}}) = I_W(\Gamma_{\{B_k\}})$ and $I_W(\Lambda_{\{A_k\}}) = I_W(\Lambda_{\{B_k\}})$.

Proof. If $\{A_k\} = \{B_k\}$ a.a.k for $k \in \mathbb{N}$, then

$$K := \{k \in \mathbb{N} : A_k \neq B_k\} \in I \quad (52)$$

Let $A \in I_W(\Gamma_{\{A_k\}})$. For each $\varepsilon > 0$ and for each $x \in X$ we have

$$\{k \in \mathbb{N} : |d(x, A_k) - d(x, A)| < \varepsilon\} \notin I, \quad (53)$$

$\forall \varepsilon > 0$. If $\{A_k\} = \{B_k\}$ I -a.a.k, then $\{k \in \mathbb{N} : |d(x, B_k) - d(x, A)| < \varepsilon\} \notin I$ which means that $A \in I_W(\Gamma_{\{B_k\}})$; hence $I_W(\Gamma_{\{A_k\}}) \subset I_W(\Gamma_{\{B_k\}})$. Similarly we can also prove that $I_W(\Gamma_{\{B_k\}}) \subset I_W(\Gamma_{\{A_k\}})$. So we have $I_W(\Gamma_{\{A_k\}}) = I_W(\Gamma_{\{B_k\}})$.

Now, we show that $I_W(\Lambda_{\{A_k\}}) = I_W(\Lambda_{\{B_k\}})$. Let $A \in I_W(\Lambda_{\{A_k\}})$. Then there exists a set $M = \{m = (m_i) : m_i < m_{i+1}, i \in \mathbb{N}\} \subset \mathbb{N}$ such that $M \notin I$ and

$$\begin{aligned} & \lim_{k \rightarrow \infty} d(x, A_{m_k}) = d(x, A), \\ & M = \{k : k \in M \text{ and } A_k \neq B_k\} \\ & \cup \{k : k \in M \text{ and } A_k = B_k\}, \end{aligned} \quad (54)$$

$M \notin I$, and hence $\{k : k \in M \text{ and } A_k = B_k\} \notin I$. Then there exists

$$P = \{p = (p_i) : p_i < p_{i+1}, i \in \mathbb{N}\} \notin I \quad (55)$$

such that

$$\lim_{k \rightarrow \infty} d(x, B_{p_k}) = d(x, A) \quad (56)$$

which means that $A \in I_W(\Lambda_{\{B_k\}})$. Similarly we can also prove that $I_W(\Lambda_{\{B_k\}}) \subset I_W(\Lambda_{\{A_k\}})$. Therefore we have $I_W(\Lambda_{\{A_k\}}) = I_W(\Lambda_{\{B_k\}})$. $\square$

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