In the present paper, we establish some new inequalities similar to Hilbert’s type inequalities. Our results provide new estimates to these types of inequalities.

1. Introduction

The well-known classical Hilbert’s double-series inequality can be stated as follows [1, page 253].

Theorem A. If \( p_1, p_2 > 1 \) such that \( 1/p_1 + 1/p_2 \geq 1 \) and \( 0 < \lambda = 2 - 1/p_1 - 1/p_2 = 1/q_1 + 1/q_2 \leq 1 \), where, as usual, \( q_1 \) and \( q_2 \) are the conjugate exponents of \( p_1 \) and \( p_2 \), respectively, then

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} \leq K \left( \sum_{m=1}^{\infty} a_m^{p_1} \right)^{1/p_1} \left( \sum_{n=1}^{\infty} b_n^{p_2} \right)^{1/p_2},
\]

where \( K = K(p_1, p_2) \) depends on \( p_1 \) and \( p_2 \) only.

In recent years, several authors [1–18] have given considerable attention to Hilbert’s double-series inequality together with its integral version, inverse version, and various generalizations. In particular, Pachpatte [11] established an inequality similar to inequality (1) as follows.

Theorem 1. Let \( p > 1 \) be constant and \( 1/p + 1/q = 1 \). If \( a(s) \) and \( b(t) \) are real-valued functions defined for \( 0, 1, \ldots, m \) and \( 0, 1, \ldots, n \), respectively, and \( a(0) = b(0) = 0 \). Moreover, define the operators \( \nabla \) by \( \nabla u(t) = u(t) - u(t - 1) \).

Then,

\[
\sum_{s=1}^{m} \sum_{t=1}^{n} \frac{|a(s)| |b(t)|}{q(s)^{p-1} + p|t|^{q-1}} \leq \frac{1}{pq} \sum_{s=1}^{m} (m - s + 1) \left( \sum_{s=1}^{m} a(s) |b(t)|^q \right)^{1/p} \times \left( \sum_{s=1}^{n} (n - t + 1) |b(t)|^p \right)^{1/q}.
\]

\( \Gamma_{pq} \) is the gamma function.

The first aim of this paper is to establish a new inequality similar to Hilbert’s type inequality. Our result provides new estimates to this type of inequality.

Theorem 2. Let \( p > 1 \) be constants, and \( 1/p + 1/q = 1 \). For \( i = 1, 2 \), let \( a_i(s_i, t_i) \) be real-valued functions defined for \( (s_i, t_i) \), where \( s_i = 1, 2, \ldots, m_i; t_i = 1, 2, \ldots, n_i \), and let \( m_i, n_i \) be natural numbers. Let \( a_i(0, t_i) = a_i(s_i, 0) = 0 \). Define the operators \( \nabla_1, \nabla_2 \) by

\[
\nabla_1 v_i(s_i, t_i) = v_i(s_i, t_i) - v_i(s_i - 1, t_i),
\]

\[
\nabla_2 v_i(s_i, t_i) = v_i(s_i, t_i) - v_i(s_i, t_i - 1).
\]

Then,

\[
\sum_{s_1=1}^{m_1} \sum_{t_1=1}^{n_1} \left( \sum_{s_2=1}^{m_2} \sum_{t_2=1}^{n_2} \left| a_1(s_1, t_1) \right|^p + \left| a_2(s_2, t_2) \right|^q \right) \times \left( \Gamma_{pq}(s_1, t_1, s_2, t_2) \right) \times \max \left\{ p(s_1)^{p/q}, q(s_2)^{q/p} \right\}^{-1}
\]

\( \Gamma_{pq} \) is the gamma function.

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On the other hand, let \( a_1(s_1,t_1) \) and \( a_2(s_2,t_2) \) change to \( a_1(s_1) \) and \( a_2(s_2) \), respectively, and, with appropriate transformation, we have

\[
\sum_{s_1=1}^{m_1} \sum_{t_1=1}^{n_1} |a_1(s_1)|^p + |a_2(s_2)|^q
\]

\[
\leq \frac{1}{pq} \left( m_1^{1/q} + m_2^{1/p} \right)
\]

\[
\times \left( \sum_{s_1=1}^{m_1} \sum_{t_1=1}^{n_1} (m_1 - s_1 + 1)(n_1 - t_1 + 1) \right)^{1/p}
\]

\[
\times |\nabla^2 a_1(s_1,t_1)|^p
\]

\[
\times \left( \sum_{s_2=1}^{m_2} \sum_{t_2=1}^{n_2} (m_2 - s_2 + 1)(n_2 - t_2 + 1) \right)^{1/p}
\]

\[
\times |\nabla^2 a_2(s_2,t_2)|^p
\]

(4)

\[
\Gamma_{p,q} (s_1,t_1,s_2,t_2)
\]

\[
= pqS \left( \sum_{s_1=1}^{m_1} \sum_{t_1=1}^{n_1} |\nabla^2 a_1(s_1,t_1)|^p \right)
\]

\[
\times \left( \sum_{s_2=1}^{m_2} \sum_{t_2=1}^{n_2} |\nabla^2 a_2(s_2,t_2)|^q \right)^{1/q}
\]

(5)

\[
S(h) = \frac{h^{1/(h-1)}}{e \log h^{1/(h-1)}}, \quad h \neq 1.
\]

(6)

Remark 3. Inequality (4) is just a similar version of the following inequality established by Pachpatte [11]:

\[
\sum_{s_1=1}^{m_1} \sum_{t_1=1}^{n_1} \left( \sum_{k=1}^{m_1} \sum_{r=1}^{n_1} |a(s,t)||b(k,r)| \right)^{1/q} \left( \sum_{s_1=1}^{m_1} \sum_{t_1=1}^{n_1} (s - k + 1)(t - r + 1) \right)^{1/p}
\]

\[
\times |\nabla^2 a(s,t)|^p \left( \sum_{s_2=1}^{m_2} \sum_{t_2=1}^{n_2} (z - k + 1)(w - r + 1) \right)^{1/q}
\]

\[
\times |\nabla^2 b(k,r)|^q
\]

(7)

Another aim of this paper is to establish a new integral inequality similar to Hilbert’s type inequality.
Theorem 5. Let \( p > 1 \), and \( 1/p + 1/q = 1 \). For \( i = 1, 2 \), let \( h_i \geq 1 \), \( f_i(s_i, t_i) \) be real-valued differentiable functions defined on \( [0, x_i] \times [0, y_i] \), where \( x_i \in (0, \infty) \), \( y_i \in (0, \infty) \), and \( f_i(0, t_i) = f_i(s_i, 0) = 0 \). As usual, partial derivatives of \( f_i \) are denoted by \( D_1 f_i, D_2 f_i, D_{12} f_i = D_{1} f_i D_{2} f_i \), and so forth. Let

\[
D_{12}^* f_i(s_i, t_i) = D_2 \left( h_i f_i^{h_i-1}(s_i, t_i) \cdot D_1 f_i(s_i, t_i) \right).
\]

Then,

\[
\int_0^{x_1} \int_0^{y_1} \left( \int_0^{x_2} \int_0^{y_2} \frac{|f^{h_1}_i(s_1, t_1)|^p + |f^{h_2}_i(s_2, t_2)|^q}{L_{pq} (s_1, t_1, s_2, t_2)} \max \left\{ p(s_1 t_1)^{p/q}, q(s_2 t_2)^{q/p} \right\} ds_1 dt_1 \right) ds_2 dt_2 
\]

\[
\leq \frac{1}{pq} (x_1 y_1)^{1/p} (x_2 y_2)^{1/p} \left( \int_0^{x_1} \int_0^{y_1} (x_1 - s_1) (y_1 - t_1) \left| D_{12} f_1(s_1, t_1) \right|^p ds_1 dt_1 \right)^{1/p} 
\]

\[
\times \left( \int_0^{x_2} \int_0^{y_2} (x_2 - s_2) (y_2 - t_2) \left| D_{12} f_2(s_2, t_2) \right|^q ds_2 dt_2 \right)^{1/q},
\]

where

\[
L_{pq} (s_1, t_1, s_2, t_2) = pqS \left( \int_0^{x_1} \int_0^{y_1} \left| D_{12} f_1(\xi_1, \eta_1) \right|^p d\eta_1 d\xi_1, \right) \int_0^{x_2} \int_0^{y_2} \left| D_{12} f_2(\xi_2, \eta_2) \right|^q d\eta_2 d\xi_2,
\]

and \( S(h) \) is as in (6).

Remark 6. Inequality (13) is just a similar version of the following inequality established by Pachpatte [11]:

\[
\int_0^{x_1} \int_0^{y_1} \left( \int_0^{x_2} \int_0^{y_2} \frac{|f_i(s_1, t_1)| |f_j(s_2, t_2)|}{q(s_1 t_1)^{p/q} + p(s_2 t_2)^{q/p}} ds_1 dt_1 \right) ds_2 dt_2 
\]

\[
\leq \frac{1}{pq} (x_1 y_1)^{1/p} (x_2 y_2)^{1/p} \left( \int_0^{x_1} \int_0^{y_1} (x_1 - s_1) (y_1 - t_1) \left| D_{12} f_i(s_1, t_1) \right|^p ds_1 dt_1 \right)^{1/p} 
\]

\[
\times \left( \int_0^{x_2} \int_0^{y_2} (x_2 - s_2) (y_2 - t_2) \left| D_{12} f_j(s_2, t_2) \right|^q ds_2 dt_2 \right)^{1/q},
\]

(15)

On the other hand, let \( f_i(s_1, t_1) \) and \( f_j(s_2, t_2) \) change to \( f_i(s_i) \) and \( f_j(s_2) \), respectively, and, with appropriate transformation, we have

\[
\int_0^{x_1} \int_0^{y_1} \left( \int_0^{x_2} \int_0^{y_2} \frac{|f_i^{h_1}(s_1)|^p + |f_j^{h_2}(s_2)|^q}{L_{pq} (s_1, s_2)} \max \left\{ p{1/\alpha}{\beta}, q{1/\beta}{\alpha} \right\} ds_1 dt_1 \right) 
\]

\[
\leq \frac{1}{pq} (x_1 y_1)^{1/p} (x_2 y_2)^{1/p} \left( \int_0^{x_1} \int_0^{y_1} (x_1 - s_1) (y_1 - t_1) \left| D_{12} f_i(s_1, t_1) \right|^p ds_1 dt_1 \right)^{1/p} 
\]

\[
\times \left( \int_0^{x_2} \int_0^{y_2} (x_2 - s_2) (y_2 - t_2) \left| D_{12} f_j(s_2, t_2) \right|^q ds_2 dt_2 \right)^{1/q},
\]

(16)

where

\[
L_{pq} (s_1, s_2) = pqS \left( \int_0^{x_1} \int_0^{y_1} \left| D_{12} f_i(\sigma_1) \right|^p d\sigma_1, \right) \int_0^{x_2} \int_0^{y_2} \left| D_{12} f_j(\sigma_2) \right|^q d\sigma_2,
\]

and this is just a similar version of inequality (11) in Theorem 4.

2. Proof of Theorems

Proof of Theorem 2. From the hypotheses of Theorem 2, we have

\[
|a_1(s_1, t_1)| \leq \sum_{\xi_1=1}^{x_1} \sum_{\eta_1=1}^{y_1} \left| \nabla V_1 a_1(\xi_1, \eta_1) \right|,
\]

(18)

\[
|a_2(s_2, t_2)| \leq \sum_{\xi_2=1}^{x_2} \sum_{\eta_2=1}^{y_2} \left| \nabla V_2 a_2(\xi_2, \eta_2) \right|.
\]

By using \( \xi_1 \) and \( \eta_1 \) inequality and noticing the reverse Young's inequality [19],

\[
s_1^{1/\alpha} s_2^{1/\beta} S \left( \frac{s_1}{s_2} \frac{1/\alpha}{1/\beta} \right) \geq \frac{s_1}{\alpha} + \frac{s_2}{\beta},
\]

(19)
for positive real numbers $s_1, s_2$ and $1/\alpha + 1/\beta = 1$, $\alpha > 1$, where $S(h)$ is as in (6). Hence,

$$
\frac{|a_1(s_1, t_1)|^p + |a_2(s_2, t_2)|^q}{\max \left\{ p(s_1 t_1)^{p/q} , q(s_2 t_2)^{q/p} \right\}}
\leq \frac{1}{p} \sum_{\xi_1 = 1}^{t_1} \sum_{\eta_1 = 1}^{t_1} |\nabla^2 V_{1} a_1 (\xi_1, \eta_1)|^p 
+ \frac{1}{q} \sum_{\xi_2 = 1}^{t_2} \sum_{\eta_2 = 1}^{t_2} |\nabla^2 V_{2} a_2 (\xi_2, \eta_2)|^q 
\leq S \left( \sum_{\xi_1 = 1}^{t_1} \sum_{\eta_1 = 1}^{t_1} |\nabla^2 V_{1} a_1 (\xi_1, \eta_1)|^p \right) 
\times \left( \sum_{\xi_1 = 1}^{t_1} \sum_{\eta_1 = 1}^{t_1} |\nabla^2 V_{1} a_1 (\xi_1, \eta_1)|^q \right)^{1/p} 
\times \left( \sum_{\xi_2 = 1}^{t_2} \sum_{\eta_2 = 1}^{t_2} |\nabla^2 V_{2} a_2 (\xi_2, \eta_2)|^q \right)^{1/q} 
\times \left( \sum_{\xi_2 = 1}^{t_2} \sum_{\eta_2 = 1}^{t_2} |\nabla^2 V_{2} a_2 (\xi_2, \eta_2)|^q \right)^{1/q}.
$$

(20)

Dividing both sides of (20) by

$$
\Gamma_{p, q} (s_1, t_1, s_2, t_2) = pqS \left( \sum_{\xi_1 = 1}^{t_1} \sum_{\eta_1 = 1}^{t_1} |\nabla^2 V_{1} a_1 (\xi_1, \eta_1)|^p \right) 
\times \left( \sum_{\xi_1 = 1}^{t_1} \sum_{\eta_1 = 1}^{t_1} |\nabla^2 V_{1} a_1 (\xi_1, \eta_1)|^q \right)^{1/p} 
\times \left( \sum_{\xi_2 = 1}^{t_2} \sum_{\eta_2 = 1}^{t_2} |\nabla^2 V_{2} a_2 (\xi_2, \eta_2)|^q \right)^{1/q},
$$

(21)

taking the sum of both sides of (20) over $t_i$ and $s_i$ from 1 to $m_i$ and $n_i$ ($i = 1, 2$), respectively, and making use of Hölder's inequality, we have

$$
\sum_{s_1 = 1}^{m_1} \sum_{t_1 = 1}^{n_1} \left( \sum_{s_2 = 1}^{m_2} \sum_{t_2 = 1}^{n_2} \left( \left| a_1(s_1, t_1) \right|^p + \left| a_2(s_2, t_2) \right|^q \right) \right) 
\times \left( \Gamma_{p, q} (s_1, t_1, s_2, t_2) \right) 
\times \left( \max \left\{ p(s_1 t_1)^{p/q} , q(s_2 t_2)^{q/p} \right\} \right)^{-1}
\leq \frac{1}{pq} \sum_{s_1 = 1}^{m_1} \sum_{t_1 = 1}^{n_1} \left( \sum_{\xi_1 = 1}^{t_1} \sum_{\eta_1 = 1}^{t_1} |\nabla^2 V_{1} a_1 (\xi_1, \eta_1)|^p \right)^{1/p} 
\times \sum_{s_2 = 1}^{m_2} \sum_{t_2 = 1}^{n_2} \left( \sum_{\xi_2 = 1}^{t_2} \sum_{\eta_2 = 1}^{t_2} |\nabla^2 V_{2} a_2 (\xi_2, \eta_2)|^q \right)^{1/q} 
\leq \frac{1}{pq} (m_1 n_1)^{1/q}
\times \left( \sum_{s_1 = 1}^{m_1} \sum_{t_1 = 1}^{n_1} \sum_{\xi_1 = 1}^{t_1} \sum_{\eta_1 = 1}^{t_1} |\nabla^2 V_{1} a_1 (\xi_1, \eta_1)|^p \right)^{1/p} 
\times \left( \sum_{s_2 = 1}^{m_2} \sum_{t_2 = 1}^{n_2} \sum_{\xi_2 = 1}^{t_2} \sum_{\eta_2 = 1}^{t_2} |\nabla^2 V_{2} a_2 (\xi_2, \eta_2)|^q \right)^{1/q}.
$$

(22)

This completes the proof.

**Proof of Theorem 5.** From the hypotheses of Theorem 5, we obtain for $i = 1, 2$:

$$
\int_0^t f_i^{\text{h}\iota}(s_i, t_i) = f_i^{\text{h}\iota}(s_i, t_i) - f_i^{\text{h}\iota}(s_i, 0) - f_i^{\text{h}\iota}(s_i, 0) + f_i^{\text{h}\iota}(0, 0) 
= \int_0^t D_i f_i^{\text{h}\iota}(\xi_i, t_i) d\xi_i - \int_0^t D_i f_i^{\text{h}\iota}(\xi_i, 0) d\xi_i 
= \int_0^t \left( D_i f_i^{\text{h}\iota}(\xi_i, t_i) - D_i f_i^{\text{h}\iota}(\xi_i, 0) \right) d\xi_i.
$$
\[
\begin{align*}
= \int_0^s \int_0 t \left. \frac{d}{dt} \left[ \frac{\partial}{\partial s} \right] \frac{1}{2} \left( \frac{\partial}{\partial t} \right) f(t,s) \right|_{t=0}^t \, ds \\
= \int_0^s \int_0 t \frac{d}{dt} \left[ \frac{d}{ds} \right] f(t,s) \, ds \\
= \int_0^s \int_0 t \frac{d}{dt} \left[ \frac{d}{ds} \right] f(t,s) \, ds.
\end{align*}
\]

From (23), Hölder's integral inequality and in view of the reverse Young's inequality (19), we have

\[
\left| f_{i}^{h} (s, t) \right|^{p} + \left| f_{i}^{h} (s, t) \right|^{q} \leq \frac{1}{p} \int_0^t \int_0 s \left| D_{ij} f_{i} (s, t) \right|^{p} \, ds dt.
\]

Integrating both sides of (24) over \( s \) and \( t \) from 1 to \( x_1 \) and \( y_1 (i = 1, 2) \), respectively, and by using Hölder's integral inequality, we arrive at

\[
\begin{align*}
\int_0^{x_1} & \int_0^{y_1} \left( \int_0^{x_2} \int_0^{y_2} \left| f_{1}^{h} (s, t) \right|^{p} + \left| f_{2}^{h} (s, t) \right|^{q} \right) ds dt \\
& \leq \frac{1}{pq} \left( \int_0^{x_1} \int_0^{y_1} \left( \int_0^{x_2} \int_0^{y_2} \left| D_{ij} f_{i} (s, t) \right|^{p} \, ds dt \right)^{1/p} \, ds dt \right)^{1/p} \\
& \times \left( \int_0^{x_1} \int_0^{y_1} \left( \int_0^{x_2} \int_0^{y_2} \left| D_{ij} f_{i} (s, t) \right|^{q} \, ds dt \right)^{1/q} \, ds dt \right)^{1/q}.
\end{align*}
\]

This completes the proof.

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References


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